Integrals evaluated in terms of Catalan's constant

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Catalan's constant, named after E. C. Catalan (1814–1894) and usually denoted by G, is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots.$$

It is, of course, a close relative of

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \tfrac{3}{4}\zeta(2) = \frac{\pi^2}{8}.$$

The numerical value is $G \approx 0.9159656$. It is not known whether G is irrational: this remains a stubbornly unsolved problem. The best hope for a solution might appear to be the method of Beukers [1] to prove the irrationality of $\zeta(2)$ directly from the series, but it is not clear how to adapt this method to G.

A remarkable assortment of seemingly very different definite integrals equate to G, or are evaluated in terms of G. A compilation of no fewer than eighty integral and series representations for G, with proofs, is given by Bradley [2]. Another compilation, based on *Mathematica*, is [3]. Here we will present a selection of some of the simpler integrals and double integrals that are evaluated in terms of G, including a few that are actually not to be found in [2] or [3].

The most basic one is

$$\int_0^1 \frac{\tan^{-1} x}{x}\,dx = G, \tag{1}$$

obtained at once by termwise integration of the series

$$\frac{\tan^{-1} x}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}.$$

Termwise integration (for those who care) is easily justified. Write

$$s_{2n}(x) = \sum_{r=0}^{n} (-1)^n \frac{x^{2r}}{2r+1}.$$

Since the series is alternating, with terms decreasing in magnitude, we have $\tan^{-1} x/x = s_{2n}(x) + r_{2n}(x)$, where $|r_{2n}(x)| \le x^{2n+2}/(2n+3)$, so that $\int_0^1 r_{2n}(x)\,dx \to 0$ as $n \to \infty$.

We now embark on a round trip of integrals derived directly from (1). Most of them can be seen in [4], but we repeat them here for completeness. First, the substitution $x = \tan\theta$

gives

$$G = \int_0^{\pi/4} \frac{\theta}{\tan\theta} \sec^2\theta \, d\theta = \int_0^{\pi/4} \frac{\theta}{\sin\theta\cos\theta} \, d\theta. \tag{2}$$

Since $\tan\theta + \cot\theta = 1/(\sin\theta\cos\theta)$, (2) can be rewritten

$$\int_0^{\pi/4} \theta(\tan\theta + \cot\theta) \, d\theta = G. \tag{3}$$

The substitution $\theta = 2\phi$ gives one of the most important equivalent forms:

$$\int_0^{\pi/2} \frac{\theta}{\sin\theta} \, d\theta = \int_0^{\pi/4} \frac{4\phi}{\sin 2\phi} \, d\phi = \int_0^{\pi/4} \frac{2\phi}{\sin\phi\cos\phi} \, d\phi = 2G. \tag{4}$$

We mention in passing that (4) can be rewritten in terms of the gamma function. Recall that by Euler's reflection formula, $\Gamma(1+x)\Gamma(1-x) = \pi x/(\sin \pi x)$. Substituting $\pi x = \theta$ in (4), we deduce:

$$\int_0^{1/2} \Gamma(1+x)\Gamma(1-x) \, dx = \frac{2G}{\pi}.$$

The substitution $x = e^{-t}$ in (1) gives at once

$$G = \int_0^{\infty} \tan^{-1}(e^{-t}) \, dt. \tag{5}$$

Next, we integrate by parts in (1), obtaining

$$G = \Big[\tan^{-1} x \log x \Big]_0^1 - \int_0^1 \frac{\log x}{1+x^2} \, dx.$$

The first term is zero, since $\tan^{-1} x \log x \sim x \log x \to 0$ as $x \to 0^+$, hence

$$\int_0^1 \frac{\log x}{1+x^2} \, dx = -G. \tag{6}$$

Now substituting $x = 1/y$, we deduce

$$\int_1^{\infty} \frac{\log x}{1+x^2} \, dx = G. \tag{7}$$

Substituting $x = e^y$ in (7), we obtain

$$G = \int_0^{\infty} \frac{y}{1+e^{2y}} e^y \, dy = \int_0^{\infty} \frac{y}{2\cosh y} \, dy,$$

hence

$$\int_0^{\infty} \frac{x}{\cosh x} \, dx = 2G. \tag{8}$$

Now substituting $x = \tan\theta$ in (6), we obtain

$$-G = \int_0^{\pi/4} \frac{\log\tan\theta}{\sec^2\theta} \sec^2\theta \, d\theta = \int_0^{\pi/4} \log\tan\theta \, d\theta. \tag{9}$$

