Cohomology of commutative Banach algebras and ℓ^1-semigroup algebras

YEMON CHOI

Thesis submitted for the degree of

Doctor of Philosophy

UNIVERSITY OF
NEWCASTLE

School of Mathematics and Statistics

University of Newcastle upon Tyne

Newcastle upon Tyne

United Kingdom

May 2006

For APS,

comrade and confidante in bygone times,

and to whom I shall always be indebted.

Audere est facere.

Acknowledgements

First and foremost I would like to thank my supervisor Michael White: for all the encouragement and guidance he has given me over the last three years, for constant generosity with his time and mathematical insight, and for being so good-humoured about my lax interpretations of the phrase "by tomorrow".

Each person should render unto Caesar what is Caesar's; this would not have been possible for me without the financial support provided by EPSRC. I am also grateful to the Product Development department of the National Extension College, Cambridge, for providing a year outside the bubble, and for showing me more about Wider Key Skills than I really wanted to know.

The production of this thesis has benefited from a wealth of other people's freely available TEXpertise. Particular thanks are due to Paul Taylor for his `diagrams.sty` macros, without which what follows would be several pages shorter and significantly less readable.

Life as a PhD student would have been much more arduous without the help and cheer provided by staff and fellow students in the School of Mathematics and Statistics. Special thanks to the 5-a-siders for putting up with my lack of first (or second) touch, and to the Jesmond Academics for reintroducing me to the simple pleasures of sacrificing on f7.

I am deeply grateful to my family, who have been ever-supportive of my studies and who have tolerated many garbled attempts at explanation over the years. Lastly, thanks to friends past and present for their patience, and for making this all worthwhile.

"What is this thing, anyway?" said the Dean, inspecting the implement in his hands.

"It's called a shovel," said the Senior Wrangler. "I've seen the gardeners use them. You stick the sharp end in the ground. Then it gets a bit technical."

– from *Reaper Man* by Terry Pratchett

Contents

Abstract

In this thesis we investigate the higher cohomology groups of various classes of Banach algebras, focusing on the ℓ^1-convolution algebras of commutative semigroups.

The properties of *amenability* and *weak amenability* have been studied by many authors. A theme of the work presented here is that for commutative Banach algebras, vanishing conditions on simplicial cohomology – which provide a spectrum of intermediate notions between weak and full amenability – yield partial results on cohomology with symmetric coefficients.

The strongest such vanishing condition is *simplicial trivality.* We show that the augmentation ideals of ℓ^1-group algebras are simplicially trivial for a wide class of groups: this class includes all torsion-free, finitely-generated word-hyperbolic groups, but not the direct product $F_2 \times F_2$. Using very different arguments we show that the ℓ^1-convolution algebras of commutative Clifford semigroups are simplicially trivial. The proof requires an inductive normalisation argument not covered by existing results in the literature.

Recent results on the simplicial cohomology of $\ell^1(\mathbb{Z}_+^k)$ are used to obtain vanishing results for the cohomology of $\ell^1(\mathbb{Z}_+^k)$ in degrees 3 and above for a restricted class of symmetric coefficient modules. In doing so we briefly investigate topological and algebraic properties of the first simplicial homology group of $\ell^1(\mathbb{Z}_+^k)$. In contrast, examples are given of weighted convolution algebras on $\mathbb{Z}$ whose third simplicial cohomology groups are non-Hausdorff (and in particular are non-zero).

We also investigate a natural extension of the cohomological notion of a *smooth commutative ring* to the setting of commutative Banach algebras, giving examples to show that the notion is perhaps too restrictive in a functional-analytic setting.

Introduction

Hochschild cohomology has proved a useful tool in studying commutative rings and group rings. The analogous theory for Banach algebras has been studied by many authors. However, there have been several technical obstacles in the Banach world that make computations more difficult (and the theory more subtle) than in the "purely algebraic" setting. For example, while the polynomial ring $\mathbb{C}[z]$ in one variable has homological dimension 1, that of its ℓ^1-completion $\ell^1(\mathbb{Z}_+)$ has nontrivial cohomology in dimension 2.

In this thesis we aim to compute the cohomology groups of various classes of Banach algebras. For the most part we restrict our attention to the ℓ^1-convolution algebras of semigroups: such algebras constitute a class wide enough to contain examples with very different behaviour, yet are nevertheless accessible to direct computation since many potential difficulties with multilinear maps on Banach spaces are rendered harmless. Some of the tools we develop are nevertheless stated in more general settings.

Chapter 1

Preliminaries

1.1 General notation and terminology

Two important conventions apply throughout. All vector spaces (and consequently all algebras, modules etc.) are assumed in this thesis to have ground field $\mathbb{C}$ *unless explicitly stated otherwise*. The symbol := is used to mean "defined to be equal to", in analogy with the programmer's use of "$a := b$" to mean "set a equal to b".

Seminormed and Banach spaces

If $(V, \|\_\_\|)$ is a seminormed vector space then we shall always equip it with the canonical topology that is induced by the pseudometric $(x, y) \mapsto \|x - y\|$. Note that this topology need not be Hausdorff; indeed, it is Hausdorff if and only if $\{0\}$ is a closed subset of V.

Just as for normed spaces, a bounded linear map between seminormed spaces is continuous. It follows that if E and F are seminormed spaces and there exist bounded linear, mutually inverse maps $S : E \to F$ and $T : F \to E$, then E and F are linearly homeomorphic. In such an instance we shall say that E and F are isomorphic as seminormed spaces.

Remark. The point of labouring this definition is that *a continuous linear bijection from one seminormed space onto another need not be a homeomorphism*, even when both spaces are complete. (In particular such a bijection is not necessarily an open map, in contrast to what happens for Banach spaces.)

An easy – albeit artifical – example is provided by the identity map $\iota : (V, \_\_) \to (V, \|\_\_\|_0)$, where $\|\_\_\|$ is a not-identically-zero seminorm on V and $\|\_\_\|_0$ denotes the zero seminorm; clearly ι is norm-decreasing and hence is continuous, but it cannot be a homeomorphism since the topology induced by $\|\_\_\|_0$ is the discrete one.

If X and Y are seminormed spaces then (following [7]) we shall write $X \underset{1}{\cong} Y$ to denote an *isometric* linear isomorphism between X and Y.

If E and F are *Banach spaces* then we shall denote the projective tensor product of E and F by $E\widehat{\otimes}F$. We expect that the reader is familiar with the definition of $\widehat{\otimes}$ and some of its basic properties: for a gentle account of such matters, see [29, Ch. 2]. Note also that if $\psi_1 : E_1 \to F_1$ and $\psi_2 : E_2 \to F_2$ are bounded linear maps between Banach spaces, then we write $\psi_1\widehat{\otimes}\psi_2$ for the bounded linear map $E_1\widehat{\otimes}E_2 \to F_1\widehat{\otimes}F_2$ that is defined by

$$(\psi_1\widehat{\otimes}\psi_2)(x_1\otimes x_2) := \psi_1(x_1)\otimes\psi_2(x_2) \qquad (x_1 \in E_1, x_2 \in E_2).$$

Modules over a Banach algebra

If A is a Banach algebra then our definition of a left Banach A-module is the standard one: we require that the action of A is continuous but do not assume that it is necessarily contractive. We shall assume the reader is familiar with the definition of left, right and two-sided Banach modules: for details see the introductory sections of [19].

One point where we have departed from the terminology used in [19] is in consistent use of the phrase "A-module map" to mean "map preserving A-module structure". *In particular, A-module maps are a priori linear throughout this thesis.* (The terminology "A-module morphism" is used in [19]; we have chosen to follow more concise terminology which can be found in many texts on ring theory and is thus an alternative standard.)

Categories and functors

We make use of basic categorical language, up to the notion of natural transformation between functors: see [25] for the relevant background.

Some of the categories that will appear in this thesis should be familiar to the reader: for instance Vect is the category of vector spaces and linear maps between them.

Other categories which will recur throughout the thesis are:

- Ban: objects are Banach spaces; morphisms are continuous linear maps.
- BAlg: objects are Banach algebras; morphisms are continuous algebra homomorphisms.
- BAlg^+: objects are unital Banach algebras; morphisms are continuous, *unital* algebra homomorphisms.

Note that in each of these three examples, morphisms are bounded linear maps and may therefore be equipped with the operator norm. Therefore, if $\mathcal{C}$ denotes one of these three categories, we may consider the subcategory $\mathcal{C}_1$ which has the same objects as $\mathcal{C}$ but where the morphisms are required to also be *contractive* linear maps. (For instance, BAlg_1 is the category of Banach algebras and contractive algebra homomorphisms between them.)

If A is a Banach algebra then we denote by ${}_A\mathsf{mod}$, mod_A the categories of left and right Banach A-modules respectively; in both cases the morphisms are take to be the bounded left (respectively right) A-module maps. If B is another Banach algebra then we let ${}_A\mathsf{mod}_B$ denote the category of Banach A-B-bimodules and A-B-bimodule maps.

If A and B are unital Banach algebras, then the corresponding categories of *unit-linked* modules and module maps will be denoted by ${}_A\mathsf{unmod}$, unmod_A and ${}_A\mathsf{unmod}_B$ respectively.

1.2 A word on units

Let A be a Banach algebra: that is, a Banach space A equipped with an associative multiplication map $A \times A \to A$ such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

Definition 1.2.1. The forced unitisation of A is the Banach algebra whose underlying Banach space is $A \oplus \mathbb{C}$ (where the norm on this sum is the ℓ^1-sum norm) and whose multiplication is defined by

$$(a, \lambda) \oplus (b, \mu) := (ab + \mu a + \lambda b, \lambda\mu) \qquad (a, b \in A; \lambda, \mu \in \mathbb{C})$$

We denote the forced unitisation of A by $A^{\#}$.

The following definition allows us to state some results more concisely.

Definition 1.2.2. Let A be a Banach algebra. We define the conditional unitisation of A to be $A^{\#}$ if A has no identity element, and A itself if A already has a unit element. The conditional unitisation of A will be denoted by A_{un}.

Definition 1.2.3. If R is a ring (not necessarily with identity element) we denote by R^{op} the opposite of R, i.e. the ring with the same underlying set and additive structure as R but with multiplication defined by

$$(r,s) \mapsto sr \qquad (r,s \in R).$$

Definition 1.2.4. The enveloping algebra of A is the Banach algebra $A_{\mathsf{un}}\widehat{\otimes}A_{\mathsf{un}}^{\mathsf{op}}$. Often we abbreviate $A_{\mathsf{un}}\widehat{\otimes}A_{\mathsf{un}}^{\mathsf{op}}$ to A^e.

