The approximate functional equation for $\zeta(s)^2$

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As usual, write
$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s).$$
One version of the approximate functional equation for $\zeta(s)^2$ (e.g. [Iv, sect. 4.2]) states:

THEOREM. *Let $\tau(n)$ be the divisor function, and write $s = \sigma + it$. Let $0 < \sigma < 1$ and $x \asymp t$, and define y by $4\pi^2 xy = t^2$. Then*
$$\zeta(s)^2 = \sum_{n \le x} \tau(n) n^{-s} + \chi(s)^2 \sum_{n \le y} \tau(n) n^{s-1} + O(t^{\frac{1}{2}-\sigma} \log t). \tag{1}$$

Here I present a relatively simple proof which I devised before discovering that a similar one had been given in [Moto]. Actually, Motohashi only states the case where $x = y$, but his method can be adapted to the general case.

We start from the approximate functional equation for $\zeta(s)$ itself (e.g. [Ti, chapter 4]):

Let $0 < \sigma < 1$, $1 < u \le t$ and $2\pi uv = t$. Then
$$\zeta(s) = \sum_{n \le u} n^{-s} + \chi(s) \sum_{n \le v} n^{s-1} + O(u^{-\sigma}) + O(t^{-1/2} u^{1-\sigma}). \tag{2}$$

We will only be interested in the case when $u \gg t^{1/2}$, in which case the error term $u^{-\sigma}$ is dominated by $t^{-1/2}u^{1-\sigma}$. Also, since $\chi(s) \ll t^{\frac{1}{2}-\sigma}$, the term with $n = v$ (if this happens to be an integer) makes a contribution of magnitude
$$t^{\frac{1}{2}-\sigma}(t/u)^{\sigma-1} = t^{-1/2}u^{1-\sigma},$$
the same as the error term. So we can replace the condition $n \le v$ by $n < v$ without change. Note also that with u taken to be $t^{1/2}$, (2) implies that $\zeta(s) \ll t^{\frac{1}{2}(1-\sigma)}$.

Let $t^{1/2} \ll u_2 < u_1$ and $2\pi u_j v_j = t$. Applying (2) to u_1 and u_2 and taking the difference, we have
$$\sum_{u_2 < n \le u_1} n^{-s} = \chi(s) \sum_{v_1 < n \le v_2} n^{s-1} + O(t^{-1/2} u_1^{1-\sigma}). \tag{3}$$
The effect is to replace the sum of terms n^{-s} with the "reflected" sum of terms n^{1-s}.

Now assume that $x \asymp t$ and $4\pi^2 x^2 y^2 = t^2$. Write
$$S_1 = \sum_{m \le x^{1/2}} m^{-s}, \qquad S_2 = \sum_{n \le y^{1/2}} n^{s-1}.$$

Note that by (2),

$$S_1 + \chi(s)S_2\zeta(s) + O(t^{-\sigma/2}).$$

Expand $\sum_{n\le x}\tau(n)n^{-s}$ by the Dirichlet hyperbola method:

$$\begin{aligned}\sum_{n\le x}\tau(n)n^{-s} &= \sum_{mn\le x} m^{-s}n^{-s}\\ &= \sum_{m\le x^{1/2}} m^{-s}\sum_{n\le x/m} n^{-s} + \sum_{n\le x^{1/2}} n^{-s}\sum_{x^{1/2}<m\le x/n} m^{-s}\\ &= S_1^2 + 2\sum_{m\le x^{1/2}} m^{-s}\sum_{x^{1/2}<n\le x/m} n^{-s}.\end{aligned}$$

Now use (3) to replace the final sum by the corresponding reflected sum. The total error introduced is estimated by

$$\sum_{m\le x^{1/2}} m^{-\sigma}\, t^{-1/2}\left(\frac{x}{m}\right)^{1-\sigma} = t^{-\frac{1}{2}}x^{1-\sigma}\sum_{m\le x^{1/2}}\frac{1}{m} \ll t^{\frac{1}{2}-\sigma}\log t$$

(note that our stated objective was to express the error term in terms of t). So we obtain

$$\sum_{n\le x}\tau(n)n^{-s} = S_1^2 + 2\chi(s)\sum_{m\le x^{1/2}} m^{-s}\sum_{\frac{tm}{2\pi x}<n\le y^{1/2}} n^{s-1} + O(t^{\frac{1}{2}-\sigma}\log t). \tag{4}$$

Now replace x by y and s by $1-s$, and multiply by $\chi(s)^2$. Note that $\chi(s)^2\chi(1-s)=\chi(s)$. Also, since $\chi(s)^2 \ll t^{1-2\sigma}$ and $t^{1-2\sigma}t^{\sigma-\frac{1}{2}} = t^{\frac{1}{2}-\sigma}$, the error term is the same as before. As justified above, we also replace $>$ by $\ge$ in the range for m. The conclusion is

$$\chi(s)^2\sum_{n\le y}\tau(n)n^{s-1} = \chi(s)^2S_2^2 + 2\chi(s)\sum_{n\le y^{1/2}} n^{s-1}\sum_{\frac{tn}{2\pi y}\le m\le x^{1/2}} m^{-s} + O(t^{\frac{1}{2}-\sigma}\log t). \tag{5}$$

The condition $n > \frac{tm}{2\pi x}$ is equivalent to $m < \frac{tn}{2\pi y}$, and hence the double sums in (4) and (5) combine to give exactly

$$2\chi(s)S_1S_2.$$

So by adding (4) and (5), we obtain

$$\sum_{n\le x}\tau(n)n^{-s} + \chi(s)^2\sum_{n\le y}\tau(n)n^{s-1} = Z(x,s)^2 + O(t^{\frac{1}{2}-\sigma}\log t), \tag{6}$$

where

$$Z(x,s) = S_1 + \chi(s)S_2 = \zeta(s) + O(t^{-\sigma/2}).$$

Since (as mentioned above) $\zeta(s) \ll t^{\frac{1}{2}(1-\sigma)}$, we have

$$Z(x,s)^2 = \zeta(s)^2 + O(t^{\frac{1}{2}-\sigma}),$$

and (1) now follows.

Comparison with Motohashi's method. As mentioned, this is only presented for the case $y = x$, but adapts easily to the general case. The main difference is that the hyperbola method is used in the form

$$\sum_{n \le x} \tau(n) n^{-s} = 2 \sum_{m \le x^{1/2}} m^{-s} \sum_{n \le x/m} n^{-s} - S_1^2.$$

The effect is to finish with

$$2\zeta(s) Z(x, s) - Z(x, s)^2$$

where we have $Z(x, s)^2$. This equates to $\zeta(s)^2 - E^2$, where E is the error term, hence to $\zeta(s)^2 + O(t^{-\sigma})$.

References

[Iv] A. Ivić, *The Riemann Zeta Function*, Wiley (1985).

[Moto] Y. Motohashi, A note on the approximate functional equation for $\zeta^2(s)$, *Proc. Japan Acad.* **59A** (1983), 392–396.

[Ti] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function*, 2nd ed., Oxford Univ. Press (1986).