An approximation to the arithmetic-geometric mean

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Given positive numbers $a > b$, consider the iteration given by $a_0 = a$, $b_0 = b$ and

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n), \qquad b_{n+1} = (a_n b_n)^{1/2}.$$

At each stage, the two new numbers are the arithmetic and geometric means of the previous two. It is easily seen that $b_n < a_n$, (a_n) is decreasing, (b_n) is increasing and $a_{n+1} - b_{n+1} < \frac{1}{2}(a_n - b_n)$, and hence that (a_n) and (b_n) converge to a common limit, which is called the *arithmetic-geometric mean* of a and b. We will denote it by $M(a,b)$.

Furthermore, the convergence is quadratic, hence very rapid. We illustrate this by the calculation of $M(100,1)$:

n	b_n	a_n
0	1	100
1	10	50.5
2	22.4722	30.25
3	26.0727	26.3611
4	26.2165	26.2169
5	26.2167	26.2167

Of course, a_5 and b_5 are not actually equal, but more decimal places would be required to show the difference; in fact, $a_5 - b_5 < 10^{-9}$.

A famous result of Gauss, dating from 1799, relates $M(a,b)$ to the integral

$$I(a,b) = \int_0^\infty \frac{1}{(x^2+a^2)^{1/2}(x^2+b^2)^{1/2}}\,dx = \int_0^{\pi/2} \frac{1}{(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}}\,d\theta. \tag{1}$$

The equivalence of the two integrals is seen by substituting $x = b\tan\theta$. Gauss's theorem states that

$$I(a,b) = \frac{\pi}{2M(a,b)}. \tag{2}$$

This result has been discussed in a number of *Gazette* articles, e.g. [1], [2], [3]. The main step of the proof is to show, in the above notation, that $I(a_1,b_1) = I(a,b)$. It then follows that $I(a,b) = I(a_n,b_n)$ for all n. By the first integral in (1),

$$I(a_n,b_n) > \int_0^\infty \frac{1}{x^2+a_n^2}\,dx = \frac{\pi}{2a_n},$$

and similarly $I(a_n,b_n) < \pi/(2b_n)$; (2) now follows. One way to prove that $I(a_1,b_1) =$

$I(a,b)$ is by the substitution $2x = t - ab/t$. This ingenious substitution seems to have been introduced by D.J. Newman in [4]; more accessible accounts of it can be seen in [1] and [5]. In [3], Nick Lord presents an interesting new method based on the area in polar coordinates.

Depending on one's starting point, Gauss's theorem can be regarded either as the evaluation of the integral $I(a,b)$ (including a quick way to calculate it in specific cases), or as a closed expression for $M(a,b)$.

Clearly, $M(ca,cb) = cM(a,b)$ and $I(ca,cb) = \frac{1}{c}I(a,b)$ for $c > 0$, so it is enough to state results in terms of $M(a,1)$ and $I(a,1)$.

The topic of this article is the fact that there are pleasantly simple formulae giving good approximations to I and M, as follows: for large a and small (positive) b,

$$aI(a,1) \approx \log 4a,$$

$$I(1,b) \approx \log \frac{4}{b},$$

$$M(a,1) \approx F(a), \quad \text{where } F(a) = \frac{\pi}{2} \frac{a}{\log 4a}.$$

The first two statements are equivalent because $aI(a,1) = I(1,\frac{1}{a})$, and the statement for $M(a,1)$ follows by Gauss's theorem. The approximation is strikingly accurate for moderately large a: for example, $F(100) = 26.2172$ to four d.p.: compare the value for $M(100,1)$ found above.

These approximations have been known for a long time: for example, a version appears in [6, p. 522], published in 1927. A variety of methods have been used, with differing estimates of the accuracy of the approximation. A very effective method was presented by Newman in [5]. However, it is sketched rather briefly there, and the statement is given only in the form $aI(a,1) = \log 4a + O(1/a^2)$. Here we develop Newman's method in more detail to establish the following more specific estimation (which is rather more precise than any version I have seen in the existing literature):

