Bundles over quantum weighted projective spaces

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References:

Examples of quantum orbifolds.

Every $n$-orbifold is a quotient of a manifold by an almost free action of a subgroup of $O(n)$.

Every orbifold Riemannian surface is a quotient of a Seifert 3-manifold by an action of $S^1$.

Illustration of noncommutative resolution of singularities.
Let $q$ be a real number, $0 < q < 1$. The coordinate algebra $\mathcal{O}(S_q^{2n+1})$ of the odd-dimensional quantum sphere is the unital complex $\ast$-algebra with generators $z_0, z_1, \ldots, z_n$ subject to the following relations:

$$z_i z_j = q z_j z_i \quad \text{for } i < j,$$
$$z_i z_j^* = q z_j^* z_i \quad \text{for } i \neq j,$$
$$z_i z_i^* = z_i^* z_i + (q^{-2} - 1) \sum_{m=i+1}^{n} z_m z_m^*, \quad \sum_{m=0}^{n} z_m z_m^* = 1.$$

The coordinate algebra $\mathcal{O}(S_q^{2n})$ of the even-dimensional quantum sphere is the unital complex $\ast$-algebra with generators $z_0, z_1, \ldots, z_n$ and above relations supplemented with $z_n^* = z_n$. 
For any $n + 1$ pairwise coprime numbers $l_0, \ldots, l_n$, $\mathcal{O}(U(1)) = \mathbb{C}[u, u^*]$ coacts on $\mathcal{O}(S_q^{2n+1})$, as

$$\varrho_{l_0, \ldots, l_n} : z_i \mapsto z_i \otimes u^{l_i}, \quad i = 0, 1, \ldots, n.$$ 

The quantum weighted projective space is the subalgebra of $\mathcal{O}(S_q^{2n+1})$ containing all elements invariant under the coaction $\varrho_{l_0, \ldots, l_n}$, i.e.

$$\mathcal{O}(\mathbb{W}P_q(l_0, l_1, \ldots, l_n)) = \{ x \in \mathcal{O}(S_q^{2n+1}) \mid \varrho_{l_0, \ldots, l_n}(x) = x \otimes 1 \}.$$ 

In the case $l_0 = l_1 = \cdots = 1$ one obtains the algebra of functions on the quantum complex projective space $\mathbb{C}P_q^n$ defined in [YaS Soibelman’man & LL Vaksman ’90].
The weighted projective line $\mathcal{O}(\mathbb{WP}_q(k, l))$ is the $*$-algebra generated by $a$ and $b$ subject to the following relations

\[
a^* = a, \quad ab = q^{-2l} ba, \quad bb^* = q^{2kl} a^k \prod_{m=0}^{l-1} (1 - q^{2m} a),
\]

\[
b^* b = a^k \prod_{m=1}^{l} (1 - q^{-2m} a).
\]

Embedding in $\mathcal{O}(S^3_q)$:

\[
a \mapsto z_1 z_1^*, \quad b \mapsto z_0^l z_1^{*k}.
\]
$O(\mathbb{WP}_q(1, 1))$ is the standard Podleś sphere.

$O(\mathbb{WP}_q(1, l))$ is the quantum teardrop.

- No repeated roots in polynomial relations.
- Classical singularities are resolved.
Theorem

Up to a unitary equivalence, the following is the list of all bounded irreducible $\ast$-representations of $\mathcal{O}(\mathbb{WP}_q(k, l))$.

