The denominators of the Bernoulli numbers

Notes by G.J.O. Jameson

One way to define the Bernoulli numbers B_n is by the power series identity

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \tag{1}$$

valid for sufficiently small $|x|$.

The first few are $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$. It is well known that $B_n = 0$ for odd $n \geq 3$, and that the sign of B_{2n} is $(-1)^{n-1}$.

As the reader may know, the Bernoulli numbers appear in the expression for the sum of powers $\sum_{r=1}^{m} r^n$ and in the evaluation of $\zeta(2n)$. They also figure in the highly effective approximation process known as Euler-Maclaurin summation. However, none of these facts will be needed in this note.

Our topic here is the expression of the B_n as rational numbers a_n/b_n. We record these expressions for even n between 2 and 30:

n	a_n	b_n
2	1	6
4	−1	30
6	1	42
8	−1	30
10	5	66
12	−691	2730
14	7	6
16	−3617	510
18	43867	798
20	−174611	330
22	854513	138
24	−236364091	2730
26	8553103	6
28	−23749461029	870
30	8615841276005	14322

At first sight, these numbers appear complicated and no pattern is transparent. Why does 6 recur among much larger numbers? In fact, the denominators are fully explained by the following remarkable theorem, proved independently by von Staudt [St] and Clausen [Cl] as long ago as 1840.

THEOREM 1. *Let $n \geq 2$ be even and let E_n be the set of primes p such that $p-1$ divides n. Then*

$$B_n = A_n - \sum_{p \in E_n} \frac{1}{p}, \tag{2}$$

where A_n is an integer. If $B_n = a_n/b_n$ in lowest terms, then $b_n = \prod_{p \in E_n} p$.

Probably the best known proof is the one given in [HWr], pp. 90-93 or [IR], pp. 234-236. Here we outline another method which, in my view, reveals very clearly why the result is true, as well as giving a formal proof. The steps are indicated as an exercise, with no details, in [Ap], pp. 274–275. They were described rather briefly in [Car], where the method is attributed to Lucas [Lu]. A variant of the same approach, rather more heavily dependent on results from elsewhere, was given in [Rz].

The method uses the following expression for B_n in terms of binomial coefficients, which is of interest in itself.

THEOREM 2. *For all $n \geq 1$,*

$$B_n = \sum_{k=1}^{n} \frac{S_{n,k}}{k+1}. \tag{3}$$

where

$$S_{n,k} = \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^n. \tag{4}$$

Numerous proofs of Theorem 2 have been given. Several are listed in [G], and another was given in [Rz]. Here we outline the pleasantly simple proof indicated in [Ap] and [Car].

LEMMA 1. *For $k \geq 1$, we have*

$$(e^x - 1)^k = (-1)^k \sum_{n=k}^{\infty} \frac{S_{n,k}}{n!} x^n, \tag{5}$$

Also, $S_{n,k} = 0$ for $n < k$.

Proof. By the binomial theorem,

$$(e^x - 1)^k = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} e^{rx}.$$

To avoid terms like 0^0, we use the fact that $\sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} = (1-1)^k = 0$ to rewrite this as

$$(e^x - 1)^k \quad = \quad \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} (e^{rx} - 1)$$

$$
\begin{aligned}
&= (-1)^k \sum_{r=0}^{k} (-1)^r \binom{k}{r} \sum_{n=1}^{\infty} \frac{r^n}{n!} x^n \\
&= (-1)^k \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n \\
&= (-1)^k \sum_{n=1}^{\infty} \frac{S_{n,k}}{n!} x^n,
\end{aligned}
$$

(note that a zero term term $r = 0$ can be included in the sum defining $S_{n,k}$).