Since $(1+\cos\theta)/(1-\cos\theta) = \cot^2\frac{\theta}{2}$, we can deduce

$$\int_0^{\pi/2} \log\frac{1+\cos\theta}{1-\cos\theta} \, d\theta = -2\int_0^{\pi/2} \log\tan\frac{\theta}{2} \, d\theta = -4\int_0^{\pi/4} \log\tan\phi \, d\phi = 4G. \tag{10}$$

This, in turn, leads to an interesting series representation. Given the series

$$\tfrac{1}{2}\log\frac{1+x}{1-x} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

and the well-known identity

$$\int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta = \int_0^{\pi/2} \sin^{2n+1}\theta \, d\theta = \frac{2.4.\ldots(2n)}{1.3.\ldots(2n+1)}, \tag{11}$$

we deduce

$$2G = \sum_{n=0}^{\infty} \frac{2.4.\ldots(2n)}{1.3.\ldots(2n-1)(2n+1)^2} = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!(2n+1)^2} = \sum_{n=0}^{\infty} \frac{2^{2n}}{\binom{2n}{n}(2n+1)^2}. \tag{12}$$

Using (11) again (but none of the earlier integrals), we now establish

$$\int_0^{\pi/2} \sinh^{-1}(\sin\theta) \, d\theta = G. \tag{13}$$

Write

$$a_n = \frac{1.3.\ldots(2n-1)}{2.4.\ldots(2n)}.$$

By the binomial series,

$$\frac{1}{(1+x^2)^{1/2}} = \sum_{n=0}^{\infty} (-1)^n a_n x^{2n}.$$

Hence

$$\sinh^{-1} x = \int_0^x \frac{1}{(1+t^2)^{1/2}} \, dt = \sum_{n=0}^{\infty} (-1)^n \frac{a_n}{2n+1} x^{2n+1}.$$

As above,

$$\int_0^{\pi/2} \sin^{2n+1}\theta \, d\theta = \frac{1}{a_n(2n+1)},$$

so by a neat cancellation of a_n,

$$\int_0^{\pi/2} \sinh^{-1}(\sin\theta) \, d\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = G.$$

We remark that the same method, applied to $\sin^{-1}(\sin\theta)$ $(= \theta)$ delivers the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \int_0^{\pi/2} \theta \, d\theta = \frac{\pi^2}{8}.$$

Integrals of logsine type and applications

A further application of (9) is to integrals of the logsine type, of which there are a rich variety. We start by stating the most basic such integral, which does not involve G:

$$\int_0^{\pi/2} \log \sin\theta \, d\theta = \int_0^{\pi/2} \log\cos\theta \, d\theta = -\frac{\pi}{2}\log 2. \tag{14}$$

Our proof follows [5, p. 246]). The substitution $\theta = \frac{\pi}{2} - \phi$ shows that the two integrals are equal: denote them both by I. Also, the substitution $\theta = \pi - \phi$ gives $\int_{\pi/2}^{\pi} \log\sin\theta \, d\theta = I$. Hence $2I = \int_0^{\pi} \log\sin\theta \, d\theta$. Now substituting $\theta = 2\phi$ and using $\sin 2\phi = 2\sin\phi\cos\phi$, we have

$$\begin{aligned} 2I &= 2\int_0^{\pi/2} \log\sin 2\phi \, d\phi \\ &= 2\int_0^{\pi/2} (\log\sin\phi \, + \log\cos\phi + \log 2) \, d\phi \\ &= 4I + \pi\log 2, \end{aligned}$$

hence (14).

Note that the equality of the two integrals in (14) (without knowing their value) implies that $\int_0^{\pi/2} \log\tan\theta \, d\theta = 0$. Also, (14) can be restated neatly as follows:

$$\int_0^{\pi/2} \log(2\sin\theta) \, d\theta = \int_0^{\pi/2} \log(2\cos\theta) \, d\theta = 0.$$

The integral (14) has numerous equivalent forms. For example, first substituting $x = \sin\theta$ and then integrating by parts, we find

$$\int_0^1 \frac{\sin^{-1} x}{x} \, dx = \int_0^{\pi/2} \theta\cot\theta \, d\theta = -\int_0^{\pi/2} \log\sin\theta \, d\theta = \frac{\pi}{2}\log 2. \tag{15}$$

Further equivalents, and a survey of Euler's work in this area, are given in [6].