THEOREM 1. *For $0 < b \le 1$ and $a \ge 1$, we have*

$$\log \frac{4}{b} < I(1,b) < (1 + \tfrac{1}{4}b^2) \log \frac{4}{b}, \tag{3}$$

$$\log 4a < aI(a,1) < \left(1 + \frac{1}{4a^2}\right) \log 4a. \tag{4}$$

Moreover, the following lower bounds also apply:

$$I(1,b) > (1 + \tfrac{1}{4}b^2) \log \frac{4}{b} - \tfrac{7}{16}b^2, \tag{5}$$

$$aI(a,1) > \left(1+\frac{1}{4a^2}\right)\log 4a - \frac{7}{16a^2}. \tag{6}$$

The pairs of statements are equivalent, because of the identity $aI(a,1) = I(1,\frac{1}{a})$. Statements (5) and (6) may look rather technical, but we include them because they emerge from the same process of reasoning, and because, together with (3) and (4), they show that

$$aI(a,1) = \left(1+\frac{1}{4a^2}\right)\log 4a + O(1/a^2),$$

and a similar statement for b (so in fact the estimation $\log 4a + O(1/a^2)$ in [5] is not quite correct). At the same time, we have gained the pleasantly simple inequality $aI(a,1) > \log 4a$.

By Gauss's theorem and the inequality $1/(1+x) > 1-x$, applied with $x = 1/(4a^2)$, statement (4) translates into the following pair of inequalities for $M(a,1)$:

COROLLARY. *Let $F(a)$ be as above. For $a \geq 1$,*

$$\left(1-\frac{1}{4a^2}\right)F(a) < M(a,1) < F(a). \tag{7}$$

This agrees well with the accuracy seen above for $a = 100$. Since $M(a,b) = b\,M(a/b,\,1)$ we can derive the following bounds for $M(a,b)$ (where $a > b$) in general:

$$\frac{\pi}{2}\frac{a}{\log(4a/b)}\left(1-\frac{b^2}{4a^2}\right) < M(a,b) < \frac{\pi}{2}\frac{a}{\log(4a/b)}. \tag{8}$$

The proofs can be set out in terms of either a or b: we shall use b. The method requires nothing more than some elementary integrals and the binomial and logarithmic series. We will also see that a weaker version of Theorem 1 can be established very quickly.

At this point, we make two preliminary observations about $I(a,b)$. Firstly, by the inequality of the means, we have

$$\frac{1}{(x^2+a^2)^{1/2}}\,\frac{1}{(x^2+b^2)^{1/2}} \leq \frac{1}{2(x^2+a^2)} + \frac{1}{2(x^2+b^2)},$$

so that

$$I(a,b) \leq \frac{\pi}{4}\left(\frac{1}{a}+\frac{1}{b}\right). \tag{9}$$

Secondly, $I(1,b)$ equates to a standard elliptic integral in the following way. Let $b^* = (1-b^2)^{1/2}$, so that $b^2 + b^{*2} = 1$. Then $\cos^2\theta + b^2\sin^2\theta = 1 - b^{*2}\sin^2\theta$, so that

$$I(1,b) = \int_0^{\pi/2} \frac{1}{(1-b^{*2}\sin^2\theta)^{1/2}}\,d\theta.$$

In the language of elliptic integrals, this is a "complete elliptic integral of the first kind", denoted by $K(b^*)$. However, we do not assume any prior knowledge of elliptic intgrals.

We now embark on the proof. Taking $a = 1$, write

$$H(x) = \frac{1}{(x^2+1)^{1/2}(x^2+b^2)^{1/2}}. \tag{10}$$

The starting point is the observation that the substitution $x = b/y$ gives

$$\int_0^{b^{1/2}} H(x)\,dx = \int_{b^{1/2}}^{\infty} H(y)\,dy,$$

so that

$$I(1,b) = 2\int_0^{b^{1/2}} H(x)\,dx. \tag{11}$$

Now $(1+y)^{-1/2} > 1 - \frac{1}{2}y$ for $0 < y < 1$, as is easily seen by multiplying out $(1+y)(1-\frac{1}{2}y)^2$, so for $0 < x < 1$,

$$(1 - \tfrac{1}{2}x^2)\frac{1}{(x^2+b^2)^{1/2}} < H(x) < \frac{1}{(x^2+b^2)^{1/2}}. \tag{12}$$

Now observe that the substitution $x = by$ gives

$$\int_0^{b^{1/2}} \frac{1}{(x^2+b^2)^{1/2}}\,dx = \int_0^{1/b^{1/2}} \frac{1}{(y^2+1)^{1/2}}\,dy = \sinh^{-1}\frac{1}{b^{1/2}}.$$