1.3 Homological algebra in normed settings

Our standard reference for this section is [19], and we shall not repeat here the definitions and basic properties of (relative) projectivity, injectivity, flatness, Tor and Ext that are developed in that book. We content ourselves with establishing notation in the following definition.

Definition 1.3.1. Let A be a Banach algebra; let X and Y be left Banach A-modules and let Z be a right Banach A-module. Let

$$0 \leftarrow X \leftarrow P_0 \leftarrow P_1 \leftarrow \dots$$

be an exact sequence in ${}_A\mathsf{mod}$ which splits in the category Ban, such that each P_n is A-projective. (We say that P_* is an admissible resolution of X by A-projective modules.)

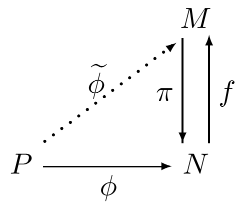
Then $\operatorname{Tor}_*^A(Z,X)$ is defined to be the homology of the chain complex $0 \leftarrow Z\underset{A}{\widehat{\otimes}}P_*$, while $\operatorname{Ext}^A(X,Y)$ is defined to be the cohomology of the cochain complex $0 \to {}_A\operatorname{Hom}(P_*,Y)$.

We recall from this definition that by construction the Tor and Ext groups are complete seminormed spaces. (Strictly speaking, they are determined up to isomorphism of seminormed spaces but not up to *isometric* isomorphism.)

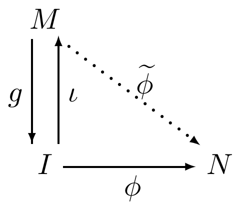
The defintions of relative projectivity, injectivity and flatness, as laid out in [19], clearly admit quantitative versions. Such versions are not defined explicitly in [19]; the clearest reference appears to be in the first half of the paper [31]. We reproduce the relevant definitions below, paraphrased slightly: our notion of "projective (resp. injective, flat) with constant C" corresponds to the notion of "C-relatively projective (resp. injective, flat)" in [31].

In what follows, fix a Banach algebra A and let $C > 0$.

Definition 1.3.2 ([31, Defn 2.2]). A left Banach A-module P is projective with constant C if, for every bounded left A-module map $\pi : M \to N$ which has a bounded linear right inverse $f : N \to M$, and every bounded left A-module map $\phi : P \to N$, there exists a bounded left A-module map $\widetilde{\phi} : P \to M$ such that $\pi\widetilde{\phi} = \phi$ and $\|\widetilde{\phi}\| \leq C\|\phi\|\|f\|$.



Definition 1.3.3 ([31, Defn 3.2]). A right Banach A-module I is injective with constant C if, for every bounded right A-module map $\iota : N \to M$ which has a bounded linear left inverse $g : M \to N$, and every bounded right A-module map $\phi : N \to I$, there exists a morphism $\widetilde{\phi} : M \to I$ such that $\widetilde{\phi}\iota = \phi$ and $\|\widetilde{\phi}\| \leq C\|\phi\|\|g\|$.



Definition 1.3.4 ([31, Defn 4.3]). A left Banach A-module M is flat with constant C if for every short exact sequence of right Banach A-module maps

$$0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0$$

where ι has a bounded linear left inverse $f : Y \to X$, the sequence

$$0 \to X \underset{A}{\widehat{\otimes}} M \xrightarrow{\iota} Y \underset{A}{\widehat{\otimes}} M \xrightarrow{\pi} Z \underset{A}{\widehat{\otimes}} M \to 0$$

is short exact and satisfies $\|\iota \underset{A}{\widehat{\otimes}} \mathsf{id}_M\| \leq C\|f\|$.

Lemma 1.3.5 ([31, Propn 4.9]). *Let M be a left Banach A-module. Then M is flat with constant C if and only if M' is injective with constant C.*

Remark. In this thesis the custom is to use the phrase "A-projective" to mean "relatively projective as a Banach A-module". We have adopted slightly different terminology from other sources, since in several results more than one Banach algebra is considered; on such occasions it is convenient to have the shorter phrase to hand.

1.4 Hochschild homology and cohomology for Banach algebras

There are several accounts of the basic definitions that we need: the canonical references are Johnson's 1972 monograph [21] and Helemskii's tome [19]. Let us briefly repeat the relevant definitions in order to set out the notational conventions which will be used throughout this thesis.

Definition 1.4.1. Let A be a Banach algebra (not necessarily unital) and let M be a Banach A-bimodule. For $n \geq 0$ we define

$$\begin{aligned} \mathcal{C}_n(A,M) &:= M \widehat{\otimes} A^{\widehat{\otimes} n} \\ \mathcal{C}^n(A,M) &:= \{\text{bounded, } n\text{-linear maps } \overbrace{A \times \ldots \times A}^{n} \to M\} \end{aligned} \tag{1.1}$$

For $0 \leq i \leq n+1$ the face maps $\partial_i^n : \mathcal{C}_{n+1}(A,M) \to \mathcal{C}_n(A,M)$ are the contractive linear maps given by

$$\partial_i^n(x \otimes a_1 \otimes \ldots \otimes a_{n+1}) = \begin{cases} x a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1} & \text{if } i = 0 \\ x \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1} & \text{if } 1 \leq i \leq n \\ a_{n+1} x \otimes a_1 \otimes \ldots \otimes a_n & \text{if } i = n+1 \end{cases}$$

and the Hochschild boundary operator $\mathsf{d}_n : \mathcal{C}_{n+1}(A,M) \to \mathcal{C}_n(A,M)$ is given by

$$\mathsf{d}_n = \sum_{j=0}^{n+1} (-1)^j \partial_j^n \ .$$

With these definitions, the Banach spaces $\mathcal{C}_n(A,M)$ assemble into a chain complex

$$\ldots \xleftarrow{\mathsf{d}_{n-1}} \mathcal{C}_n(A,M) \xleftarrow{\mathsf{d}_n} \mathcal{C}_{n+1}(A,M) \xleftarrow{\mathsf{d}_{n+1}} \ldots$$

called the Hochschild chain complex of (A,M).

Dually, the Banach spaces $\mathcal{C}^n(A,M)$ assemble into a cochain complex

$$\ldots \xrightarrow{\delta^{n-1}} \mathcal{C}^n(A,M) \xrightarrow{\delta^n} \mathcal{C}^{n+1}(A,M) \xrightarrow{\delta^{n+1}} \ldots$$

(the Hochschild cochain complex of (A, M)), where the Hochschild coboundary operator δ is given by

$$\delta^n\psi(a_1, \ldots, a_{n+1} = \begin{cases} a_1\psi a_2, \ldots, a_{n+1}) \\ +\sum_{j=1}^{n}(-1)^j\psi(a_1, \ldots, a_ja_{j+1}, \ldots, a_{n+1}) \\ +(-1)^{n+1}\psi(a_1, \ldots, a_n)a_{n+1} \end{cases}$$

We let

$$\begin{aligned} \mathcal{Z}_n(A, M) &:= \ker \mathsf{d}_{n-1} && \text{(the space of } n\text{-cycles)} \\ \mathcal{B}_n(A, M) &:= \operatorname{im} \mathsf{d}_n && \text{(the space of } n\text{-boundaries)} \\ \mathcal{H}_n(A, M) &:= \frac{\mathcal{Z}_n(A, M)}{\mathcal{B}_n(A, M)} && \text{(the } n\text{th Hochschild homology group)} \end{aligned} \tag{1.2}$$

Similarly,

$$\begin{aligned} \mathcal{Z}^n(A, M) &:= \ker \delta_n && \text{(the space of } n\text{-cocycles)} \\ \mathcal{B}^n(A, M) &:= \operatorname{im} \delta_{n-1} && \text{(the space of } n\text{-coboundaries)} \\ \mathcal{H}^n(A, M) &:= \frac{\mathcal{Z}^n(A, M)}{\mathcal{B}^n(A, M)} && \text{(the } n\text{th Hochschild cohomology group)} \end{aligned} \tag{1.3}$$

Remark. In the literature the spaces defined above are often referred to as the space of *bounded* n-cycles, *continuous* n-cocycles, etc. and the resulting homology and cohomology groups are then called the continuous Hochschild homology and cohomology groups, respectively, of (A, M). We have chosen largely to omit these adjectives as we never deal with the purely algebraic Hochschild cohomology of Banach algebras; if A is a Banach algebra and M a Banach A-bimodule then there will be no ambiguity in this thesis about what the Hochschild cohomology of (A, M) means.

Theorem 1.4.2 ([19, Thms III.4.6, III.4.25]). *Let A be a Banach algebra and X a Banach A-bimodule. We may regard the conditional unitisation A_{un} as a left A^e-module via the action*

$$(a_1 \otimes a_2) \cdot b := a_1ba_2 \qquad (a_1 \otimes a_2 \in A^e; b \in A_{\mathsf{un}})$$

We may regard X as a right A^e-module $\widetilde{X}_R$ via the action

$$x \cdot (a_1 \otimes a_2) := a_2xa_1 \qquad (a_1 \otimes a_2 \in A^e, x \in X)$$

and also as a left A^e-module via $_L\widetilde{X}$ via the action

$$(a_1 \otimes a_2) \cdot x := a_1xa_2 \qquad (a_1 \otimes a_2 \in A^e, x \in X)$$

Then there are isomorphisms of seminormed spaces

$$\mathcal{H}_n(A,X) \cong \mathrm{Tor}_n^{A^e}(\widetilde{X}_R, A_{\mathsf{un}}) \qquad (n \in \mathbb{Z}_+)$$

and

$$\mathcal{H}^n(A,X) \cong \mathrm{Ext}^n_{A^e}(A_{\mathsf{un}}, {}_L\widetilde{X}) \qquad (n \in \mathbb{Z}_+)$$

At this point it is useful to fix an important convention. When we talk about the seminorm on a Hochschild homology or cohomology group, we shall *always* mean the canonical quotient seminorm inherited from the norm on the Hochschild chain or cochain complex, respectively. In this sense, relative Tor and Ext only calculate Hochschild homology and cohomology up to topological (linear) isomorphism of seminormed spaces; if we wish to determine the Hochschild groups up to *isometric (linear) isomorphism* then more direct calculations are required.

K-relative (co)homology

In later sections we shall briefly make use of K-relative homology groups $\mathcal{H}^n_K(A,X)$. More common – and arguably easier to get a feel for – are the K-relative cohomology groups, which we now define.

Definition 1.4.3. Let A be a Banach algebra, K a subalgebra, X a Banach A-bimodule. An n-cochain $T \in \mathcal{C}^n(A,X)$ is said to be K-balanced or K-normalised if it has the following properties:

- $T(ca_1, a_2, \ldots, a_n) = cT(a_1, \ldots, a_n)$
- $T(a_1, \ldots, a_j c, a_{j+1}, \ldots, a_n) = T(a_1, \ldots, a_j, ca_{j+1}, \ldots, a_n)$ for $j = 1, \ldots, n-1$
- $T(a_1, \ldots, a_n c) = T(a_1, \ldots, a_n)c$

for all $c \in K$ and $a_1, \ldots, a_n \in A$. We denote the subspace of all such cochains by $\mathcal{C}^n_K(A,X)$.