(i) One dimensional representation

$$\varphi_0 : a \mapsto 0, \quad b \mapsto 0.$$ 

(ii) Infinite dimensional faithful representations

$$\varphi_s : \mathcal{O}(\mathbb{WP}_q(k, l)) \to \text{End}(V_s), \ s = 1, 2, \ldots, l. \ V_s \cong l^2(\mathbb{N})$$ has orthonormal basis $e_p^s, \ p \in \mathbb{N}, \ \varphi_s(b)e_0^s = 0$ and

$$\varphi_s(a)e_p^s = q^{2(lp+s)}e_p^s,$$

$$\varphi_s(b)e_p^s = q^{k(lp+s)} \prod_{r=1}^{l} \left(1 - q^{2(lp+s-r)}\right)^{1/2} e_{p-1}^s.$$
**Theorem**

Let $\mathcal{K}_s$ denote the algebra of all compact operators on the Hilbert space $V_s$. There is a split-exact sequence of $C^*$-algebra maps

$$0 \to \bigoplus_{s=1}^{l} \mathcal{K}_s \to C(\mathbb{WP}_q(k, l)) \to \mathbb{C} \to 0.$$ 

Consequently,

$$K_0\left(C(\mathbb{WP}_q(k, l))\right) = \mathbb{Z}^{l+1}, \quad K_1\left(C(\mathbb{WP}_q(k, l))\right) = 0.$$
Quantum principal bundles

Definition

Let $H$ be a Hopf algebra with bijective antipode and let $A$ be a right $H$-comodule algebra with coaction $\varrho : A \to A \otimes H$. Let $B = A^{coH} := \{ b \in A \mid \varrho(b) = b \otimes 1 \}$. $A$ is a principal $H$-comodule algebra if:

(a) the coaction $\varrho$ is free, that is the canonical map

$$\text{can} : A \otimes_B A \to A \otimes H, \quad a \otimes a' \mapsto a \varrho(a'),$$

is bijective (the Hopf-Galois condition);

(b) there exists a strong connection in $A$, that is

$$B \otimes A \to A, \quad b \otimes a \mapsto ba,$$

splits as a left $B$-module and right $H$-comodule map (the equivariant projectivity).
A right $H$-comodule algebra $A$ with coaction $\varrho : A \to A \otimes H$ is principal if and only if it admits a strong connection form, that is if there exists a map $\omega : H \to A \otimes A$, such that

$$\omega(1) = 1 \otimes 1,$$

$$\mu \circ \omega = \eta \circ \varepsilon,$$

$$(\omega \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varrho) \circ \omega,$$

$$(S \otimes \omega) \circ \Delta = (\sigma \otimes \text{id}) \circ (\varrho \otimes \text{id}) \circ \omega.$$

Here $\mu : A \otimes A \to A$ denotes the multiplication map, $\eta : \mathbb{C} \to A$ is the unit map, $\Delta : H \to H \otimes H$ is the comultiplication, $\varepsilon : H \to \mathbb{C}$ counit and $S : H \to H$ the (bijective) antipode of the Hopf algebra $H$, and $\sigma : A \otimes H \to H \otimes A$ is the flip.
Theorem

The algebra $\mathcal{O}(S^3_q)$ is a principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{WP}_q(k, l))$ by the coaction $\varrho_{k,l}$ if and only if $k = l = 1$ (the quantum Hopf fibration).

Remark

The coaction $\varrho_{1,l}$ is almost free, meaning that the cokernel of the canonical map can is a finitely generated left $\mathcal{O}(S^3_q)$-module.
Carving out the total space

- $\mathbb{C}[Z_l]$ coacts on $\mathcal{O}(S^3_q)$ by

$$z_0 \mapsto z_0 \otimes w, \quad z_1 \mapsto z_1 \otimes 1.$$  

Coinvariants are the quantum lens space $\mathcal{O}(L_q(l; 1, l))$ [JH Hong & W Szymański ’03]. It is generated by $c = z_0^l$ and $d = z_1$, and admits an $\mathcal{O}(U(1))$-coaction

$$c \mapsto c \otimes u, \quad d \mapsto d \otimes u.$$  

- $\mathcal{O}(L_q(l; 1, l))$ is a non-cleft principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{WP}_q(1, l))$. Strong connection form:

$$\omega(u^n) = c^*\omega(u^{n-1})c - \sum_{m=1}^{l} (-)^m q^{-m(m+1)} \binom{l}{m}^q d^* m d^{m-1} \omega(u^{n-1})d.$$  