Catalan's constant enters the scene when we integrate from 0 to $\frac{\pi}{4}$ instead of $\frac{\pi}{2}$. Let

$$I_S = \int_0^{\pi/4} \log\sin\theta \, d\theta, \qquad I_C = \int_0^{\pi/4} \log\cos\theta \, d\theta.$$

Substituting $\theta = \frac{\pi}{2} - \phi$, we have $I_C = \int_{\pi/4}^{\pi/2} \log\sin\theta \, d\theta$. So by (14), $I_S + I_C = \int_0^{\pi/2} \log\sin\theta \, d\theta = -\frac{\pi}{2}\log 2$. Meanwhile by (9), $I_S - I_C = \int_0^{\pi/4} \log\tan\theta \, d\theta = -G$. So we conclude

$$I_S = -\tfrac{1}{2}G - \frac{\pi}{4}\log 2, \qquad I_C = \tfrac{1}{2}G - \frac{\pi}{4}\log 2. \tag{16}$$

Again there is a neat restatement:

$$\int_0^{\pi/4} \log(2\sin\theta) \, d\theta = -\tfrac{1}{2}G, \qquad \int_0^{\pi/4} \log(2\cos\theta) \, d\theta = \tfrac{1}{2}G.$$

However, (16) will be more useful in the ensuing applications.

Alternatively, (14) and (16) can be derived from the series $\log(2\sin\theta) = -\sum_{n=1}^{\infty} \frac{1}{n}\cos 2n\theta$, $\log(2\cos\theta) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\cos 2n\theta$ (e.g. see [7]); however, justification of termwise convergence is more delicate in this case.

The substitution $x = \tan\theta$ delivers the following reformulation of (16) in terms of the log function:

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2}\,dx = \int_0^{\pi/4} \log\sec^2\theta\,d\theta = -2\int_0^{\pi/4}\log\cos\theta\,d\theta = \frac{\pi}{2}\log 2 - G. \tag{17}$$

One can easily check that by substituting $x = 1/y$ and combining with (7), one obtains

$$\int_1^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx = G + \frac{\pi}{2}\log 2. \tag{18}$$

Now integrate by parts in (16): we find

$$\begin{aligned}\int_0^{\pi/4} \log\cos\theta\,d\theta &= \Big[\theta\log\cos\theta\Big]_0^{\pi/4} + \int_0^{\pi/4} \theta\frac{\sin\theta}{\cos\theta}\,d\theta \\ &= -\frac{\pi}{8}\log 2 + \int_0^{\pi/4}\theta\tan\theta\,d\theta,\end{aligned}$$

so that

$$\int_0^{\pi/4}\theta\tan\theta\,d\theta = \tfrac{1}{2}G - \frac{\pi}{8}\log 2, \tag{19}$$

an identity not given in [2] or [3]. Similarly, or by (19) combined with (3), we have $\int_0^{\pi/4}\theta\cot\theta\,d\theta = \frac{1}{2}G + \frac{\pi}{8}\log 2$.

Next, write

$$I = \int_0^{\pi/2}\log(1+\sin\theta)\,d\theta = \int_0^{\pi/2}\log(1+\cos\theta)\,d\theta.$$

By (16) and the identity $1+\cos\theta = 2\cos^2\frac{1}{2}\theta$, we obtain

$$\begin{aligned} I &= \int_0^{\pi/2}(\log 2 + 2\log\cos\tfrac{1}{2}\theta)\,d\theta \\ &= \frac{\pi}{2}\log 2 + 4\int_0^{\pi/4}\log\cos\phi\,d\phi \\ &= \frac{\pi}{2}\log 2 + 2G - \pi\log 2 \\ &= 2G - \frac{\pi}{2}\log 2, \end{aligned} \tag{20}$$

another integral not given in [2] or [3]. Combining (20) and (14), we have the pleasingly simple result

$$\int_0^{\pi/2} \log(1+\operatorname{cosec}\theta)\,d\theta = \int_0^{\pi/2} \Big(\log(1+\sin\theta)-\log\sin\theta\Big)\,d\theta = 2G, \tag{21}$$

and of course the same applies with $\operatorname{cosec}\theta$ replaced by $\sec\theta$.