It is easily checked that if T is K-normalised then so is δT; hence $\mathcal{C}^*_K(A,X)$ is a subcomplex of $\mathcal{C}^*(A,X)$, and we define

$$\begin{aligned}
\mathcal{Z}^n_K(A,X) &:= \mathcal{Z}^n(A,X) \cap \mathcal{C}^n_K(A,X) \\
\mathcal{B}^n_K(A,X) &:= \delta_n\left(\mathcal{C}^{n+1}_K(A,X)\right) \\
\mathcal{H}^n_K(A,X) &:= \frac{\mathcal{Z}^n_K(A,X)}{\mathcal{B}^n_K(A,X)}
\end{aligned}$$

These spaces are, respectively, the space of K-normalised n-cocycles, the space of K-normalised n-coboundaries and the nth K-relative cohomology group of A with coefficients in X.

Less common in the context of Banach algebras is the (pre)dual notion of K-relative homology groups. The definition is as follows.

Definition 1.4.4. Let A be a Banach algebra, K a subalgebra, X a Banach A-bimodule. The space of K-normalised n-chains on A with coefficients in X is the Banach space $X \underset{K^e}{\widehat{\otimes}} \left(A^{\widehat{\otimes}_K n} \right)$, and is denoted by $\mathcal{C}_n^K(A, X)$.

More explicitly, $\mathcal{C}_n^K(A, X)$ is defined to be the quotient of $X \widehat{\otimes} A^{\widehat{\otimes} n}$ by the closed subspace $\mathcal{N}_n(K)$, where $\mathcal{N}_n$ is the closed linear span of all tensors of the form

$$
\begin{aligned}
& xc\otimes a_1 \otimes \dots a_n - x \otimes c a_1 \dots a_n \\
\text{or } & x \otimes a_1 \dots \otimes a_j c \otimes a_{j+1} \otimes \dots a_n - x \otimes a_1 \dots \otimes a_j \otimes c a_{j+1} \otimes \dots a_n \qquad (1 \leq j \leq n-1) \\
\text{or } & x \otimes a_1 \otimes \dots \otimes a_n c - cx \otimes a_1 \otimes \dots \otimes a_n
\end{aligned}
$$

where $x \in X$, $c \in K$ and $a_1, \dots, a_n \in A$.

By considering the action of face maps on each such tensor, one sees that $\mathsf{d}_{n-1}(\mathcal{N}_n(K)) \subseteq \mathcal{N}_{n-1}(K)$, and so $\mathcal{N}_\bullet(K)$ is a subcomplex of the Hochschild chain complex $\mathcal{C}_*(A, X)$. Hence by a standard diagram chase the quotient spaces $\mathcal{C}_*^K(A, X)$ form a quotient complex of $\mathcal{C}_*(A, X)$, whose homology groups are the K-relative homology groups of A with coefficients in X.

The K-relative (co)homology groups of A may be easier to compute than the "full" (co)homology groups, but it is not always clear how to relate the two families. Usually one starts with A and tries to choose K such that the natural maps $\mathcal{H}_K^* \to \mathcal{H}^*$ and $\mathcal{H}_* \to \mathcal{H}_*^K$ are isomorphisms; we shall only require the following well-known instance of this phenomenon.

Definition 1.4.5. A Banach algebra K is said to be contractible if there exists $\Delta \in K \widehat{\otimes} K$ satisfying the following conditions:

(i) $x \cdot \Delta = \Delta \cdot x$ for all $x \in K$;

(ii) $x\pi(\Delta) = x = \pi(\Delta)x$ for all $x \in K$

where $\pi_K : K \widehat{\otimes} K \to K$ is the product map. Such a Δ, if it exists, is called a diagonal for K.

Let $\mathcal{B}$ be a unital Banach algebra and let K be a unital, contractible subalgebra. As a special case of standard results, we can "normalise cochains on $\mathcal{B}$ with respect to K". Later on, in Chapter 5, we will need to know a quantitative version of this fact, and it seems worth isolating the precise statement here.

The following result is stated in the simplest version suitable for our future purposes, and not in its full generality (for more on this, please see Appendix A). In particular we shall only state the result for chains and not for cochains as this is the setting in which we shall need it.

Theorem 1.4.6. *Let A be a unital Banach algebra and K a finite-dimensional, unital subalgebra.*

Suppose that K is contractible, with diagonal Δ, and let X be a Banach A-bimodule. Then there exists a chain map $\alpha : \mathcal{C}_*(A,X) \to \mathcal{C}_*(A,X)$ with the following properties:

(a) each α_n is K-normalised, i.e. factors through the quotient map $\mathcal{C}_n(A,X) \to \mathcal{C}_n^K(A,X)$;

(b) there exists a chain homotopy from id to α, given by bounded linear maps $t_n : \mathcal{C}_n(A,X) \to \mathcal{C}_{n+1}(A,X)$ satisfying $\mathsf{d}_n t_n + t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \alpha_n$ for all n;

(c) the norm of each t_n is bounded above by some constant depending only on n and $\|\Delta\|$.

Consequently, for each n the canonical map $\mathcal{H}_n(A,X) \to \mathcal{H}_n^K(A,X)$ is an isomorphism of seminormed spaces.

I have not been able to find a reference which gives this theorem in full detail: the problem is part (c), which is implicitly known in the field, but is merely hinted at or sketched in the references I could find. Therefore a proof is provided in Appendix A.

Symmetric coefficients

Finally, we make a seemingly trivial observation which will be exploited in later chapters. The idea to introduce this extra structure is not at all original, but there seems to have been no systematic pursuit of this line of enquiry in the Banach-algebraic setting.

Proposition 1.4.7. *Let A be a commutative Banach algebra and let M be a symmetric Banach A-bimodule. For each n, regard $\mathcal{C}_n(A,M)$ and $\mathcal{C}^n(A,M)$ as left Banach A-modules, via the actions*

$$c\cdot(m\otimes a_1\otimes\ldots\otimes a_n) := (c\cdot m)\otimes a_1\otimes\ldots\otimes a_n \qquad (m\in M; c,a_1,\ldots,a_n\in A)$$

and

$$(c\cdot T)(a_1,\ldots,a_n) := c\cdot[T(a_1,\ldots,a_n)]$$

respectively. Then the boundary maps $\mathsf{d}_n : \mathcal{C}_{n+1}(A,M)\to\mathcal{C}_n(A,M)$ and the coboundary maps $\delta_n : \mathcal{C}^n(A,M)\to\mathcal{C}^{n+1}(A,M)$ are A-module maps.

In particular, the Hochschild chain complex

$$\mathcal{C}_0(A,A_{\mathsf{un}}) \longleftarrow \mathcal{C}_1(A,A_{\mathsf{un}}) \longleftarrow \mathcal{C}_2(A,A_{\mathsf{un}}) \longleftarrow \ldots$$

is a complex of Banach A-modules, and we have the following isometric isomorphisms of chain complexes:

$$\mathcal{C}_*(A,M) \underset{1}{\cong} M_R\widehat{\otimes}_{A_{\mathsf{un}}}\mathcal{C}_*(A,A_{\mathsf{un}})$$
$$\mathcal{C}^*(A,M) \underset{1}{\cong} {}_{(A_{\mathsf{un}})}\mathrm{Hom}\,(\mathcal{C}_*(A,A_{\mathsf{un}}),M_L)$$

where M_L and M_R are the one-sided modules obtained by restricting the action on M to left and right actions respectively.

The proposition is really just a statement about the boundary and coboundary operators, and its proof is immediate from their definition.

Remark. We can make the chain isomorphisms above explicit: the one for homology is given in degree n on elementary tensors by

$$m\otimes a_1\otimes\ldots a_n \longleftrightarrow m\underset{A_{\mathsf{un}}}{\otimes}\mathbf{1}\otimes a_1\otimes\ldots\otimes a_n$$

and that for cohomology is given in degree n by

$$\theta^n : {}_{(A_{\mathsf{un}})}\mathrm{Hom}\,(\mathcal{C}_*(A,A_{\mathsf{un}}),M_L)\to\mathcal{C}^n(A,M)$$

where $\theta^n\psi(a_1,\ldots,a_n) := \psi(\mathbf{1}\otimes a_1\otimes\ldots\otimes a_n)$.

1.5 The "Hodge decomposition": basic definitions

The "Hodge decomposition" of the title gives a decomposition of the Hochschild homology and cohomology of a complex algebra. It was first introduced in Gerstenhaber and Schack's paper [10]; for some of the history and context behind that paper, the reader is recommended to consult Gerstenhaber's excellent survey article [9].

We shall follow the exposition in [30, §9.4.3] which provides a terse guide. More details can be found in Loday's book [23].

Remark. The material in this section is largely a presentation of standard knowledge from commutative algebra, with the adjectives "Banach" or "bounded" inserted in the obvious places. However, there do not seem to be any explicit references for the Banach-algebraic case. We shall therefore endeavour to give precise statements, even when the proofs are trivial; the alternative approach would have led to tiresome repetition of the phrase "just as in the purely algebraic case, the reader may check that ...".

Let us start in the setting of $\mathbb{C}$-algebras. Fix $n \in \mathbb{N}$: then for any $\mathbb{C}$-vector space V the permutation group S_n acts on $V^{\otimes n}$. This induces an action of the group algebra $\mathbb{Q}S_n$ on the vector space $V^{\otimes n}$: we shall identify elements of $\mathbb{Q}S_n$ with the linear maps $V^{\otimes n} \to V^{\otimes n}$ that they induce.

With this notation, we can now state the so-called "Hodge decomposition" of Gerstenhaber and Schack in a form convenient for us.