LEMMA 1. *We have*

$$\log\frac{2}{b^{1/2}} < \sinh^{-1}\frac{1}{b^{1/2}} < \log\frac{2}{b^{1/2}} + \tfrac{1}{4}b. \tag{13}$$

Proof. Recall that $\sinh^{-1} x = \log[x + (x^2+1)^{1/2}]$, which is clearly greater than $\log 2x$. Also, since $(1+b)^{1/2} < 1 + \frac{1}{2}b$,

$$\frac{1}{b^{1/2}} + \left(\frac{1}{b}+1\right)^{1/2} = \frac{1}{b^{1/2}}[1 + (1+b)^{1/2}] < \frac{1}{b^{1/2}}(2 + \tfrac{1}{2}b) = \frac{2}{b^{1/2}}(1 + \tfrac{1}{4}b).$$

The right-hand inequality in (13) now follows from the fact that $\log(1+x) \le x$. □

This is already enough to prove the following interim version of Theorem 1:

THEOREM 1A. *For* $0 < b \le 1$ *and* $a \ge 1$,

$$\log\frac{4}{b} - \tfrac{1}{2}b < I(1,b) < \log\frac{4}{b} + \tfrac{1}{2}b, \tag{14}$$

$$\log 4a - \frac{1}{2a} < aI(a,1) < \log 4a + \frac{1}{2a}. \tag{15}$$

Proof. The two statements are equivalent, since $aI(a,1) = I(1, \frac{1}{a})$. By (11) and (12),

$$2\sinh^{-1}\frac{1}{b^{1/2}} - R_1(b) < I(1,b) < 2\sinh^{-1}\frac{1}{b^{1/2}},$$

where

$$R_1(b) = \int_0^{b^{1/2}} \frac{x^2}{(x^2+b^2)^{1/2}}\,dx. \tag{16}$$

Since $(x^2+b^2)^{1/2} > x$, we have

$$R_1(b) < \int_0^{b^{1/2}} \frac{x^2}{x}\,dx = \int_0^{b^{1/2}} x\,dx = \tfrac{1}{2}b.$$

Both inequalities in (14) now follow from (13), since $2\log\frac{2}{b^{1/2}} = \log\frac{4}{b}$. □

Theorem 1A is sufficient to show that $M(a,1) - F(a) \to 0$ as $a \to \infty$. To see this, write $aI(a,1) = \log 4a + r(a)$. Then

$$\frac{1}{I(a,1)} - \frac{a}{\log 4a} = -\frac{ar(a)}{\log 4a[\log 4a + r(a)]},$$

which tends to 0 as $a \to \infty$, since $|ar(a)| \le \frac{1}{2}$.

Some readers may be content with Theorem 1A, but for those with the appetite for it, we now refine the method to establish the full strength of Theorem 1. All we need to do is improve our estimations by the insertion of further terms. The upper estimate in (12) only used $(1+y)^{-1/2} < 1$. Instead, we now use the stronger bound

$$\frac{1}{(1+y)^{1/2}} < 1 - \tfrac{1}{2}y + \tfrac{3}{8}y^2$$

for $0 < y < 1$. These are the first three terms of the binomial expansion, and the stated inequality holds because the terms of the expansion alternate in sign and decrease in magnitude. So we have instead of (12):

$$(1 - \tfrac{1}{2}x^2)\frac{1}{(x^2+b^2)^{1/2}} < H(x) < (1 - \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4)\frac{1}{(x^2+b^2)^{1/2}}. \tag{17}$$

We also need a further degree of accuracy in the estimates for $\sinh^{-1}\frac{1}{b^{1/2}}$ and $R_1(b)$.

LEMMA 2. *For $0 < b \le 1$, we have*

$$\log\frac{2}{b^{1/2}} + \tfrac{1}{4}b - \tfrac{3}{32}b^2 < \sinh^{-1}\frac{1}{b^{1/2}} < \log\frac{2}{b^{1/2}} + \tfrac{1}{4}b - \tfrac{1}{16}b^2 + \tfrac{1}{32}b^3. \tag{18}$$

Proof. Again by the binomial series, we have

$$1 + \tfrac{1}{2}b - \tfrac{1}{8}b^2 < (1+b)^{1/2} < 1 + \tfrac{1}{2}b - \tfrac{1}{8}b^2 + \tfrac{1}{16}b^3. \tag{19}$$

