The approximate functional equation for $\zeta(s)^2$

Notes by Tim Jameson

As usual, write

$$\chi(s) = \frac{\zeta(s)}{\zeta(1 - s)} = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1 - s).$$

One version of the approximate functional equation for $\zeta(s)^2$ (e.g. [Iv, sect. 4.2]) states:

THEOREM. Let $\tau(n)$ be the divisor function, and write $s = \sigma + it$. Let $0 < \sigma < 1$ and $x \asymp t$, and define $y$ by $4\pi^2xy = t^2$. Then

$$\zeta(s)^2 = \sum_{n \leq x} \tau(n)n^{-s} + \chi(s)^2 \sum_{n \leq y} \tau(n)n^{s-1} + O(t^{1/2 - \sigma} \log t). \quad (1)$$

Here I present a relatively simple proof which I devised before discovering that a similar one had been given in [Moto]. Actually, Motohashi only states the case where $x = y$, but his method can be adapted to the general case.

We start from the approximate functional equation for $\zeta(s)$ itself (e.g. [Ti, chapter 4]):

Let $0 < \sigma < 1$, $1 < u \leq t$ and $2\pi uv = t$. Then

$$\zeta(s) = \sum_{n \leq u} n^{-s} + \chi(s) \sum_{n \leq v} n^{s-1} + O(u^{-\sigma}) + O(t^{-1/2}u^{1-\sigma}). \quad (2)$$

We will only be interested in the case when $u \gg t^{1/2}$, in which case the error term $u^{-\sigma}$ is dominated by $t^{-1/2}u^{1-\sigma}$. Also, since $\chi(s) \ll t^{1-\sigma}$, the term with $n = v$ (if this happens to be an integer) makes a contribution of magnitude

$$t^{1/2} - \sigma \left(\frac{t}{u}\right)^{\sigma-1} = t^{-1/2}u^{1-\sigma},$$

the same as the error term. So we can replace the condition $n \leq v$ by $n < v$ without change. Note also that with $u$ taken to be $t^{1/2}$, (2) implies that $\zeta(s) \ll t^{1/2}(1-\sigma)$.

Let $t^{1/2} \ll u_2 < u_1$ and $2\pi u_j v_j = t$. Applying (2) to $u_1$ and $u_2$ and taking the difference, we have

$$\sum_{u_2 < u \leq u_1} n^{-s} = \chi(s) \sum_{v_1 < n \leq v_2} n^{s-1} + O(t^{-1/2}u_1^{1-\sigma}). \quad (3)$$

The effect is to replace the sum of terms $n^{-s}$ with the “reflected” sum of terms $n^{1-s}$.

Now assume that $x \asymp t$ and $4\pi^2x^2y^2 = t^2$. Write

$$S_1 = \sum_{m \leq x^{1/2}} m^{-s}, \quad S_2 = \sum_{n \leq y^{1/2}} n^{s-1}.$$
Note that by (2),

\[ S_1 + \chi(s)S_2\zeta(s) + O(t^{-\sigma/2}). \]

Expand \( \sum_{n \leq x} \tau(n)n^{-s} \) by the Dirichlet hyperbola method:

\[
\sum_{n \leq x} \tau(n)n^{-s} = \sum_{mn \leq x} m^{-s}n^{-s} = \sum_{m \leq x^{1/2}} m^{-s} \sum_{n \leq x/m} n^{-s} + \sum_{n \leq x^{1/2}} n^{-s} \sum_{x^{1/2} < m \leq x/n} m^{-s} = S_1^2 + 2 \sum_{m \leq x^{1/2}} m^{-s} \sum_{n \leq x/m} n^{-s}.
\]

Now use (3) to replace the final sum by the corresponding reflected sum. The total error introduced is estimated by

\[
\sum_{m \leq x^{1/2}} m^{-\sigma}t^{-1/2}(\frac{x}{m})^{1-\sigma} = t^{1/2}x^{1-\sigma} \sum_{m \leq x^{1/2}} \frac{1}{m} \ll t^{1/2-\sigma} \log t
\]

(note that our stated objective was to express the error term in terms of \( t \)). So we obtain

\[
\sum_{n \leq x} \tau(n)n^{-s} = S_1^2 + 2\chi(s) \sum_{m \leq x^{1/2}} m^{-s} \sum_{\frac{im}{2\pi x} < n \leq y^{1/2}} n^{s-1} + O(t^{1/2-\sigma} \log t). \tag{4}
\]

Now replace \( x \) by \( y \) and \( s \) by \( 1-s \), and multiply by \( \chi(s)^2 \). Note that \( \chi(s)^2\chi(1-s) = \chi(s) \).

Also, since \( \chi(s)^2 \ll t^{1-2\sigma} \) and \( t^{1-2\sigma}t^{\sigma-\frac{1}{2}} = t^{\frac{1}{2}-\sigma} \), the error term is the same as before. As justified above, we also replace \( > \) by \( \geq \) in the range for \( m \). The conclusion is

\[
\chi(s)^2 \sum_{n \leq y} \tau(n)n^{s-1} = \chi(s)^2 S_2^2 + 2\chi(s) \sum_{n \leq y^{1/2}} n^{s-1} \sum_{\frac{im}{2\pi x} \leq m \leq x^{1/2}} m^{-s} + O(t^{1/2-\sigma} \log t). \tag{5}
\]

The condition \( n > \frac{im}{2\pi x} \) is equivalent to \( m < \frac{in}{2\pi y} \), and hence the double sums in (4) and (5) combine to give exactly

\[ 2\chi(s)S_1S_2. \]

So by adding (4) and (5), we obtain

\[
\sum_{n \leq x} \tau(n)n^{-s} + \chi(s)^2 \sum_{n \leq y} \tau(n)n^{s-1} = Z(x, s)^2 + O(t^{1/2-\sigma} \log t), \tag{6}
\]

where

\[ Z(x, s) = S_1 + \chi(s)S_2 = \zeta(s) + O(t^{-\sigma/2}). \]

Since (as mentioned above) \( \zeta(s) \ll t^{\frac{1}{2}(1-\sigma)} \), we have

\[ Z(x, s)^2 = \zeta(s)^2 + O(t^{1/2-\sigma}), \]

and (1) now follows.
Comparison with Motohashi’s method. As mentioned, this is only presented for the case \( y = x \), but adapts easily to the general case. The main difference is that the hyperbola method is used in the form

\[
\sum_{n \leq x} \tau(n)n^{-s} = 2 \sum_{m \leq x^{1/2}} m^{-s} \sum_{n \leq x/m} n^{-s} - S_1^2.
\]

The effect is to finish with

\[
2\zeta(s)Z(x, s) - Z(x, s)^2
\]

where we have \( Z(x, s)^2 \). This equates to \( \zeta(s)^2 - E^2 \), where \( E \) is the error term, hence to \( \zeta(s)^2 + O(t^{-\sigma}) \).

References