Theorem 1.5.1 (Hodge decomposition for commutative $\mathbb{C}$-algebras). *Let B be a commutative $\mathbb{C}$-algebra. For each $n \geq 1$ there are pairwise orthogonal idempotents in $\mathbb{Q}S_n$, denoted $e_n(1), e_n(2), \ldots$, which satisfy*

(i) $e_n(j) = 0$ *for all* $j > n$;

(ii) $\sum_i e_n(i) = 1_{\mathbb{Q}S_n}$;

and are such that for each $i \in \mathbb{N}$, $\mathsf{id}{\otimes}e_\bullet(i)$ acts a chain map on $\mathcal{C}_n^{\mathsf{alg}}(B, B_{\mathsf{un}})$, i.e. the diagram

$$\begin{array}{ccc}
\mathcal{C}_{n-1}^{\mathsf{alg}}(B, B_{\mathsf{un}}) & \xleftarrow{\mathsf{d}_{n-1}} & \mathcal{C}_{n}^{\mathsf{alg}}(B, B_{\mathsf{un}}) \\
\downarrow{\scriptstyle \mathsf{id}\otimes e_{n-1}(i)} & & \downarrow{\scriptstyle \mathsf{id}\otimes e_{n}(i)} \\
\mathcal{C}_{n-1}^{\mathsf{alg}}(B, B_{\mathsf{un}}) & \xleftarrow[\mathsf{d}_{n-1}]{} & \mathcal{C}_{n}^{\mathsf{alg}}(B, B_{\mathsf{un}})
\end{array}$$

commutes for each $i, n \in \mathbb{N}$.

Proof. See [30, §9.4.3]. □

Remark. The idempotents $e_n(i)$ are also known in other branches of algebra as the Eulerian idempotents, although this does not seem to have been noted when Gerstenhaber and Schack's paper [10] was written. We have followed the notation from [10]; what we have written as $e_n(i)$ is often denoted elsewhere in the literature by $e_n^{(i)}$.

The following is then obvious, and is stated for reference.

Theorem 1.5.2 (Hodge decomposition for commutative Banach algebras). *Let A be a commutative Banach algebra. Each $\mathsf{id}\widehat{\otimes}e_n(i)$ acts as a bounded linear projection on $\mathcal{C}_n(A, A_{\mathsf{un}})$; moreover, for fixed i the family $\mathsf{id}\widehat{\otimes}e_*(i)$ acts as a chain map on $\mathcal{C}_*(A, A_{\mathsf{un}})$.*

Proof. It is clear that $\mathsf{id}\widehat{\otimes}e_n(i)$ acts boundedly on the Banach space $\mathcal{C}_n(A, A_{\mathsf{un}})$ – and that the norm of the induced linear projection is bounded by some constant depending only on i and n.

The remaining properties follow now by continuity, using Theorem 1.5.1 and the density of the algebraic tensor product inside the projective tensor product. ☐

In the algebraic case we could have replaced B_{un} with any symmetric B-bimodule. The same is true in the Banach context:

Corollary 1.5.3. *Let A be a commutative Banach algebra and let M be a symmetric Banach A-bimodule. Then for each i, $\mathsf{id}_M\widehat{\otimes}e_*(i)$ is a bounded chain projection on $\mathcal{C}_*(A, M)$, and "pre-composition with $e_*(i)$" is a bounded chain projection on $\mathcal{C}^*(A, M)$.*

Proof. Since $e_n(i)$ acts as a bounded linear projection on $A^{\widehat{\otimes}n}$, it is immediate that $\mathsf{id}_M\widehat{\otimes}e_n(i)$ acts as a bounded linear projection on $M\widehat{\otimes}A^{\widehat{\otimes}n} = \mathcal{C}_n(A, M)$ and that pre-composition with $e_n(i)$ acts as a bounded linear projection on $\mathcal{C}^n(A, M)$. Therefore it only remains to show that these two maps are chain maps on the Hochschild chain and cochain complexes respectively.

This is essentially a trivial deduction from the case where $M = A_{\mathsf{un}}$. In more detail: recall (Proposition 1.4.7) that there are isomorphisms of Banach complexes

$$\mathcal{C}_*(A, M) \underset{1}{\cong} M\widehat{\otimes}_{A_{\mathsf{un}}}\mathcal{C}_*(A, A_{\mathsf{un}}) \tag{1.4a}$$

$$\mathcal{C}^*(A, M) \underset{1}{\cong} {}_{(A_{\mathsf{un}})}\mathrm{Hom}\left(\mathcal{C}_*(A, A_{\mathsf{un}}), M\right) \ . \tag{1.4b}$$

We have seen that for each n and each i there is a commuting diagram

$$\begin{array}{ccc} \mathcal{C}_{n-1}(A, A_{\mathsf{un}}) & \xleftarrow{\mathsf{d}_{n-1}} & \mathcal{C}_n(A, A_{\mathsf{un}}) \\ \Big\downarrow {\scriptstyle \mathsf{id}\widehat{\otimes} e_{n-1}(i)} & & \Big\downarrow {\scriptstyle \mathsf{id}\widehat{\otimes} e_n(i)} \\ \mathcal{C}_{n-1}(A, A_{\mathsf{un}}) & \xleftarrow[\mathsf{d}_{n-1}]{} & \mathcal{C}_n(A, A_{\mathsf{un}}) \end{array} \tag{1.5}$$

in which all arrows are unit-linked, continuous A_{un}-module maps. Hence applying the functor $M \mathbin{\widehat{\otimes}}_{A_{\mathsf{un}}} \_\_$ and applying Equation (1.4a) yields a commuting diagram of Banach spaces:

$$\begin{array}{ccc} \mathcal{C}_{n-1}(A, M) & \xleftarrow{\mathsf{d}_{n-1}} & \mathcal{C}_n(A, M) \\ \Big\downarrow {\scriptstyle \mathsf{id}\widehat{\otimes} e_{n-1}(i)} & & \Big\downarrow {\scriptstyle \mathsf{id}\widehat{\otimes} e_n(i)} \\ \mathcal{C}_{n-1}(A, M) & \xleftarrow[\mathsf{d}_{n-1}]{} & \mathcal{C}_n(A, M) \end{array}$$

as required. Similarly, applying the functor ${}_{A_{\mathsf{un}}}\mathrm{Hom}(\_\_, M)$ to Diagram 1.5 and applying Equation (1.4b) gives a commuting diagram

$$\begin{array}{ccc} \mathcal{C}^{n-1}(A, M) & \xrightarrow{\delta^{n-1}} & \mathcal{C}^n(A, M) \\ \Big\downarrow {\scriptstyle e_{n-1}(i)^*} & & \Big\downarrow {\scriptstyle e_n(i)^*} \\ \mathcal{C}^{n-1}(A, M) & \xrightarrow[\delta^{n-1}]{} & \mathcal{C}^n(A, M) \end{array}$$

and the proof is complete. □

Definition 1.5.4 (Notation). Let A be a commutative Banach algebra and M a symmetric Banach A-bimodule. For $n \in \mathbb{N}$ and $i = 1, \ldots, n$ we follow the notation of [10] and write

$$\mathcal{C}_{i,n-i}(A, M) := (\mathsf{id}\widehat{\otimes} e_n(i))\mathcal{C}_n(A, M)$$
$$\mathcal{C}^{i,n-i}(A, M) := e_n(i)^* \mathcal{C}^n(A, M)$$

where $e_n(i)^*$ is defined to be "pre-composition with $e_n(i)$".

Given a chain or cochain, we shall sometimes say that it is of BGS type $(i, n-i)$ if it lies in the corresponding summand $\mathcal{C}_{i,n-i}$ or $\mathcal{C}^{i,n-i}$. We shall also sometimes refer to the projections $\mathsf{id}\widehat{\otimes} e_n(i)$ and $e_n(i)^*$ as the BGS projections on homology and cohomology respectively. (This terminology comes from the survey article [9]; the acronym "BGS" is for Barr-Gerstenhaber-Schack.)

Since $\mathsf{id}\widehat{\otimes}e(i)$ is a chain projection for each i, we have a decomposition of the chain complex $\mathcal{C}_*(A,M)$ into orthogonal summands; dually, the chain projections $(e(i)^*)_{i\geq 1}$ yield a decomposition of the cochain complex $\mathcal{C}^*(A,M)$ into orthogonal summands. (For both homology and cohomology the decomposition has n summands in degree n). This might reasonably be called the "BGS decomposition" of homology and cohomology, but in deference to the original authors we shall persevere with the terminology Hodge decomposition.

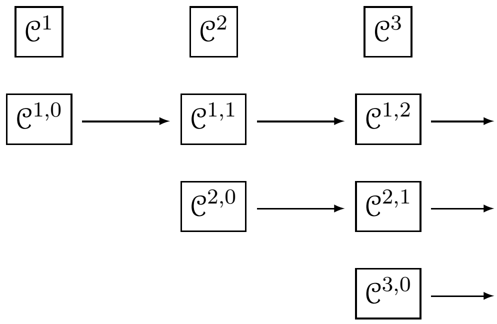
Note that in the proof of Corollary 1.5.3, it was shown *en passant* that there are chain isomorphisms

$$\mathcal{C}_{i,*}(A,M) \underset{1}{\cong} M\widehat{\otimes}_{A_{\mathsf{un}}}\mathcal{C}_{i,*}(A,A_{\mathsf{un}}) \tag{1.6a}$$

$$\mathcal{C}^{i,*}(A,M) \underset{1}{\cong} {}_{(A_{\mathsf{un}})}\mathrm{Hom}\left(\mathcal{C}_{i,*}(A,A_{\mathsf{un}}),M\right) \ . \tag{1.6b}$$

for any i (the case $i=1$ will be used in some later calculations).

Remark. At first glance the various indices, subscript and superscripts may cloud one's picture of what is going on. It is therefore useful to have in mind the following schematic diagram (for cohomology):



Definition 1.5.5. The complex $\mathcal{C}^{1,*}$ is called the Harrison summand of the Hochschild chain complex, and its cohomology is called the Harrison cohomology. Dually, the complex $\mathcal{C}_{1,*}$ is the Harrison summand of the Hochschild chain complex, and its homology is called the Harrison homology.

In later sections where we focus on the Harrison summand and not on the Hodge decomposition in general, we shall adopt the alternative notation $\mathcal{H}ar\mathcal{C}^n := \mathcal{C}^{1,n-1}$, $\mathcal{H}ar\mathcal{C}_n := \mathcal{C}_{1,n-1}$, etc.

1.6 Harrison and Lie (co)homology

The Hodge decomposition of Hochschild (co)homology respects long exact sequences. The precise formulation is as follows:

Lemma 1.6.1 (Long exact sequences for Harrison (co)homology). *Let A be a commutative Banach algebra, and let $L \to M \to N$ be a short exact sequence of symmetric Banach A-bimodules which is split exact in* Ban*. Then there are long exact sequences of Harrison homology*

$$0 \leftarrow \mathcal{H}ar\mathcal{H}_1(A,N) \leftarrow \mathcal{H}ar\mathcal{H}_1(A,M) \leftarrow \mathcal{H}ar\mathcal{H}_1(A,L) \leftarrow \mathcal{H}ar\mathcal{H}_2(A,N) \leftarrow \dots$$

and Harrison cohomology

$$0 \to \mathcal{H}ar\mathcal{H}^1(A,L) \to \mathcal{H}ar\mathcal{H}^1(A,M) \to \mathcal{H}ar\mathcal{H}^1(A,N) \to \mathcal{H}ar\mathcal{H}^2(A,L) \to \dots$$

Proof. We shall give the proof for Harrison homology and omit that for cohomology since the proof technique is identical.