(Non-cleft because multiples of 1 are the only invertible elements of $\mathcal{O}(S^3_q)$.)
For any integer $n$, modules of sections on line bundles over $\mathcal{O}(\mathbb{WP}_q(1, l))$ (associated to the principal comodule algebra $\mathcal{O}(L_q(l; 1, l))$) are defined as

$$\mathcal{L}[n] := \{ x \in \mathcal{O}(L_q(l; 1, l)) \mid \varrho_l(x) = x \otimes u^n \}.$$ 

Each $\mathcal{L}[n]$ is a projective left $\mathcal{O}(\mathbb{WP}_q(1, l))$-module with an idempotent $E[n]$ given as follows. Write

$$\omega(u^n) = \sum_i \omega(v^n)^{[1]}_i \otimes \omega(u^n)^{[2]}_i.$$ 

Then

$$E[n]_{ij} = \omega(u^n)^{[2]}_i \omega(u^n)^{[1]}_j \in \mathcal{O}(\mathbb{WP}_q(1, l)).$$
Theorem

For every $s = 1, 2, \ldots, l$, $(\varphi_s, \oplus \varphi_0)$ is a 1-summable Fredholm module over $\mathcal{O}(\mathbb{WP}_q(k, l))$ with the Chern character

$$\tau_s(a^m b^n) = \frac{q^{2ms}}{1 - q^{2ml}},$$

for $n = 0$, $m \neq 0$, and 0 otherwise.

Theorem

For all $s = 1, 2, \ldots, l$,

$$\tau_s(\text{Tr } E[1]) = 1.$$

Consequently, the left $\mathcal{O}(\mathbb{WP}_q(1, l))-module L[1]$ is not free.
Quantum spheres are right comodule algebras over the Hopf algebra $O(\mathbb{Z}_2)$ generated by a self-adjoint grouplike element $\nu$ satisfying $\nu^2 = 1$. The coaction is

$$Z_i \mapsto Z_i \otimes \nu.$$ 

$O(U(1))$ is a left $O(\mathbb{Z}_2)$-comodule algebra with coaction:

$$u \mapsto \nu \otimes u.$$ 

Quantum spheres can be prolonged to algebras

$$O(\Sigma_{q+1}^m) := O(S_q^m) \Box_{O(\mathbb{Z}_2)} O(U(1)).$$
The structure of $\mathcal{O}(\Sigma^m_q)$

**Theorem (TB & BP Zieliński)**

1. $\mathcal{O}(\Sigma^{2n+2}_q)$ is an algebra isomorphic to $\mathcal{O}(S^{2n+1}_q) \otimes \mathcal{O}(U(1))$.

2. $\mathcal{O}(\Sigma^{2n+1}_q)$ is isomorphic to a polynomial $\ast$-algebra generated by $\zeta_0, \zeta_1, \ldots, \zeta_n$ and a central unitary $\xi$ subject to the following relations:

   $$
   \zeta_i \zeta_j = q \zeta_j \zeta_i \quad \text{for } i < j, \quad \zeta_i \zeta^*_j = q \zeta^*_j \zeta_i \quad \text{for } i \neq j,
   $$

   $$
   \zeta_i \zeta^*_i = \zeta^*_i \zeta_i + (q^{-2} - 1) \sum_{m=i+1}^{n} \zeta_m \zeta^*_m, \quad \sum_{m=0}^{n} \zeta_m \zeta^*_m = 1,
   $$

   $$
   \zeta^*_n = \zeta_n \xi.
   $$

3. For all $m > 1$, $\Sigma^{m+1}_q$ are non-trivial quantum $U(1)$-principal bundles over $\mathbb{RP}^m_q$.  

Brzeziński  Quantum weighted projective spaces
Choose \( n + 1 \) pairwise coprime numbers \( l_0, \ldots, l_n \).