Writing $(1+\sin\theta)(1-\sin\theta)=\cos^2\theta$, we deduce further

$$\begin{aligned}\int_0^{\pi/2} \log(1-\sin\theta)\,d\theta &= 2\int_0^{\pi/2}\log\cos\theta\,d\theta - \int_0^{\pi/2}\log(1+\sin\theta)\,d\theta \\ &= -2G - \frac{\pi}{2}\log 2. \end{aligned} \tag{22}$$

Integrating by parts in (20), we have

$$\int_0^{\pi/2} \log(1+\sin\theta)\,d\theta = \Big[\theta\log(1+\sin\theta)\Big]_0^{\pi/2} - J = \frac{\pi}{2}\log 2 - J,$$

where

$$J = \int_0^{\pi/2} \frac{\theta\cos\theta}{1+\sin\theta}\,d\theta,$$

hence

$$J = \pi\log 2 - 2G. \tag{23}$$

Furthermore, the substitution $x=\sin\theta$ gives

$$\int_0^1 \frac{\sin^{-1}x}{1+x}\,dx = J. \tag{24}$$

A further deduction from (16) is derived using the identity $\cos\theta+\sin\theta=\sqrt{2}\cos(\theta-\frac{\pi}{4})$:

$$\begin{aligned}\int_0^{\pi/2}\log(\cos\theta+\sin\theta)\,d\theta &= \int_0^{\pi/2}\Big(\tfrac{1}{2}\log 2+\log\cos(\theta-\tfrac{\pi}{4})\Big)\,d\theta \\ &= \frac{\pi}{4}\log 2 + \int_{-\pi/4}^{\pi/4}\log\cos\phi\,d\phi \\ &= \frac{\pi}{4}\log 2 + G - \frac{\pi}{2}\log 2 \\ &= G - \frac{\pi}{4}\log 2. \end{aligned} \tag{25}$$

Alternatively, (25) can be deduced from (20) and the identity $(\cos\theta+\sin\theta)^2 = 1+\sin 2\theta$. By (25) and (14),

$$\begin{aligned}\int_0^{\pi/2}\log(1+\tan\theta)\,d\theta &= \int_0^{\pi/2}\Big(\log(\cos\theta+\sin\theta)-\log\cos\theta\Big)\,d\theta \\ &= G - \frac{\pi}{4}\log 2 + \frac{\pi}{2}\log 2 \\ &= G + \frac{\pi}{4}\log 2, \end{aligned} \tag{26}$$

and hence also, with the substitution $x = \tan\theta$,

$$\int_0^\infty \frac{\log(1+x)}{1+x^2}\, dx = G + \frac{\pi}{4}\log 2, \tag{27}$$

which is proved by a rather longer method in [2].

Of course, $\tan\theta$ can be replaced by $\cot\theta$ in (26). With (20) and (21), this means that we have obtained the values of $\int_0^{\pi/2} \log[1 + f(\theta)]\, d\theta$, where $f(\theta)$ can be any one of the six trigonometric functions.

Yet further integrals can be derived by integrating by parts in (25) and (26) (rather better with $\cot\theta$): we leave it to the reader to explore this..

Double integrals

An obvious double-integral representation of G, not involving any trigonometric or logarithmic functions, follows at once from the geometric series. For $x, y \in [0,1)$, we have $1/(1+x^2y^2) = \sum_{n=0}^{\infty} (-1)^n x^{2n} y^{2n}$. Since

$$\int_0^1 \int_0^1 x^{2n} y^{2n}\, dx\, dy = \frac{1}{(2n+1)^2},$$

termwise integration gives

$$\int_0^1 \int_0^1 \frac{1}{1+x^2y^2}\, dx\, dy = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = G. \tag{28}$$

Termwise integration is justified as in (1). (Alternatively, one stage of integration immediately equates the double integral to (1)).

Substituting $x = e^{-u}$ and $y = e^{-v}$, we deduce

$$G = \int_0^\infty \int_0^\infty \frac{e^{-u}e^{-v}}{1+e^{-2u}e^{-2v}}\, du\, dv = \int_0^\infty \int_0^\infty \frac{1}{2\cosh(u+v)}\, du\, dv. \tag{29}$$

(This is really a pair of successive single-variable substitutions, not a full-blooded two-variable one.)