Since $L \to M \to N$ is split in Ban, so is the induced short exact sequence

$$L\widehat{\otimes}A^{\widehat{\otimes}n} \to M\widehat{\otimes}A^{\widehat{\otimes}n} \to N\widehat{\otimes}A^{\widehat{\otimes}n}$$

and it remains split if we apply the BGS idempotent $\mathsf{id}\widehat{\otimes}e_n(1)$ to each term in the sequence. But by the definition of Harrison homology the resulting split exact sequence of Banach spaces is just

$$\mathcal{H}ar\mathcal{C}_n(A,L) \to \mathcal{H}ar\mathcal{C}_n(A,M) \to \mathcal{H}ar\mathcal{C}_n(A,N)$$

Thus we have a short exact sequence of complexes

$$\mathcal{H}ar\mathcal{C}_*(A,L) \to \mathcal{H}ar\mathcal{C}_*(A,M) \to \mathcal{H}ar\mathcal{C}_*(A,N)$$

and the standard diagram chase allows us to construct from this a long exact sequence of homology.

Furthermore, in the portion of the long exact sequence which goes

$$0 \leftarrow \mathcal{H}ar\mathcal{H}_0(A,N) \leftarrow \mathcal{H}ar\mathcal{H}_0(A,M) \xleftarrow{\iota_0} \mathcal{H}ar\mathcal{H}_0(A,L) \xleftarrow{\text{conn}} \mathcal{H}ar\mathcal{H}_1(A,N) \leftarrow \dots$$

we observe that $\mathcal{H}ar\mathcal{H}_0(A,X) = \mathcal{H}_0(A,X) = X$ for any symmetric A-bimodule X. Hence ι_0 is just the inclusion of L into M and is in particular injective; we deduce that the connecting map $\text{conn} : \mathcal{H}ar\mathcal{H}_1(A,N) \to \mathcal{H}ar\mathcal{H}_0(A,L)$ is zero, and so our long exact sequence starts

$$0 \xleftarrow{\text{conn}} \mathcal{H}ar\mathcal{H}_1(A,N) \leftarrow \mathcal{H}ar\mathcal{H}_1(A,M) \leftarrow \dots$$

as claimed. □

Remark. It is clear that similar long exact sequences exist for each summand $\mathcal{C}^{i,*}$ in the Hodge decomposition of cohomology, and for each summand $\mathcal{C}_{i,*}$ in the Hodge decomposition of homology. We omit the details since they will not be needed in what follows.

The "Lie component" of the Hodge decomposition

Explicit formulas for the idempotents $e_n(i)$ are hard to work with in general, but importantly the idempotents $e_n(n)$ turn out to be familiar and tractable. As is pointed out in [10], the key is to combine the observation that

$$\mathsf{d}(\mathsf{id}\widehat{\otimes}e_n(n)) = (\mathsf{id}\widehat{\otimes}e_{n-1}(n))\mathsf{d} = 0$$

together with a lemma of Barr.

Lemma 1.6.2 (Barr's lemma). *Let $n \geq 1$. Define the* signature idempotent *to be*

$$\epsilon_n := \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\sigma \in \mathbb{Q}S_n$$

Then for any commutative $\mathbb{Q}$-algebra R, and any $r_0, \ldots, r_n \in \mathsf{R}$,

$$\mathsf{d}(\mathsf{id}\otimes\epsilon_n)(r_0\otimes r_1\otimes\ldots\otimes r_n) = 0$$

Moreover, if $u \in \mathbb{Q}S_n$ has the property that

$$\mathsf{d}(\mathsf{id}\otimes u)(r_0\otimes r_1\otimes\ldots\otimes r_n) = 0$$

for all $r_0, \ldots, r_n$ in an arbitrary *commutative $\mathbb{Q}$-algebra R, then $u = \epsilon_n$.*

Proof. This is [3, Propn 2.1]. See also [30, Lemma 9.4.9] for a slightly more concise account. □

In particular we may deduce that

$$\boxed{e_n(n) = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^\sigma\sigma \in \mathbb{Q}S_n}$$

and thus the summands $\mathcal{C}_{n,0}(A, M)$ and $\mathcal{C}^{n,0}(A, M)$ turn out to be the spaces of alternating chains and cochains.

Remark. Since $e_2(1) + e_2(2) = \mathsf{id}$, we see that in degree 2 the Hodge decomposition coincides with the decomposition of (co)homology into symmetric and anti-symmetric summands (with the symmetric part being the Harrison summand).

Corollary 1.6.3. *Suppose ψ is both an alternating n-cochain and an n-coboundary. Then $\psi = 0$.*

Proof. Since ψ is alternating, $e_n(n)^*\psi = \psi$; and since $\psi = \delta\phi$ for some ϕ, the chain property of the maps $e(n)^*$ implies that

$$e_n(n)^*\psi = e_n(n)^*\delta\phi = \delta e_{n-1}(n)^*\phi = 0$$

□

Definition 1.6.4. The Lie component of degree n is the space $\mathcal{C}^{n,0}(A, M)$ of continuous, *alternating* n-cochains from A to M.

The name "Lie component" follows the discussion in [10, Thms 5.9,5.10] which loosely says that for a commutative algebra A and symmetric bimodule M, $\mathcal{H}^{n,0}_{\mathsf{alg}}(A, M)$ is isomorphic to the Lie algebra cohomology of the Lie algebra A and module M.

We shall not say much about the Lie component, since it was rediscovered (under a different name) in Johnson's paper [22]. The central notion of that paper was a definition of n-dimensional weak amenability; in the language adopted here, a commutative Banach algebra A is k-dimensionally weakly amenable if $\mathcal{H}^{n,0}(A, M) = 0$ for all $n > k$.

I could not find a reference in [10] for the following: it was proved in [22] for the case of Banach algebras and bimodules, but the proof works without any topological conditions.

Proposition 1.6.5 (c.f. [22, Thm 2.3]). *Let R be a commutative $\mathbb{Q}$-algebra, M a symmetric R-bimodule, and let $\psi \in \mathcal{C}^n_{\mathsf{alg}}(\mathsf{R}, \mathsf{M})$. Then*

(i) if ψ is a derivation in each variable, it is an n-cocycle.

(ii) if ψ is an alternating n-cocycle, then it is a derivation in each variable.

1.7 Semigroups and convolution algebras

A semigroup is a set S, equipped with a binary operation $S \times S \to S, (s,t) \mapsto s \cdot t$ that is associative, i.e.

$$(r \cdot s) \cdot t = r \cdot (s \cdot t) \qquad \forall\, r, s, t \in S$$

The operation $(s,t) \mapsto s \cdot t$ is called multiplication. We often suppress the symbol $\cdot$ and write st rather than $s \cdot t$, etc.

Definition 1.7.1. Let S be a semigroup. An identity for S is an element $1 \in S$ such that $1 \cdot s = s = s \cdot 1$ for all $s \in S$. If S has an identity element then we say that S is a monoid.

It is trivial to show that if 1 and $1'$ are identities for S then $1 = 1'$: hence if S is a monoid we may speak of *the* identity element of S without ambiguity.

For more detailed information about semigroups and their structure theory, the reader may consult [20]. We shall not say much here as our main interest in semigroups is in the Banach algebras they give rise to. Before constructing these we need a definition.

Definition 1.7.2. Let S be a semigroup. A weight function or weight on S is a function $\omega : S \to (0, \infty)$ such that

$$\omega(st) \leq \omega(s)\omega(t) \qquad \forall\, s, t \in S$$

If S has a identity element 1 then it is easy to see that $\omega(1) \geq 1$; in this case it is usual to normalise our weight so that $\omega(1) = 1$.

Definition 1.7.3 (Beurling algebras). Let S be a semigroup and ω a weight on it. We define $\ell^1(S, \omega)$ to be the Banach space of all functions $a : S \to \mathbb{C}$ such that

$$\|a\|_\omega := \sum_{s \in S} |a(s)|\omega(s) < \infty$$

If $a, b \in \ell^1(S, \omega)$ we define $ab : S \to \mathbb{C}$ by

$$(ab)(s) := \sum_{(t,u) \in S \times S \,:\, tu = s} a(t)b(u)$$

it is then routine to verify that the sum on the RHS is absolutely convergent and that moreover $ab \in \ell^1(S; \omega)$.

The product map thus defined on $\ell^1(S, \omega)$ is in fact associative, and satisfies $\|ab\|_\omega \leq \|a\|_\omega \|b\|_\omega$ for all $a, b \in \ell^1(S, \omega)$. Hence $\ell^1(S, \omega)$ is a Banach algebra, called the weighted ℓ^1-convolution algebra of the pair (S, ω), or the Beurling algebra on S with weight ω.

Chapter 2

Aspects of simplicial triviality

This chapter has two main results. The first provides some new examples of simplicially trivial Banach algebras; these examples are noncommutative, and our discussion of them is somewhat tangential to the main themes of this thesis. The second result shows that a condition slightly stronger than simplicial triviality has strong consequences for *commutative* Banach algebras.

2.1 Simplicial triviality and related homological conditions

The concepts of contractibility, amenability and weak amenability have been studied by many authors. We have already met contractibility (Definition 1.4.5); it admits an equivalent characterisation in terms of Hochschild cohomology, as follows.

A Banach algebra A is contractible if it satisfies either of the following equivalent conditions:

(C1) $\mathcal{H}^n(A, X) = 0$ for every Banach A-bimodule X and all $n \geq 1$;

(C2) $\mathcal{H}^1(A, X) = 0$ for every Banach A-bimodule X.

Similarly, A is amenable if it satisfies either of the following equivalent conditions:

(A1) $\mathcal{H}^n(A, X') = 0$ for every Banach A-bimodule X and all $n \geq 1$;

(A2) $\mathcal{H}^1(A, X') = 0$ for every Banach A-bimodule X.

(The equivalence of (C1) and (C2), and the equivalence of (A1) and (A2), are usually proved using a dimension-shift argument: see [21, §1] for details.)

Finally, a Banach algebra A is said to be weakly amenable if $\mathcal{H}^1(A, A') = 0$ (this is Johnson's extension of the original definition made by Bade, Curtis and Dales).

We are interested in a notion intermediate between amenability and weak amenability, given by the following definition.

Definition 2.1.1. Let A be a Banach algebra. We say that A is simplicially trivial if either one of the the following *equivalent* conditions is satisfied:

(1) $\mathcal{H}^n(A, A') = 0$ for all $n \geq 1$;

(2) $\mathcal{H}_n(A, A) = 0$ for all $n \geq 1$ and $\mathcal{H}_0(A, A)$ is Hausdorff.