But also

$$
(e^x - 1)^k = \left(\sum_{r=1}^{\infty} \frac{x^r}{r!} \right)^k ,
$$

which is of the form $\sum_{n=0}^{\infty} c_{n,k} x^n$, with $c_{n,k} = 0$ for $n < k$ and $c_{k,k} = 1$. Equating coefficients, we see that $S_{n,k} = 0$ for $n < k$. Note that also $S_{k,k} = (-1)^k k!$. □

Proof of Theorem 2. We will obtain another power series for $x/(e^x - 1)$ and compare it with (1). To do this, we write x as $\log[1 + (e^x - 1)]$ and use the series $\frac{1}{y} \log(1 + y) = \sum_{k=0}^{\infty} (-1)^k y^k/(k + 1)$. For sufficiently small $|x|$ (so that $|e^x - 1| < 1$), substituting (5), we obtain

$$
\begin{aligned}
\frac{x}{e^x - 1} &= \frac{1}{e^x - 1} \log\Big(1 + (e^x - 1)\Big) \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k + 1} (e^x - 1)^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{1}{k + 1} \sum_{n=k}^{\infty} \frac{S_{n,k}}{n!} x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{k=1}^{n} \frac{S_{n,k}}{k + 1}.
\end{aligned}
$$

Equating the coefficients to those in (1), we obtain (3). □

The reversal of summation (for those who care) can be justified as follows. The coefficients in $(e^x - 1)^k$ are positive, hence $(-1)^k S_{n,k} \geq 0$. Now using the series for $\frac{1}{y} \log(1 - y)$, we see that the double series with $S_{n,k}$ replaced by $|S_{n,k}|$ converges when $|e^x - 1| < 1$.

We now consider the terms $S_{n,k}/(k + 1)$, showing that in many cases they are integers. A further essential step is the next Lemma. Here we part from [Car] and [Rz] to give a direct proof that does not depend on other results.

LEMMA 2. *For all $k \geq 1$ and $n \geq 1$, $S_{n,k}$ is a multiple of $k!$.*

Proof. We prove this by induction on n, as a statement for all $k \geq 1$. Note that the case $k = 1$ is trivial for all n. For $n = 1$, we have $S_{1,k} = 0$ for $k \geq 2$. The induction step follows from the Pascal-like identity

$$S_{n+1,k} = k(S_{n,k} - S_{n,k-1}).$$

To prove this, we use the elementary identities $\binom{k}{r} = \binom{k-1}{r} + \binom{k-1}{r-1}$ and $r\binom{k}{r} = k\binom{k-1}{r-1}$. We have

$$\begin{aligned} S_{n+1,k} &= \sum_{r=1}^{k} (-1)^r \binom{k}{r} r^{n+1} \\ &= k \sum_{r=1}^{k} (-1)^r \binom{k-1}{r-1} r^n \\ &= k \sum_{r=1}^{k} (-1)^r \left[\binom{k}{r} - \binom{k-1}{r} \right] r^n \\ &= k(S_{n,k} - S_{n,k-1}). \qquad \square \end{aligned}$$

The numbers $\frac{(-1)^k}{k!} S_{n,k}$ are called the *Stirling numbers of the second kind*. They describe the number of partitions of $\{1, 2, \ldots, n\}$ into k non-empty subsets. See [Cam], pp. 80–83.

LEMMA 3. *If $k \geq 4$ and $k + 1$ is composite, then $k + 1$ divides $k!$.*

Proof. First, suppose $k + 1 = ab$, where $a \geq 2$, $b \geq 2$ and $a \neq b$. Then $a \leq k$, so a, and similarly b, is a term in the product defining $k!$. Now suppose that $k + 1 = a^2$, where $a \geq 3$. Then $k = a^2 - 1 > 2a$, so a and $2a$ are terms in the product for $k!$. $\square$

So in (3), the terms $S_{n,k}/(k + 1)$ are integers for all such k. We now consider k of the form $p - 1$, where p is prime.

LEMMA 4. *Let p be prime and $n \geq 2$. Then*

$$S_{n,p-1} \equiv \begin{cases} -1 \bmod p & \text{if } p-1 \text{ divides } n \\ 0 \bmod p & \text{if } p-1 \text{ does not divide } n \end{cases}$$

Proof. Recall that

$$S_{n,p-1} = \sum_{r=1}^{p-1} (-1)^r \binom{p-1}{r} r^n.$$

If $p - 1$ divides n, then, by Fermat's theorem, $r^n \equiv 1 \bmod p$ for $1 \leq r \leq p - 1$, so

$$S_{n,p-1} \equiv \sum_{r=1}^{p-1} (-1)^r \binom{p-1}{r} \mod p.$$

But $\sum_{r=1}^{p-1} (-1)^r \binom{p-1}{r} = (1 - 1)^{p-1} - 1 = -1$.