Next, we establish a much less transparent double-integral representation:

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{1+\cos\theta\cos\phi}\, d\theta\, d\phi = 2G. \tag{30}$$

This, again, is not in [2] or [3]. It was given in [4], possibly its first appearance. Here we present a proof based on (2). We actually show

$$\int_0^{\pi/4} \int_0^{\pi/4} \frac{1}{1+\cos 2\theta\cos 2\phi}\, d\theta\, d\phi = \tfrac{1}{2}G, \tag{31}$$

from which (30) follows by substituting $\theta = 2\theta'$ and $\phi = 2\phi'$. Write

$$J(\theta) = \int_0^{\pi/4} \frac{1}{1 + \cos 2\theta \cos 2\phi}\, d\phi.$$

Now

$$\begin{aligned} 1 + \cos 2\theta \cos 2\phi &= (\cos^2\theta + \sin^2\theta)(\cos^2\phi + \sin^2\phi) + (\cos^2\theta - \sin^2\theta)(\cos^2\phi - \sin^2\phi) \\ &= 2(\cos^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi). \end{aligned} \tag{32}$$

The substitution $t = \tan\phi$ gives

$$\int_0^{\pi/4} \frac{1}{a^2\cos^2\phi + b^2\sin^2\phi}\, d\phi = \frac{1}{ab}\tan^{-1}\frac{b}{a}.$$

Applied with $a = \cos\theta$ and $b = \sin\theta$, this gives

$$J(\theta) = \frac{\theta}{2\sin\theta\cos\theta}.$$

By (2), it follows that $\int_0^{\pi/4} J(\theta)\, d\theta = \frac{1}{2}G$, which is (31).

Alternatively, substitute $x = \tan\theta$ and $y = \tan\phi$ in (28). Again, (31) is obtained after applying (32). (This is the method of [4].)

We now establish two further double integrals:

$$\int_0^1 \int_0^1 \frac{1}{2 - x^2 - y^2}\, dx\, dy = G, \tag{33}$$

$$\int_0^1 \int_0^1 \frac{1}{(x+y)(1-x)^{1/2}(1-y)^{1/2}}\, dx\, dy = 4G. \tag{34}$$

Result (34) is identity (44) in [2], achieved with some effort; we present a somewhat simpler proof. The substitution $x = 1 - u^2$, $y = 1 - v^2$ reduces (34) to (33). Denote the integral in (33) by I. Since the integrand satisfies $f(y,x) = f(x,y)$, the contributions with $x \le y$ and $y \le x$ are the same, hence

$$I = 2\int_0^1 \int_0^x \frac{1}{2 - x^2 - y^2}\, dy\, dx.$$

Transform to polar coordinates:

$$I = \int_0^{\pi/4} \int_0^{\sec\theta} \frac{2r}{2 - r^2}\, dr\, d\theta.$$

Now

$$\int_0^{\sec\theta} \frac{2r}{2 - r^2}\, dr = \Big[-\log(2 - r^2) \Big]_0^{\sec\theta} = \log\frac{2}{2 - \sec^2\theta}$$

and

$$\frac{2}{2-\sec^2\theta} = \frac{2}{1-\tan^2\theta} = \frac{\tan 2\theta}{\tan\theta},$$

so

$$I = \int_0^{\pi/4} (\log\tan 2\theta - \log\tan\theta)\, d\theta.$$

As seen in (14), $\int_0^{\pi/4} \log\tan 2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2} \log\tan\phi\, d\phi = 0$. By (9), we deduce that $I = G$.

The discussion site [8] gives the following variant of (33):

$$\int_0^1 \int_0^1 \frac{1}{4-x^2-y^2}\, dx\, dy = \tfrac{1}{3}G.$$

The method is similar, but the writer resorts to *Mathematica* for the final stage, the integral $\int_0^{\pi/4} \log[4/(4-\sec^2\theta)]\, d\theta$; with a bit of effort, this can be deduced from our results above using the identity $4/(4-\sec^2\theta) = 2\sin 2\theta\cos\theta/(\sin 3\theta)$.

Applications to elliptic integrals

The "complete elliptic integral of the first kind" is defined, for $0 \le t < 1$, by

$$K(t) = \int_0^{\pi/2} \frac{1}{(1-t^2\sin^2\theta)^{1/2}}\, d\theta.$$

The value of $K(t)$ is given explicitly by a theorem of Gauss:

$$K(t) = \frac{\pi}{2M(1,t^*)},$$

where $M(a,b)$ is the arithmetic-geometric mean of a and b, and $t^* = (1-t^2)^{1/2}$. Two quite different proofs of this theorem can be seen in [9] and [10]. However, without any reference to Gauss's theorem, we can apply (4) to evaluate $\int_0^1 K(t)\, dt$. Indeed, reversing the implied double integral, we have $\int_0^1 K(t)\, dt = \int_0^{\pi/2} F(\theta)\, d\theta$, where

$$F(\theta) = \int_0^1 \frac{1}{(1-t^2\sin^2\theta)^{1/2}}\, dt.$$

Substituting $t\sin\theta = \sin\phi$, we have

$$F(\theta) = \int_0^{\theta} \frac{1}{\cos\phi}\frac{\cos\phi}{\sin\theta}\, d\phi = \frac{\theta}{\sin\theta}.$$

Hence, by (4),

$$\int_0^1 K(t)\, dt = 2G. \tag{35}$$

Of course, this really amounts to another double-integral representation of G.