(The equivalence of conditions (1) and (2) is the case $m = 1$ of [21, Coroll 1.3].)

Remark. Note that in this definition we use what are sometimes called the "naive simplicial cohomology groups" of A. Some authors define simplicial cohomology of A in a functorial way, so as to ensure that A and its forced unitisation $A^{\#}$ always have isomorphic cohomology in degrees ≥ 1.

The implications

$$\text{contractible} \overset{(a)}{\Longrightarrow} \text{amenable} \overset{(b)}{\Longrightarrow} \text{weakly amenable}$$

follow immediately from the definitions. Simplicial triviality fits into this picture as follows:

$$\begin{array}{ccccc}
\text{contractible} & \overset{(a)}{\Longrightarrow} & \text{amenable} & \overset{(b)}{\Longrightarrow} & \text{weakly amenable}\\
\Big\Downarrow{\scriptstyle (c)} & & \Big\Downarrow{\scriptstyle (d)} & & \Big\Uparrow{\scriptstyle (e)}\\
\text{biprojective} & \underset{(f)}{\Longrightarrow} & \text{biflat} & \underset{(g)}{\Longrightarrow} & \textit{simplicially trivial}
\end{array}$$

It is not easy to find a definitive reference for the implication (g). I would like to thank Dr. Zinaida Lykova for pointing out that it can be deduced easily from the expression of Hochschild cohomology via Ext, as follows: if A is biflat then A' is injective as an A-bimodule, which is the same as saying that it is A^e-injective; therefore

$$\mathcal{H}^n(A, A') \cong \operatorname{Ext}^n_{A^e}(A^{\#}, A') \qquad \text{which is zero for all } n > 0.$$

A slightly more direct proof is given in Appendix B.

Simplicial triviality appears to have received relatively little attention in the founding years of Banach-algebraic cohomology theory. More recently it has been used as a stepping stone towards vanishing results in *cyclic cohomology* (by virtue of the Connes-Tsygan exact sequence) but we shall not discuss such links in this thesis. In the next section we shall exhibit a new class of simplicially trivial Banach algebras.

2.2 Augmentation ideals of discrete groups and their cohomology

In this section we show that for a wide class of discrete groups the augmentation ideals are simplicially trivial. This work generalises some of the work of Grønbæk and Lau ([18]) on weak amenability of such ideals.

2.2.1 Introduction and terminology

We start with a small observation. The original version of the following lemma was stated in the special case of augmentation ideals in group algebras: thanks to Prof. Niels Grønbæk for pointing out that a more general result holds.

Lemma 2.2.1. *Let A be a unital Banach algebra which has a character $\varphi : A \to \mathbb{C}$; let $I = \ker(\varphi)$. Then the following are equivalent:*

(i) I is simplicially trivial;.

(ii) $\mathcal{H}^n(A, I') = 0$ for all $n \geq 1$;

(iii) for each $n \geq 1$, the canonical map

$$\mathcal{H}^n(A, \mathbb{C}_\varphi) \xrightarrow{\varphi^*} \mathcal{H}^n(A, A')$$

that is induced by the inclusion $\mathbb{C} \to A', 1 \mapsto \varphi$, is an isomorphism.

Proof. The implications $(i) \iff (ii)$ are immediate from the observation that $A \cong I^\#$ and that $\mathcal{H}^n(B^\#, M) \cong \mathcal{H}^n(B, M)$ for any Banach algebra B and Banach B-bimodule M.

To get the implications $(ii) \iff (iii)$, consider the long exact sequence of cohomology associated to the short exact sequence $0 \to \mathbb{C}_\varphi \to A' \to I' \to 0$, *viz.*

$$\ldots \mathcal{H}^n(A, \mathbb{C}_\varphi) \xrightarrow{\varphi^*} \mathcal{H}^n(A, A') \xrightarrow{\rho} \mathcal{H}^n(A, I') \to \mathcal{H}^{n+1}(A, \mathbb{C}_\varphi) \to \ldots$$

Claim: the map $\mathcal{H}^0(A, A') \xrightarrow{\rho} \mathcal{H}^0(A, I')$ is surjective.

[Recall that for any A-bimodule X, $\mathcal{H}^0(A, X)$ is just the centre $Z(X)$ of X, so that $\rho : Z(A') \to Z(I')$ is just the restriction of a trace on A to the ideal I; hence to prove the claim it suffices to show that every element of $Z(I')$ extends to a trace on A. But this is easy: if $\psi \in I'$ and $\psi \cdot a = a \cdot \psi$ for all $a \in A$, then the functional $a \mapsto \psi(a - \varphi(a)1_A)$ gives such a trace, and thus the claim is proved.]

Using the claim we deduce that the long exact sequence of cohomology has the form

$$0 \to \mathcal{H}^1(A, \mathbb{C}_\varphi) \xrightarrow{\varphi^*} \mathcal{H}^1(A, A') \xrightarrow{\rho} \mathcal{H}^1(A, I') \to \mathcal{H}^2(A, \mathbb{C}_\varphi) \to \dots$$

from which the equivalence of (*ii*) and (*iii*) is immediate. ☐

Remark. We shall use Lemma 2.2.1 in the form (*ii*) $\implies$ (*i*). The remaining implications have been added only for sake of completeness.

Now let us specialise to group algebras. Throughout G will denote a *discrete* group, $\ell^1(G)$ its convolution algebra and $I_0(G)$ the augmentation ideal in $\ell^1(G)$ – that is, the kernel of the augmentation character ε which sends each standard basis vector of $\ell^1(G)$ to 1.

Definition 2.2.2. A group G is said to be commutative-transitive if each element of $G \setminus \{1_G\}$ has an abelian centraliser.

It is not immediately clear that there exist any nonabelian, commutative-transitive groups: examples can be found in Chapter I of [24] (see Proposition 2.19 and the remarks afterwards). Let us just mention one family of examples.

Theorem 2.2.3 (Gromov, [16]; see also [1, Propn 3.5]). *Any torsion-free word-hyperbolic group is commutative-transitive.*

The arguments given for this in [16] are scattered over several sections and are not easily assembled into a proof. The simplest and clearest account appears in Chapter 3 of the survey article [1] (I would like to thank Keith Goda for drawing these notes to my attention).

Remark. It is often observed that direct products of hyperbolic groups need not be hyperbolic, the standard example being $F_2 \times F_2$ where F_2 denotes the free group on two generators. In the current context it is worth pointing out that clearly $F_2 \times F_2$ is not commutative-transitive (since the centraliser of $(1, x)$ always contains a copy of $F_2 \times \{1\}$).

Theorem 2.2.4. *Let G be a commutative-transitive, discrete group. Then for each $n \geq 1$, $\mathcal{H}^n(\ell^1(G), I_0(G)') = 0$.*

The key idea of the proof is simple, and uses the following well-known idea: when we pass to a conjugation action, $I_0(G)$ decomposes as an ℓ^1-direct sum of modules of the form $\ell^1(\mathcal{C}\ell_x)$, where $\mathcal{C}\ell_x$ denotes the conjugacy class of x. Hence there is an isomorphism of cochain complexes

$$\mathcal{C}^*(\ell^1(G), I_0(G)') \underset{1}{\cong} \overset{(\infty)}{\bigoplus}_{x\in\mathbb{I}} \mathcal{C}^*(\ell^1(G), \ell^1(\mathcal{C}\ell_x)')$$

where $\mathbb{I}$ is a set of representatives for each conjugacy class in $G \setminus \{1_G\}$. Our theorem will now follow from a computation of the cohomology on the RHS.

Note that for each *summand* on the RHS, the cohomology groups can be reduced to certain bounded cohomology groups: in the Banach-algebraic context this is due to A. Pourabbas and M.C. White, who observe in [26] that for each x we have isomorphisms

$$\begin{aligned} \mathcal{H}^*(\ell^1(G), \ell^1(\mathcal{C}\ell_x)') &\cong \mathrm{Ext}^*_{\ell^1(G)}(\ell^1(\mathcal{C}\ell_x), \mathbb{C}) \\ &\cong \mathrm{Ext}^*_{\ell^1(C_x)}(\mathbb{C}, \mathbb{C}) \qquad \cong \mathcal{H}^*(\ell^1(C_x), \mathbb{C}) \end{aligned} \tag{2.1}$$

where C_x denotes the centraliser of x. It is tempting to deduce that the cohomology of the cochain complex $\overset{(\infty)}{\bigoplus}_{x\in\mathbb{I}} \mathcal{C}^*(\ell^1(G), \ell^1(C_x)')$ is isomorphic to $\overset{(\infty)}{\bigoplus}_{x\in\mathbb{I}} \mathcal{H}^*(\ell^1(C_x), \mathbb{C})$, from which Theorem 2.2.4 would follow immediately. However, it is *not* in general true that the cohomology of an ℓ^∞-sum is the ℓ^∞-sum of the cohomology of the summands, so this argument contains a gap.

Remark. In the special case where G is commutative-transitive, each C_x is abelian, hence amenable, and so for each x the cochain complex $\mathcal{C}^*(\ell^1(C_x), \mathbb{C})$ has a contractive linear splitting.

Therefore, to deduce that the cochain complex

$$\overset{(\infty)}{\bigoplus}_{x\in\mathbb{I}} \mathcal{C}^*(\ell^1(G), \ell^1(C_x)')$$

splits, it would suffice to prove that the isomorphisms of Equation (2.1) are induced by chain homotopies with norm control independent of x. This is implictly done in [26, §4], but only for second-degree cohomology.

Rather than follow the approach outlined in the previous remark, we instead generalise the result of [26] so that it applies to any left G-set S (i.e. we drop their hypothesis that the action is transitive). Since our hypotheses are weaker, we are not able to deduce isomorphism of cohomology groups as in [26]; however, our weaker conclusion suffices to prove Theorem 2.2.4.