So, as in Lemma 1,

$$\frac{1}{b^{1/2}} + \left(\frac{1}{b}+1\right)^{1/2} < \frac{1}{b^{1/2}}(2+\tfrac{1}{2}b-\tfrac{1}{8}b^2+\tfrac{1}{16}b^3) = \frac{2}{b^{1/2}}(1+\tfrac{1}{4}b-\tfrac{1}{16}b^2+\tfrac{1}{32}b^3).$$

The right-hand inequality in (18) now follows from $\log(1+x) \le x$. Also,

$$\frac{1}{b^{1/2}} + \left(\frac{1}{b}+1\right)^{1/2} > \frac{1}{b^{1/2}}(2+\tfrac{1}{2}b-\tfrac{1}{8}b^2) = \frac{2}{b^{1/2}}(1+B),$$

where $B = \frac{1}{4}b - \frac{1}{16}b^2$ (so $B < \frac{1}{4}b$). By the log series, $\log(1+x) > x - \frac{1}{2}x^2$ for $0 < x < 1$, so

$$\log(1+B) > B - \tfrac{1}{2}B^2 > \tfrac{1}{4}b - \tfrac{1}{16}b^2 - \tfrac{1}{32}b^2 = \tfrac{1}{4}b - \tfrac{3}{32}b^2. \qquad \square$$

LEMMA 3. *For $0 < b \le 1$, we have:*

$$R_1(b) < \tfrac{1}{2}b + \tfrac{1}{4}b^2 - \tfrac{1}{2}b^2 \log\frac{2}{b^{1/2}}, \tag{20}$$

$$R_1(b) > \tfrac{1}{2}b + \tfrac{1}{4}b^2 - \tfrac{1}{2}b^2 \log\frac{2}{b^{1/2}} - \tfrac{3}{16}b^3, \tag{21}$$

$$R_1(b) < \tfrac{1}{2}b - \tfrac{1}{5}b^2. \tag{22}$$

Proof: We can evaluate $R_1(b)$ explicitly by the substitution $x = b\sinh t$ in (16):

$$R_1(b) = \int_0^{c(b)} b^2\sinh^2 t\,dt = \tfrac{1}{2}b^2\int_0^{c(b)}(\cosh 2t - 1)\,dt = \tfrac{1}{4}b^2\sinh 2c(b) - \tfrac{1}{2}b^2 c(b),$$

where $c(b) = \sinh^{-1}(1/b^{1/2})$. Now

$$\sinh 2c(b) = 2\sinh c(b)\cosh c(b) = \frac{2}{b^{1/2}}\left(\frac{1}{b}+1\right)^{1/2} = \frac{2}{b}(1+b)^{1/2},$$

so

$$R_1(b) = \tfrac{1}{2}b(1+b)^{1/2} - \tfrac{1}{2}b^2 c(b).$$

Statement (20) now follows from $(1+b)^{1/2} < 1 + \frac{1}{2}b$ and $c(b) > \log\frac{2}{b^{1/2}}$. Statement (21) follows from the left-hand inequality in (19) and the right-hand one in (13).

Unfortunately, (22) does not quite follow from (20). We prove it directly from the integral, as follows. Write $x^2/(x^2+b^2)^{1/2} = g(x)$. Since $g(x) < x$, we have $\int_b^{b^{1/2}} g(x)\,dx < \frac{1}{2}(b-b^2)$. For $0 < x < b$, we have by the binomial expansion again

$$g(x) = \frac{x^2}{b(1+x^2/b^2)^{1/2}} \le \frac{x^2}{b}\left(1 - \frac{x^2}{2b^2} + \frac{3x^4}{8b^4}\right),$$

and hence

$$\int_0^b g(x)\,dx < (\tfrac{1}{3} - \tfrac{1}{10} + \tfrac{3}{56})b^2 < \tfrac{3}{10}b^2.$$

Together, these estimates give $R_1(b) < \frac{1}{2}b - \frac{1}{5}b^2$. $\square$

Completion of the proof of Theorem 1: lower bounds. By (17), (18) and (22),

$$\begin{aligned} I(1,b) &> 2\sinh^{-1}\frac{1}{b^{1/2}} - R_1(b) \\ &> \log\frac{4}{b} + \tfrac{1}{2}b - \tfrac{3}{16}b^2 - (\tfrac{1}{2}b - \tfrac{1}{5}b^2) \\ &> \log\frac{4}{b}, \end{aligned}$$