Now suppose that $p-1$ does not divide n. Then $n = k(p-1)+n'$, where k is an integer and $1 \le n' \le p-2$, and by Fermat's theorem again, $r^n \equiv r^{n'} \bmod p$ for each r. So $S_{n,p-1} \equiv S_{n',p-1} \bmod p$. But we saw in Lemma 1 that $S_{n',p-1} = 0$. □

Remark. It is elementary that $\binom{p-1}{r} \equiv (-1)^r \bmod p$. So an immediate consequence of Lemma 4 is that if $p-1$ does not divide n, then $\sum_{r=1}^{p-1} r^n$ is a multiple of p, a result that is usually proved by number-theoretic methods (e. g. [HWr], Theorem 119, where this result is one of the ingredients used in the proof of Theorem 1).

Completion of the proof of Theorem 1. First, we must deal with the case $k = 3$, left over from Lemma 3. Now $S_{n,3} = -3 + 3.2^n - 3^n$. Since n is even, $3^n \equiv 1 \bmod 4$, so $3^n + 3$, hence also $S_{n,3}$, is a multiple of 4.

So the terms in (3) are integers except when $k = p-1$, where p is prime and $p-1$ divides n. For such p, we have $S_{n,p-1} = mp - 1$ for some integer m, so

$$\frac{S_{n,p-1}}{p} = m - \frac{1}{p}.$$

Identity (2) follows.

For distinct primes $p_1, p_2, \ldots, p_k$,

$$\sum_{j=1}^{k} \frac{1}{p_j} = \frac{N}{p_1 p_2 \ldots p_k},$$

where $N \equiv p_2 \ldots p_k \bmod p_1$, so is not a multiple of p_1, or any other p_j. So this fraction is in its lowest terms. □

We mention some consequences and special cases, verifying values seen in the numerical table.

For every even n, we note that 1 and 2 are divisors of n, so 2 and 3 belong to E_n. Hence b_n is always a multiple of 6.

The case $n = 2q$, where $q \ge 3$ is prime. The divisors of $2q$ are 1, 2, q and $2q$. So if $2q+1$ is composite (which will occur, for example, if q is congruent to 1 mod 3 or to 2 mod 5), then 2 and 3 are the only members of E_n, so that $b_n = 6$ and $B_n = A'_n + \frac{1}{6}$ for an integer A'_n $(= A_n - 1)$. Examples of such n are 14, 26, 34, 38.

If $n = 2q$ and $2q+1$ is prime, then $E_n = \{2, 3, 2q+1\}$. So, for example, $E_6 = \{2, 3, 7\}$, $E_{10} = \{2, 3, 11\}$, and $E_{22} = \{2, 3, 23\}$.

Further, if $q > 3$ and $n = 2q^2$, then the only new even divisor is $2q^2$, and $2q^2 + 1$ is a multiple of 3, so E_n is the same as E_{2q}.

Example: $n = 12$. The even divisors of 12 are 2, 4, 6, 12. Hence $E_{12} = \{2, 3, 5, 7, 13\}$ and $b_{12} = 2730$.

Example: $n = 30$. Even divisors 2, 6, 10, 30. Hence $E_{30} = \{2, 3, 7, 11, 31\}$ and $b_{30} = 14322$.

Some concluding remarks. (1) It is worth noting how this reasoning is compatible with the fact that $B_n = 0$ (so is an integer) for odd $n \geq 3$. Theorem 2 and Lemma 2 still apply. For odd n, the only member of E_n is 2, and $\frac{1}{2}S_{n,1} = -\frac{1}{2}$. Meanwhile, the reasoning for the case $k = 3$ now shows that $S_{n,3} \equiv 2 \bmod 4$, so $\frac{1}{4}S_{n,3} = m + \frac{1}{2}$ for an integer m.

(2) The method of Theorem 2 delivers other identities for $S_{n,k}$. Applied with the geometric series for $(1 + y)^{-1}$, it leads to $\sum_{k=1}^{n} S_{n,k} = (-1)^n$. Applied with the series for $\log(1 + y)$, it gives $\sum_{k=1}^{n} S_{n,k}/k = 0$ for $n \geq 2$.

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