2.2.2 Disintegration over stabilisers

The promised generalisation goes as follows:

Theorem 2.2.5. *Let G be a discrete group acting from the left on a set S, and let $S = \coprod_{x\in\mathbb{I}} \mathsf{Orb}_x$ be the partition into G-orbits. Let $H_x := \mathrm{Stab}_G(x)$. Then for each n the Hochschild cohomology group $\mathcal{H}^n(\ell^1(G), \ell^1(S)')$ is isomorphic to the nth cohomology group of the complex*

$$0 \longrightarrow \bigoplus_{x\in\mathbb{I}}^{(\infty)} \mathcal{C}^0(\ell^1(H_x), \mathbb{C}) \longrightarrow \bigoplus_{x\in\mathbb{I}}^{(\infty)} \mathcal{C}^1(\ell^1(H_x), \mathbb{C}) \longrightarrow \dots$$

Corollary 2.2.6. *Let G, S be as above, and assume that each stabiliser subgroup H_x is amenable. Then $\mathcal{H}^n(\ell^1(G), \ell^1(S)') = 0$ for all $n \geq 1$.*

Proof of corollary. If each H_x is amenable then the cochain complex $\mathcal{C}^1(\ell^1(H_x), \mathbb{C})$ admits a *contractive* linear splitting in degrees ≥ 1. Therefore the chain complex

$$\bigoplus_{x\in\mathbb{I}}^{(\infty)} \mathcal{C}^*(\ell^1(H_x), \mathbb{C})$$

is also split in degrees ≥ 1 by linear contractions, and is in particular exact in degree n. □

Remark. As mentioned in the proof of [26, Thm 6.1], the heart of the result lies in facts about "induced modules". We shall not pursue this explicitly as the full generality is not needed.

Proof of 2.2.5, assuming Theorem 2.2.6. By the remarks preceding [21, Thm 2.5]

$$\mathcal{H}^n(\ell^1(G), I_0(G)') \cong \mathcal{H}^n\left(\ell^1(G), \widetilde{I_0(G)}'\right)$$

where $\widetilde{I_0(G)}$ is the $\ell^1(G)$-bimodule with underlying space $I_0(G)$ but with trivial left action and the conjugation right action.

Let $S = G \setminus \{1_G\}$, regarded as a G-set via conjugation action. If we pass to the conjugation action of G on $\ell^1(G)$, there is clearly a direct sum decomposition of modules $\widetilde{\ell^1(G)} = \mathbb{C} \oplus \ell^1(S)$ where $\mathbb{C}$ is the point module with trivial action. Composing the truncation map $\ell^1(G) \to \ell^1(S)$ with the inclusion map $I_0(G) \to \ell^1(G)$ gives a linear isomorphism $I_0(G) \to \ell^1(S)$, and this is also a G-module map (for the conjugation action). So for this action $\widetilde{I_0(G)} \cong \ell^1(S)$ as G-modules, and therefore

$$\mathcal{H}^n(\ell^1(G), I_0(G)') \cong \mathcal{H}^n(\ell^1(G), \ell^1(S)')$$

Write S as the disjoint union $S = \coprod_{x\in\mathbb{I}} \mathcal{C}\ell_x$ of conjugacy classes. The corresponding stabiliser subgroups are precisely the centralisers C_x of each $x \in \mathbb{I}$; since G is assumed to be commutative-transitive and $1_G \notin S$, each C_x is commutative (hence amenable) and applying Corollary 2.2.6 completes the proof. □

We shall break the proof of Theorem 2.2.5 into a succesion of small lemmas: each is to some extent standard knowledge, but for our purposes we need to make explicit certain uniform bounds and linear splittings for which I can find no precise reference. Indeed, the theorem itself may well be known to specialists, although as far as I know the application to cohomology of augmentation ideals is new.

In what follows we shall abuse notation slightly: if H is a subgroup of G and M and N are, respectively, right and left Banach $\ell^1(H)$-modules, then we shall write $M \widehat{\otimes}_H N$ for the Banach tensor product of M and N over $\ell^1(H)$. This temporary notation is adopted purely to make various formulas more legible.

Lemma 2.2.7 (Factorisation of functors). *Let B be a closed unital subalgebra of a unital Banach algebra A, and regard A as a right B-module via the inclusion homomorphism $B \hookrightarrow A$. Then:*

(i) we have a natural isometric isomorphism of functors

$$A \widehat{\otimes}_B (B \widehat{\otimes} \_\_) \underset{1}{\cong} A \widehat{\otimes} \_\_ \tag{2.2}$$

where $B\widehat{\otimes}\_\_$ and $A\widehat{\otimes}\_\_$ are the free functors from Ban *(to* ${}_B$unmod *and* ${}_A$unmod *respectively;*

(ii) we have a (natural) isometric isomorphism of functors

$$ {}_B\mathrm{Hom}(\_\_, \mathbb{C}) \underset{1}{\cong} {}_A\mathrm{Hom}\left({}_A\mathrm{Hom}\left(A \widehat{\otimes}_B \_\_ \right), \mathbb{C} \right) \tag{2.3}$$

where both sides are functors ${}_B$unmod $\to$ Ban.

The proof is clear.

Lemma 2.2.8 (A little more than flatness). *Let H be any subgroup of G and let $G/H = \{gH \ : \ g \in G\}$ be the space of left cosets. Then we have a (natural) isometric isomorphism of functors*

$$\mathcal{U}_G\left(\ell^1(G)\widehat{\underset{H}{\otimes}}\_\_\right) \underset{1}{\cong} \ell^1(G/H)\widehat{\otimes}(\mathcal{U}_H\_\_) \tag{2.4}$$

where $\mathcal{U}_G$ and $\mathcal{U}_H$ are the forgetful functors to Ban *(from the categories* ${}_{\ell^1(G)}\mathsf{unmod}$ *and* ${}_{\ell^1(H)}\mathsf{unmod}$ *respectively).*

Proof. Choose a transversal for G/H, that is, a function $\tau : G/H \to G$ such that $\tau(\mathfrak{J}) \in \mathfrak{J}$ for all $\mathfrak{J} \in G/H$. (Equivalently, $\tau(\mathfrak{J})H = \mathfrak{J}$ for all $\mathfrak{J}$). This transversal yields a function $\eta : G \to H$ such that

$$g = \tau(gH) \cdot \eta(g) \qquad \text{for all } g \in G$$

(In words: η tells us where in the coset gH the element g lives.) Note that for every $g \in G$ and $h \in H$,

$$\eta(gh) = \eta(g) \cdot h$$

If E is a unit-linked left $\ell^1(H)$-module, define a contractive linear map $\ell^1(G)\widehat{\otimes}E \to \ell^1(G/H)\widehat{\otimes}E$ by

$$e_g\widehat{\otimes}v \mapsto e_{gH}\otimes(\eta(g) \cdot v) \quad ;$$

now observe that this map factors through the quotient map $q : \ell^1(G)\widehat{\otimes}E \to \ell^1(G)\widehat{\underset{H}{\otimes}}E$, and so induces a linear contraction

$$T_E : \ell^1(G)\widehat{\underset{H}{\otimes}}E \to \ell^1(G/H)\widehat{\otimes}E$$

where $T(e_g\underset{H}{\otimes}v) := e_{gH}\otimes(\eta(g) \cdot v)$.

On the other hand, the composite map

$$R_E : \ell^1(G/H)\widehat{\otimes}E \xrightarrow{\tau\widehat{\otimes}\mathsf{id}_E} \ell^1(G)\widehat{\otimes}E \xrightarrow{\ q\ } \ell^1(G)\widehat{\underset{H}{\otimes}}E$$

is a linear contraction, defined by the formula $R(e_{\mathfrak{J}}\otimes v) := e_{\tau(\mathfrak{J})}\underset{H}{\otimes}v$. R_E is the composition of two maps which are natural in E, hence is itself natural in E. By direct checking on elementary tensors, it is clear that R_E and T_E are mutually inverse maps. Hence R is a natural isomorphism from $\mathcal{U}_G\left(\ell^1(G)\widehat{\underset{H}{\otimes}}\_\_\right)$ to $\ell^1(G/H)\widehat{\otimes}(\mathcal{U}_H\_\_)$ as required. ☐

Lemma 2.2.9. *Let X be a left Banach $\ell^1(G)$-module. Regard it as a $\ell^1(G)$-bimodule X_ε by defining the right G-action on X to be trivial (i.e. augmentation). Then for all n there is an isomorphism*

$$\mathcal{H}^n(\ell^1(G), X'_\varepsilon) \cong \operatorname{Ext}^n_{\ell^1(G)}(X, \mathbb{C})$$

Proof. This is a special case of the isomorphisms

$$\mathcal{H}^*(A, \mathcal{L}(E,F)) \cong \operatorname{Ext}^*_{A^e}(A_{\mathsf{un}}, \mathcal{L}(E,F)) \cong \operatorname{Ext}^*_{A_{\mathsf{un}}}(E,F)$$

valid for any Banach algebra A and any left A-modules E and F. (See [19, Thm III.4.12].) ☐

Proof of Theorem 2.2.5. First observe that by Lemma 2.2.9,

$$\mathcal{H}^n(\ell^1(G), \ell^1(S)') \cong \operatorname{Ext}^n_{\ell^1(G)},(\ell^1(S), \mathbb{C})$$

Since Ext may be calculated up to isomorphism using any admissible projective resolution of the first variable, it therefore suffices to construct an admissible $\ell^1(G)$-projective resolution

$$\ell^1(S) \longleftarrow P_0 \longleftarrow P_1 \longleftarrow \dots$$

with the following property:

$\boxed{(*)}$ *the cochain complex*

$$0 \longrightarrow {}_{\ell^1(G)}\operatorname{Hom}(P_*, \mathbb{C})$$

is isomorphic as a chain complex of $\ell^1(G)$-modules to

$$0 \longrightarrow \bigoplus^{(\infty)}_{x\in\mathbb{I}} \mathcal{C}^*(\ell^1(H_x), \mathbb{C}) \ .$$

We do this as follows. For each $x \in \mathbb{I}$, let $\mathbb{C} \leftarrow P_*(x)$ denote the 1-sided bar resolution of $\mathbb{C}$ by left $\ell^1(H_x)$-projectives, i.e.

$$\mathbb{C} \xleftarrow{\varepsilon_x} \ell^1(H_x) \xleftarrow{d_0^x} \ell^1(H_x)^{\widehat{\otimes}2} \xleftarrow{d_1^x} \dots$$

where ε_x is the augmentation character and $d_n^x : \ell^1(H_x)^{\widehat{\otimes}n+2} \to \ell^1(H_x)^{\widehat{\otimes}n+1}$ is given by

$$d_n^x(e_{h(0)}\otimes\dots\otimes e_{h(n+1)}) = \begin{cases} \displaystyle\sum_{j=0}^{n}(-1)^j e_{h(0)}\otimes\dots\otimes e_{h(j)h(j+1)}\otimes\dots\otimes e_{h(n+1)} \\ \qquad +(-1)^{n+1}e_{h(0)}\otimes\dots\otimes e_{h(n)} \end{cases}$$

$(h(0), h(1), \ldots, h(n+1) \in H_x)$. Since the complex $\mathbb{C} \leftarrow P_*(x)$ is 1-split in Ban it remains 1-split after we apply $\ell^1(G/H_x)\widehat{\otimes}\_\_$ to it. Furthermore, the ℓ^1-direct sum of 1-split complexes is itself 1-split; hence the complex of Banach $\ell^1(G)$-modules

$$\bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(G/H_x) \longleftarrow \bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(G/H_x)\widehat{\otimes}\ell^1(H_x) \longleftarrow \bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(G/H_x)\widehat{\otimes}\ell^1(H_x)^{\widehat{\otimes}2} \longleftarrow \cdots \tag{2.5}$$

is 1-split as a complex in Ban.