The "complete elliptic integral of the second kind" is

$$E(t) = \int_0^{\pi/2} (1 - t^2 \sin^2 \theta)^{1/2} \, d\theta.$$

In the same way, we have $\int_0^1 E(t)\, dt = \int_0^{\pi/2} G(\theta)\, d\theta$, where

$$\begin{aligned} G(\theta) &= \int_0^1 (1 - t^2 \sin^2 \theta)^{1/2} \, dt \\ &= \int_0^\theta \cos\phi \, \frac{\cos\phi}{\sin\theta} \, d\phi \\ &= \frac{1}{2\sin\theta} \int_0^\theta (1 + \cos 2\phi) \, d\phi \\ &= \frac{\theta}{2\sin\theta} + \tfrac{1}{2}\cos\theta, \end{aligned}$$

hence

$$\int_0^1 E(t)\, dt = G + \tfrac{1}{2}. \tag{36}$$

A generalisation: the function $\mathrm{Ti}_2(x)$

Going back to the original integral in (1), we can define a function of x by considering the integral on $[0, x]$. The more-or-less standard notation is

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\tan^{-1} t}{t} \, dt.$$

Clearly, $G = \mathrm{Ti}_2(1)$. Integrating termwise as in (1), we have for $|x| \le 1$,

$$\mathrm{Ti}_2(x) = x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}. \tag{37}$$

This can be compared with "dilogarithm" function $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$. We remark that G appears in the evaluation $\mathrm{Li}_2(i) = -\frac{1}{8}\zeta(2) + iG$.

We can use (37) to calculate values for $|x| \le 1$, for example $\mathrm{Ti}_2(\frac{1}{2}) \approx 0.48722$.

Since $\tan(\frac{\pi}{2} - \theta) = 1/\tan\theta$ for $0 < \theta < \frac{\pi}{2}$, we have

$$\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$$

for $x > 0$. This translates into a corresponding functional equation for $\mathrm{Ti}_2(x)$:

$$\mathrm{Ti}_2(x) - \mathrm{Ti}_2\left(\frac{1}{x}\right) = \frac{\pi}{2} \log x. \tag{38}$$

It is sufficient to prove this for $x \geq 1$. Write $\mathrm{Ti}_2(x) - \mathrm{Ti}_2(1/x) = F(x)$. Then

$$\begin{aligned} F(x) = \int_{1/x}^{x} \frac{\tan^{-1} t}{t}\, dt &= \int_{1/x}^{x} \frac{1}{u} \tan^{-1} \frac{1}{u}\, du \\ &= \int_{1/x}^{x} \frac{1}{u} \left(\frac{\pi}{2} - \tan^{-1} u \right) du \\ &= \pi \log x - F(x). \end{aligned}$$

So for $x > 1$, we have the following series expansion in powers of $1/x$:

$$\mathrm{Ti}_2(x) = \frac{\pi}{2} \log x + \frac{1}{x} - \frac{1}{3^2 x^3} + \frac{1}{5^2 x^5} - \cdots,$$

showing very clearly the nature of $\mathrm{Ti}_2(x)$ for large x.

Of course, the procedures that gave equivalent expressions for G can be also applied to $\mathrm{Ti}_2(x)$. For example, the substitution $t = \tan\theta$ leads to

$$\mathrm{Ti}_2(x) = \tfrac{1}{2} \int_0^{2\tan^{-1} x} \frac{\theta}{\sin\theta}\, d\theta.$$

Integration by parts and then again the substitution $t = \tan\theta$ gives

$$\begin{aligned} \mathrm{Ti}_2(x) &= \tan^{-1} x \log x - \int_0^x \frac{\log t}{1+t^2}\, dt \\ &= \tan^{-1} x \log x - \int_0^{\tan^{-1} x} \log\tan\theta\, d\theta. \end{aligned}$$

As with the function $\mathrm{Li}_2(x)$, not many other explicit values of $\mathrm{Ti}_2(x)$ are known. One, proved by ingenious methods in [2] (formula (33)) is: $\mathrm{Ti}_2(2-\sqrt{3}) = \frac{2}{3}G - \frac{1}{12}\pi\log(2+\sqrt{3})$.

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