(with a spare term $\frac{1}{80}b^2$). Of course, the key fact is the cancellation of the term $\frac{1}{2}b$. Also, using (20) instead of (22), we have

$$\begin{aligned} I(1,b) &> \log\frac{4}{b} + \tfrac{1}{2}b - \tfrac{3}{16}b^2 - (\tfrac{1}{2}b + \tfrac{1}{4}b^2 - \tfrac{1}{4}b^2\log\frac{4}{b}) \\ &= (1+\tfrac{1}{4}b^2)\log\frac{4}{b} - \tfrac{7}{16}b^2. \end{aligned}$$

Upper bound: By (17),

$$I(1,b) < 2\sinh^{-1}\frac{1}{b^{1/2}} - R_1(b) + \tfrac{3}{4}R_2(b),$$

where

$$R_2(b) = \int_0^{b^{1/2}} \frac{x^4}{(x^2+b^2)^{1/2}}\,dx < \int_0^{b^{1/2}} x^3\,dx = \tfrac{1}{4}b^2.$$

So by (18) and (21),

$$\begin{aligned} I(1,b) &< \log\frac{4}{b} + \tfrac{1}{2}b - \tfrac{1}{8}b^2 + \tfrac{1}{16}b^3 - (\tfrac{1}{2}b + \tfrac{1}{4}b^2 - \tfrac{1}{4}b^2\log\frac{4}{b} - \tfrac{3}{16}b^3) + \tfrac{3}{16}b^2 \\ &= (1+\tfrac{1}{4}b^2)\log\frac{4}{b} - \tfrac{3}{16}b^2 + \tfrac{1}{4}b^3. \end{aligned}$$

If $b < \frac{3}{4}$, then $\frac{3}{16}b^2 \geq \frac{1}{4}b^3$, so $I(1,b) < (1+\frac{1}{4}b^2)\log\frac{4}{b}$.

For $b \geq \frac{3}{4}$, we reason as follows. By (9), $I(1,b) \leq \frac{\pi}{4}(1+\frac{1}{b})$. Write $h(b) = (1+\frac{1}{4}b^2)\log\frac{4}{b} - \frac{\pi}{4}(1+\frac{1}{b})$. We find that $h(\frac{3}{4}) \approx 1.9094 - 1.8326 > 0$. One verifies by differentiation that $h(b)$ is increasing for $0 < b \leq 1$ (we omit the details), hence $h(b) > 0$ for $\frac{3}{4} \leq b \leq 1$. $\square$

Note: We mention very briefly an older, perhaps better known, approach (cf. [7, p. 21–22]). Use the identity $I(1,b) = K(b^*)$, and note that $b^* \to 1$ when $b \to 0$. Define

$$A(b^*) = \int_0^{\pi/2} \frac{b^*\sin\theta}{(1-b^{*2}\sin^2\theta)^{1/2}}\,d\theta$$

and $B(b^*) = K(b^*) - A(b^*)$. One shows by direct integration that $A(b^*) = \log[(1+b^*)/b]$ and $B(1) = \log 2$, so that when $b \to 0$, we have $A(b^*) - \log\frac{2}{b} \to 0$ and hence $K(b^*) - \log\frac{4}{b} \to 0$.

This method can be developed to give an estimation with error term of the form $O(b^2 \log \frac{1}{b})$, [8, p. 355–357]. The author has refined it a little further, but the details are distinctly more laborious than the method we have given here, and the end result still does not achieve the accuracy of our Theorem 1.

Application to the calculation of logarithms and π. In my view, Theorem 1 is interesting enough in its own right. However, it also has pleasing applications to the calculation of logarithms (assuming π known) and π (assuming logarithms known), exploiting the rapid convergence of the agm iteration. Consider first the problem of calculating $\log x$ (where $x > 1$). Choose n, in a way to be discussed below, and let $4a = x^n$, so that $n \log x = \log 4a$, which is approximated by $aI(a, 1)$. We calculate $M = M(a, 1)$, and hence $I(a, 1) = \pi/(2M)$. Then $\log x$ is approximated by $(a/n)I(a, 1)$.