\( \mathcal{O}(\Sigma^2_q) \) is a right \( \mathcal{O}(U(1)) \)-comodule algebra with coaction

\[
\varrho_{l_0, \ldots, l_n} : \xi \mapsto \xi \otimes u^{-2l_n}, \quad \zeta_i \mapsto \zeta_i \otimes u^{l_i}.
\]

Define

\[
\mathcal{O}(\mathbb{RP}_q(l_0, l_1, \ldots, l_n)) := \{ x \in \mathcal{O}(\Sigma^2_q) \mid \varrho_{l_0, \ldots, l_n}(x) = x \otimes 1 \}.
\]
The even case.

\( \mathcal{O}(\mathbb{RP}_q(2s, l)) \) is generated by \( a = \zeta_1^2 \xi \) and \( c_+ = \zeta_0^l \xi^s \), which satisfy the following relations: \( a^* = a \),

\[
ac_+ = q^{-2l} c_+ a, \quad c_+ c_+^* = \prod_{m=0}^{l-1} (1 - q^{2m} a), \quad c_+^* c_+ = \prod_{m=1}^{l} (1 - q^{-2m} a).
\]

A \( \ast \)-algebra generated by \( a, c_+ \) and above relations (with \( l \) odd) is denoted by \( \mathcal{O}(\mathbb{RP}_q^2(l; +)) \).
The odd case.

$\mathcal{O}(\mathbb{R}P_q(2s - 1, l))$ is generated by $a = \zeta_1^2 \xi$, $b = \zeta_0\zeta_1\xi^s$ and $c_- = \zeta_0^{-2l}\xi^k$, which satisfy the following relations: $a^* = a$,

$$ab = q^{-2l}ba, \quad ac_- = q^{-4l}c_- a, \quad b^2 = q^{3l}ac_-, \quad bc_- = q^{-2l}c_- b,$$

$$bb^* = q^{2l}a \prod_{m=0}^{l-1} (1 - q^{2m}a), \quad b^* b = a \prod_{m=1}^{l} (1 - q^{-2m}a),$$

$$b^* c_- = q^{-l} \prod_{m=1}^{l} (1 - q^{-2m}a)b, \quad c_- b^* = q^l b \prod_{m=0}^{l-1} (1 - q^{2m}a),$$

$$c_- c^* = \prod_{m=0}^{2l-1} (1 - q^{2m}a), \quad c^* c_- = \prod_{m=1}^{2l} (1 - q^{-2m}a).$$

A $*$-algebra generated by $a, b, c_-$ and above relations is denoted by $\mathcal{O}(\mathbb{R}P^2_q(l; -))$. 
Observations

- $\mathbb{RP}_q(1; +)$ is the quantum disc.
- $\mathbb{RP}_q(1; -)$ is the quantum real projective plane $\mathbb{RP}_q^2$ [PM Hajac, R Matthes, W Szymański].
- No multiple poles in polynomial relations for $\mathbb{RP}_q(l; \pm)$; the classical singularities are resolved.
Bounded representations of $\mathcal{O}(\mathbb{RP}_q^2(I; +))$

There is a family of one-dimensional representations of $\mathcal{O}(\mathbb{RP}_q^2(I; +))$ labelled by $\theta \in [0, 1)$ and given by

$$
\pi_{\theta}(a) = 0, \quad \pi_{\theta}(c_+) = e^{2\pi i \theta}.
$$

All other representations are infinite dimensional, labelled by $r = 1, \ldots, l$, and given by

$$
\pi_r(a) e^r_n = q^{2(ln+r)} e^r_n, \quad \pi_r(c_+) e^r_n = \prod_{m=1}^{l} \left( 1 - q^{2(ln+r-m)} \right)^{1/2} e^r_{n-1},
$$

where $e^r_n, n \in \mathbb{N}$ is an orthonormal basis for the representation space $\mathcal{H}_r \cong l^2(\mathbb{N})$. 
A family of one-dimensional representations labelled by $\theta \in [0, 1)$:

$$\pi_\theta(a) = 0, \quad \pi_\theta(b) = 0, \quad \pi_\theta(c_-) = e^{2\pi i \theta}.$$ 