How accurate is this approximation? By Theorem 1, $\log 4a = aI(a, 1) - r(a)$, where $0 < r(a) < \frac{1}{4a^2} \log 4a$. Hence

$$\log x = \frac{a}{n} I(a, 1) - \frac{r(a)}{n},$$

in which

$$0 < \frac{r(a)}{n} < \frac{\log x}{4a^2} = \frac{4 \log x}{x^{2n}}.$$

We choose n so that this is within the desired degree of accuracy. It might seem anomalous that $\log x$, which we are trying to calculate, appears in this error estimation, but a rough estimate for it is always readily available.

For illustration, we apply this to the calculation of $\log 2$, taking $n = 12$, so that $a = 2^{10} = 1024$. We find that $M = M(1024, 1) \approx 193.38065$, so our approximation is

$$\frac{\pi}{2} \frac{1024}{12M} \approx 0.6931474,$$

while in fact $\log 2 = 0.6931472$ to seven d.p. The discussion above (just taking $\log 2 < 1$) shows that the approximation overestimates $\log 2$, with error no more than $4/2^{24} = 1/2^{20} < 1/10^6$.

A variation (cf. [9, p. 220]) is as follows: choose large a (e.g. $a = 10^n$ for suitable n). Then $aI(a, 1) \approx \log 4a$ and $axI(ax, 1) \approx \log 4ax$, so $\log x \approx axI(ax, 1) - aI(a, 1)$. Clearly, this method requires two agm iterations. The error is $r(a) - r(ax)$, and we have again $0 < r(a) < r^*(a)$, where $r^*(a) = \frac{1}{4a^2} \log 4a$. Now $\log y/y^2$ is decreasing for $y > e^{1/2}$, so $r^*(ax) < r^*(a)$, hence the magnitude of the error is no more than $r^*(a)$ itself; this can be compared with the error $\frac{1}{4a^2} \log x$ in the previous method.

We now turn to the question of approximating π. The simple-minded method is as follows. Choose a suitably large a. By Gauss's theorem and Theorem 1,

$$\frac{\pi}{2} = M(a,1)I(a,1) = M(a,1)\frac{\log 4a}{a} + s(a),$$

where

$$s(a) = M(a,1)\frac{r(a)}{a} < M(a,1)\frac{\log 4a}{4a^3}.$$

So an approximation to π is $\frac{2}{a}M(a,1)\log 4a$, with error no more than $2s(a)$. Using our original example with $a = 100$, this gives 3.14153 to five d.p., with the error estimated as no more than 8×10^{-5} (the actual error is about 6.4×10^{-5}).

For a better approximation, one would repeat with a larger a. However, there is a way to generate a sequence of increasingly close approximations from a single agm iteration, which we now describe (cf. [8, Proposition 3]).

In the original agm iteration, let $c_n = (a_n^2 - b_n^2)^{1/2}$. Note that c_n converges rapidly to 0. By Gauss's theorem, $\frac{\pi}{2} = M(a_0, c_0)I(a_0, c_0)$. To apply Theorem 1, we need to equate $I(a_0, c_0)$ to an expression of the form $I(d_n, e_n)$, where e_n/d_n tends to 0. This is achieved as follows. We have

$$c_{n+1}^2 = a_{n+1}^2 - b_{n+1}^2 = \tfrac{1}{4}(a_n + b_n)^2 - a_n b_n = \tfrac{1}{4}(a_n - b_n)^2,$$

so $c_{n+1} = \frac{1}{2}(a_n - b_n)$, and hence

$$a_n = a_{n+1} + c_{n+1}, \quad b_n = a_{n+1} - c_{n+1}$$

(this shows how to derive a_n and b_n from a_{n+1} and b_{n+1}, reversing the agm iteration). Hence $c_n^2 = a_n^2 - b_n^2 = 4a_{n+1}c_{n+1}$, so the arithmetic mean of $2a_{n+1}$ and $2c_{n+1}$ is a_n, while the geometric mean is c_n. Therefore $I(a_n, c_n) = I(2a_{n+1}, 2c_{n+1})$, and hence

$$I(a_0, c_0) = I(2^n a_n, 2^n c_n) = \frac{1}{2^n a_n} I\left(1, \frac{c_n}{a_n}\right).$$

Since c_n/a_n tends to 0, Theorem 1 (or even Theorem 1A) applies to show that

$$I(a_0, c_0) = \lim_{n\to\infty} \frac{1}{2^n a_n} \log \frac{4a_n}{c_n}.$$

Now take $a_0 = 1$ and $b_0 = \frac{1}{\sqrt{2}}$. Then $c_0 = b_0$, so $M(a_0, b_0) = M(a_0, c_0)$, and (a_n) converges to this value. Since $\pi = 2M(a_0, c_0)I(a_0, c_0)$, we have established the following:

THEOREM 2. *With a_n, b_n, c_n defined in this way, we have*

$$\pi = \lim_{n\to\infty} \frac{1}{2^{n-1}} \log \frac{4a_n}{c_n}. \qquad \square \tag{23}$$

Denote this approximation by p_n. With enough determination, one could derive an error estimation, but we will just illustrate the speed of convergence by performing the first three steps. Values are rounded to six significant figures. To calculate c_n, we use $c_n = c_{n-1}^2/(4a_n)$ rather than $\frac{1}{2}(a_{n-1} - b_{n-1})$, since this would require much greater accuracy in the values of a_{n-1} and b_{n-1}.

n	a_n	b_n	c_n	p_n
0	1	0.707106	0.707106	
1	0.853553	0.840896	0.146447	3.14904
2	0.847225	0.847201	0.00632849	3.14160
3	0.847213	0.847213	0.0000118181	3.14159

More accurate calculation gives the value $p_3 = 3.14159265$, showing that it actually agrees with π to eight decimal places.

While this iteration does indeed converge rapidly to π, it has the disadvantage that logarithms must be calculated, or assumed known. This is overcome by the Gauss-Brent-Salamin algorithm, which also uses the agm iteration, but depends on identities involving integrals of the kind

$$\int_0^{\pi/2} \frac{\sin^2\theta}{(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}}\, d\theta$$

instead of Theorem 1. A user-friendly account of this algorithm, with care taken to use the most elementary methods available, is given in [1].

Series expressions for $I(1,b)$. Recall that $K(b) = \int_0^{\pi/2}(1-b^2\sin^2\theta)^{-1/2}\,d\theta$ and $I(1,b) = K(b^*)$. By integrating the binomial expansion, one finds that $K(b) = \frac{\pi}{2}\sum_{n=0}^{\infty} d_{2n}b^{2n}$ for $0 < b < 1$, where $d_0 = 1$ and

$$d_{2n} = \frac{1^2.3^2 \ldots (2n-1)^2}{2^2.4^2 \ldots (2n)^2}.$$

By more elaborate methods, identifying and solving a second-order differential equation satisfied by $I(1,b)$, one can establish the following identity [9, p. 9]:

$$I(1,b) = K(b^*) = \frac{2}{\pi}K(b)\log\frac{4}{b} - 2\sum_{n=1}^{\infty} d_{2n}e_{2n}b^{2n}, \tag{24}$$

where $e_{2n} = \sum_{r=1}^{n}(\frac{1}{2r-1} - \frac{1}{2r})$. Together with the stated series for $K(b)$, this gives

$$I(1,b) = \log\frac{4}{b} + \sum_{n=1}^{\infty} d_{2n}\left(\log\frac{4}{b} - 2e_{2n}\right)b^{2n}. \tag{25}$$

Let us compare the information delivered by these series with our Theorem 1. Write $I(1,b) = \log\frac{4}{b} + r_1(b)$. The first term of the series in (25) is $\frac{1}{4}b^2(\log\frac{4}{b} - 1)$, so we can state

$$r_1(b) = \tfrac{1}{4}b^2\left(\log\frac{4}{b} - 1\right) + O\left(b^4\log\frac{1}{b}\right), \tag{26}$$

which is a little more accurate than our estimate. For the agm, this translates into

$$M(a,1) = F(a) - \frac{F(a)}{4a^2}\frac{\log 4a - 1}{\log 4a} + O(1/a^3) \quad \text{as } a \to \infty,$$

An actual lower bound can be derived as follows. We have $e_{2n} < \log 2$ for all n, since (e_{2n}) is increasing with limit $\log 2$. Hence $\log \frac{4}{b} - 2e_{2n} > 0$, so from (25),

$$r_1(b) > \tfrac{1}{4}b^2 \left(\log \frac{4}{b} - 1\right), \tag{27}$$

which is slightly stronger than (5). However, some quite delicate calculations are needed to recapture our upper bound in (3).

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