All other representations are infinite dimensional, labelled by $r = 1, \ldots, l$:

$$\pi_r(a)e^r_n = q^{2(ln+r)}e^r_n, \quad \pi_r(c_-)e^r_n = \prod_{m=1}^{2l} \left(1 - q^{2(ln+r-m)}\right)^{1/2} e^r_{n-2},$$

$$\pi_r(b)e^r_n = q^{ln+r} \prod_{m=1}^l \left(1 - q^{2(ln+r-m)}\right)^{1/2} e^r_{n-1},$$

where $e^r_n, n \in \mathbb{N}$, is an orthonormal basis for the representation space $\mathcal{H}_r \cong l^2(\mathbb{N})$. 
Let $J_{\pm}$ be ideals of $C(\mathbb{R}P_q(l; \pm))$ generated by $a; p_{\pm}: C(\mathbb{R}P_q(l; \pm)) \to C(\mathbb{R}P_q(l; \pm))/J_{\pm}$.

$C(\mathbb{R}P_q(l; \pm))/J_{\pm}$ is generated by $p_{\pm}(c_{\pm})$ and $C(\mathbb{R}P_q(l; \pm))/J_{\pm} \cong C(S^1)$.

There are short exact sequences:

$$0 \longrightarrow \bigoplus_{r=1}^l K_r \xrightarrow{i_{\pm}} C(\mathbb{R}P_q(l; \pm)) \xrightarrow{p_{\pm}} C(S^1) \longrightarrow 0.$$ 

The six-term sequences of the $K$-groups lead to

$$0 \longrightarrow K_1(C(\mathbb{R}P_q(l; \pm))) \longrightarrow \mathbb{Z} \xrightarrow{\delta_{\pm}} \mathbb{Z}^l$$

$$K_0(i_{\pm}) \xrightarrow{} K_0(C(\mathbb{R}P_q(l; \pm))) \xrightarrow{K_0(p_{\pm})} \mathbb{Z} \longrightarrow 0.$$
The connecting map $\delta_+$ comes out as

$$\delta_+: \mathbb{Z} \rightarrow \mathbb{Z}^l, \quad m \mapsto (m, m, \ldots, m).$$

Since $\delta_+$ is injective, $K_1(C(\mathbb{RP}_q(l; +))) = 0$. Furthermore,

$$K_0(C(\mathbb{RP}_q(l; +))) = \text{Im}(K_0(i_+)) \oplus \mathbb{Z} \cong \text{coker}(\delta_+) \oplus \mathbb{Z} \cong \mathbb{Z}^l,$$

The connecting map $\delta_-$ comes out as

$$\delta_-: \mathbb{Z} \rightarrow \mathbb{Z}^l, \quad m \mapsto (2m, 2m, \ldots, 2m).$$

Since $\delta_-$ is injective, $K_1(C(\mathbb{RP}_q(l; -))) = 0$. Furthermore,

$$K_0(C(\mathbb{RP}_q(l; -))) \cong \text{coker}(\delta_-) \oplus \mathbb{Z} \cong \mathbb{Z}_2 \oplus \mathbb{Z}^l,$$
Starting from

\[
0 \longrightarrow K \longrightarrow T \xrightarrow{\sigma} C(S^1) \longrightarrow 0,
\]

one finds

\[
C(\mathbb{RP}_q(l; +)) \cong \underbrace{T \oplus \sigma T \oplus \sigma \cdots \oplus \sigma T}_{l\text{-times}}
\]

\[
\cong \underbrace{C(D_q) \oplus \sigma C(D_q) \oplus \sigma \cdots \oplus \sigma C(D_q)}_{l\text{-times}}.
\]

The symbol map \(\sigma\) can be understood as a projection induced by the inclusion of the classical circle as the boundary of the quantum disc.

\(C(\mathbb{RP}_q(l; +))\) is a quantum double suspension of \(l\) points [Hong-Szymański].
Starting from

\[
0 \to \mathcal{K} \to C(\mathbb{RP}^2_q) \xrightarrow{\bar{\sigma}} C(S^1) \to 0
\]

[Hajac-Matthes-Szymański], one finds

\[
C(\mathbb{RP}^l_q(I; -)) \cong C(\mathbb{RP}^2_q) \oplus \bar{\sigma} C(\mathbb{RP}^2_q) \oplus \bar{\sigma} \cdots \oplus \bar{\sigma} C(\mathbb{RP}^2_q),
\]

where the map \(\bar{\sigma}\) is induced from the projection that corresponds to the inclusion of the classical circle as the equator of the quantum sphere \(S^2_q\).
The algebra $\mathcal{O}(\Sigma^3_q)$ is a principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{RP}_q(k, l))$ by the coaction $\rho_{k, l}$ if and only if $k = l = 1$ (the circle bundle over $\mathbb{RP}^2_q$).

Remark

For $\mathcal{O}(\mathbb{RP}_q(l; \pm))$, the cokernel of the canonical map

$$\text{can} : \mathcal{O}(\Sigma^3_q) \otimes \mathcal{O}(\mathbb{RP}_q(l; \pm)) \mathcal{O}(\Sigma^3_q) \to \mathcal{O}(\Sigma^3_q) \otimes \mathcal{O}(U(1)),$$

(corresponding to $\rho_{k, l}$, $k = 1, 2$) is a finitely generated left $\mathcal{O}(\Sigma^3_q)$-module.
Carving out total spaces

- $\mathbb{C}[\mathbb{Z}_l]$ coacts on $\mathcal{O}(\Sigma_q^3)$ by

$$\zeta_0 \mapsto \zeta_0 \otimes w, \quad \zeta_1 \mapsto \zeta_1 \otimes 1, \quad \xi \mapsto \xi \otimes 1.$$

- Algebra of coinvariants $\mathcal{O}(\Sigma_q^3, l) - \mathcal{O}(\Sigma_q^3, l^-)$ is generated by $\alpha = \zeta_0$ and $\zeta_1, \xi, \zeta_1, \xi$, and admits an $\mathcal{O}(U(1))$-coaction

$$\alpha \mapsto \alpha \otimes u, \quad \zeta_1 \mapsto \zeta_1 \otimes u, \quad \xi \mapsto \xi \otimes u^*.$$

- $\mathcal{O}(\Sigma_q^3, l^-)$ is a non-cleft principal $\mathcal{O}(U(1))$-comodule algebra over $\mathcal{O}(\mathbb{RP}_q(l; -))$. Strong connection form:

$$\omega(u^n) = \alpha^* \omega(u^{n-1})\alpha - \sum_{m=1}^{l} (-1)^m q^{-m(m+1)} \binom{l}{m} q^{-2} \zeta_1^{2m-1} \xi^m \omega(u^{n-1})\zeta_1.$$
Carving out total spaces

- $\mathbb{C}[\mathbb{Z}_{2l}]$ ($l$-odd) coacts on $O(\Sigma^3_q)$ by

\[ \zeta_0 \mapsto \zeta_0 \otimes w^2, \quad \zeta_1 \mapsto \zeta_1 \otimes w^l, \quad \xi \mapsto \xi \otimes 1. \]

- Algebra of coinvariants $O(\Sigma^3_q, l)$ is generated by $\alpha = \zeta_0$, $\beta = \zeta_1^2$ and $\xi$ and admits an $O(U(1))$-coaction

\[ \alpha \mapsto \alpha \otimes u, \quad \beta \mapsto \beta \otimes u, \quad \xi \mapsto \xi \otimes u^*. \]

- $O(\Sigma^3_q, l)$ is a cleft (trivial) principal $O(U(1))$-comodule algebra over $O(\mathbb{RP}_q(l; -))$. The cleaving map is:

\[ O(U(1)) \rightarrow O(\Sigma^3_q, l), \quad u \mapsto \xi^*. \]
Weighted circle actions on the Heegaard-type quantum sphere.

Interpretation of weighted projective spaces in terms of graph $C^*$-algebras.

Development of differential structures on weighted projective spaces.

Spectral geometry of quantum weighted projective spaces.