There is an isomorphism of $\ell^1(G)$-modules

$$\ell^1(S) = \ell^1\left(\coprod_{x\in\mathbb{I}} \mathsf{Orb}_x\right) \underset{1}{\cong} \bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(\mathsf{Orb}_x) \underset{1}{\cong} \bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(G/H_x)$$

where in the last step we identified the orbit of x with the coset space G/H_x via the correspondence $g \cdot x \leftrightarrow gH_x$. Let

$$P_* := \bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(G) \underset{H_x}{\widehat{\otimes}} P_*(x)$$

and write $\widetilde{\varepsilon}$ for the $\ell^1(G)$-module map

$$\bigoplus^{(1)}_{x\in\mathbb{I}} \mathsf{id}_{\ell^1(G)} \underset{H_x}{\widehat{\otimes}} \varepsilon_x \ .$$

By Lemma 2.2.8, the complex of $\ell^1(G)$-module maps

$$\bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(G/H_x) \xleftarrow{\ \widetilde{\varepsilon}\ } P_*$$

is isometrically isomorphic *as a complex in* Ban to the complex (2.5); since the latter is split in Ban, so is the former. That is, $\ell^1(S) \leftarrow P_*$ is an admissible complex of Banach $\ell^1(G)$-modules.

Moreover, for each $x \in \mathbb{I}$ and $n \geq 0$, Lemma 2.2.7 provides an isometric isomorphism of left $\ell^1(G)$-modules

$$\ell^1(G) \underset{H_x}{\widehat{\otimes}} P_n(x) \underset{1}{\cong} \ell^1(G)\widehat{\otimes}\ell^1(H_x)^{\widehat{\otimes}n}$$

and taking the ℓ^1-direct sum over all x yields isometric isomorphisms of left $\ell^1(G)$-modules

$$P_n = \bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(G) \underset{H_x}{\widehat{\otimes}} P_n(x) \underset{1}{\cong} \bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(G)\widehat{\otimes}\ell^1(H_x)^{\widehat{\otimes}n} \underset{1}{\cong} \ell^1(G)\widehat{\otimes}\left(\bigoplus^{(1)}_{x\in\mathbb{I}} \ell^1(H_x)^{\widehat{\otimes}n}\right)$$

from which we see that each P_n is free – and hence projective – as an $\ell^1(G)$-module.

Combining the previous two paragraphs we see that $\ell^1(S) \leftarrow P_*$ is an admissible resolution of $\ell^1(S)$ by $\ell^1(G)$-projective modules.

It remains to verify the condition **(∗)**. Observe that for each x

$$ {}_{\ell^1(H_x)}\mathrm{Hom}(P_*(x), \mathbb{C}) \underset{1}{\cong} \mathcal{C}^*(\ell^1(H_x), \mathbb{C}) \quad ; $$

hence by Lemma 2.2.7 we have

$$ \mathcal{C}^*(\ell^1(H_x), \mathbb{C}) \underset{1}{\cong} {}_{\ell^1(G)}\mathrm{Hom}\left(\ell^1(G) \underset{H_x}{\widehat{\otimes}} P_*(x),\, \mathbb{C}\right) , $$

and taking the ℓ^∞-sum over all x yields

$$ \begin{aligned} \bigoplus_{x\in\mathbb{I}}^{(\infty)} \mathcal{C}^*(\ell^1(H_x), \mathbb{C}) &\underset{1}{\cong} \bigoplus_{x\in\mathbb{I}}^{(1)} {}_{\ell^1(G)}\mathrm{Hom}\left(\ell^1(G) \underset{H_x}{\widehat{\otimes}} P_*(x),\, \mathbb{C}\right) \\ &\underset{1}{\cong} {}_{\ell^1(G)}\mathrm{Hom}(P_*, \mathbb{C}) \end{aligned} $$

as required. □

2.2.3 Corollaries and remarks

Corollary 2.2.10. *Let G be a commutative-transitive, discrete group. Then $I_0(G)$ is simplicially trivial.*

Proof. This is immediate from combining Lemma 2.2.1 and Theorem 2.2.4. □

Remark. Recalling that biflat Banach algebras are simplicially trivial, it is natural to enquire if our result might follow from biflatness of $I_0(G)$. To see that this is not always the case, observe that if $I_0(G)$ is biflat then $\mathcal{H}^3(I_0(G), \mathbb{C}_{\mathrm{ann}}) = 0$ (biflat Banach algebras have weak bidimension ≤ 2), while it is known that

$$ \mathcal{H}^3(I_0(G), \mathbb{C}_{\mathrm{ann}}) \cong \mathcal{H}^3(\ell^1(F_2), \mathbb{C}) \neq 0 \; . $$

While it is known that $I_0(G)$ is amenable if and only if G is, I have not found a similar characterisation of precisely when $I_0(G)$ is biflat.

Question 2.2.11. Let G be a discrete group. If $I_0(G)$ is biflat, is G amenable?

Remark. The arguments above show that $I_0(F_2 \times F_2)$ is *not* simplicially trivial, since its second simplicial cohomology will contain a copy of the second bounded cohomology of $C_{(x,1)}$ where $x \in F_2 \setminus \{1\}$. (To see that $\mathcal{H}^2(\ell^1(C_{(x,1)}), \mathbb{C})$ is non-zero, observe that $C_{(x,1)} \cong C_x \times F_2$ is the direct product of a commutative group with F_2, hence has the same bounded cohomology as F_2; by [21, Propn 2.8] $\mathcal{H}^2(\ell^1(F_2), \mathbb{C}) \neq 0$.)

The question of what happens for augmentation ideals in *non-discrete*, locally compact groups is much trickier since measure-theoretic considerations come into play. Johnson and White have shown that the augmentation ideal of $PSL_2(\mathbb{R})$ is not even weakly amenable; in contrast, $PSL_2(\mathbb{Z})$ is known to be commutative-transitive and so by our results its augmentation ideal is simplicially trivial.

2.3 Simplicial triviality for commutative Banach algebras

In this section we take a different direction, and show that if a commutative Banach algebra is simplicially trivial then – under some additional side conditions – its cohomology with symmetric coefficients vanishes in degrees 1 and above. Our approach is motivated by standard techniques from ring theory, which we now describe.

2.3.1 Acyclic base-change: motivation from ring theory

In this motivating section we consider "purely algebraic" Hochschild (co)homology and "purely algebraic" relative Tor, Ext, etc, and assume the reader has some knowledge of the basic concepts. To avoid conflict of notation we denote the purely algebraic Tor and Ext by Tor and Ext.

Our starting point is a standard piece of machinery from ring theory (see e.g. [27, Thm 11.65]).

Proposition 2.3.1 ("Base change for Ext"). *Let R, T be unital rings. If $R \xrightarrow{\varphi} T$ is a unital ring homomorphism and $B \in R - \mathsf{mod}$, $M \in T - \mathsf{mod}$, then there is a spectral sequence*

$$\mathsf{Ext}^p_T\left(\mathsf{Tor}^R_q(T_\varphi, B), M\right) \Rightarrow_p \mathsf{Ext}^{p+q}_R(B, {}_\varphi M)$$

Note that in this setup we are using the *right* R-module structure on T (hence the notation T_φ) and the *left* R-module structure induced by φ on M (hence the notation ${}_\varphi M$)).

Application to Hochschild cohomology. Let A be a k-algebra and $\mathsf{A}^e := \mathsf{A}_{\mathsf{un}} \otimes \mathsf{A}^{\mathsf{op}}_{\mathsf{un}}$ its enveloping algebra. Recall that Hochschild cohomology can be computed as k-relative Ext, *viz.*

$$\mathcal{H}^n_{\mathsf{alg}}(\mathsf{A}, M) \cong \mathsf{Ext}^n_{\mathsf{A}^e;k}(\mathsf{A}_{\mathsf{un}}, M)$$

and Hochschild homology as k-relative Tor, *viz.*

$$\mathcal{H}_n^{\mathsf{alg}}(\mathsf{A}, M) \cong \mathsf{Tor}_n^{\mathsf{A}^e;k}(M, \mathsf{A}_{\mathsf{un}}) \ .$$

Suppose now that A is k-projective (this is guaranteed when k is a field). Then we may replace the *relative* Tor and Ext above with *absolute* Tor and Ext, i.e. we have isomorphisms

$$\mathcal{H}^n_{\mathsf{alg}}(\mathsf{A}, M) \cong \mathsf{Ext}^n_{\mathsf{A}^e}(\mathsf{A}_{\mathsf{un}}, M) \quad \text{and} \quad \mathcal{H}_n^{\mathsf{alg}}(\mathsf{A}, M) \cong \mathsf{Tor}_n^{\mathsf{A}^e}(M, \mathsf{A}_{\mathsf{un}})$$

(see e.g. [30, Coroll 9.1.5]). We may then apply the "change-of-rings" spectral sequence with $R = \mathsf{A}^e$ and $B = \mathsf{A}_{\mathsf{un}}$, to get

$$\mathsf{Ext}^p_T\left(\mathcal{H}_q^{\mathsf{alg}}(\mathsf{A}, T_\varphi),\ M\right) \Rightarrow_p \mathcal{H}^{p+q}_{\mathsf{alg}}(\mathsf{A}, {}_\varphi M)$$

whenever there is a unital homomorphism $\mathsf{A}^e \xrightarrow{\varphi} T$ and M is a left T-module.

Corollary 2.3.2. *Let* A*,* T *and* M *be as above. Suppose that* $\mathcal{H}_n^{\mathsf{alg}}(\mathsf{A}, T_\varphi) = 0$ *for all* $n \geq 1$*. Then we have isomorphisms*

$$\mathsf{Ext}^n_T\left(\mathcal{H}_0^{\mathsf{alg}}(\mathsf{A}, T_\varphi),\ M\right) \cong \mathcal{H}^n_{\mathsf{alg}}(\mathsf{A}, {}_\varphi M)$$

for all n*.*

Now, if one attempts to set up the change-of-base spectral sequence for Banach algebras and Banach modules, problems arise since the putative formula

$$\mathrm{Ext}^p_T\left(\mathrm{Tor}^R_q(T_\varphi, B), {}_\varphi M\right) \Rightarrow_p \mathrm{Ext}^{p+q}(B, {}_\varphi M)$$

only makes sense if the Tor term is Banach rather than merely seminormed. However we can seek to generalise the special case of Corollary 2.3.2 to the Banach setting, and this will be done in the following section.

2.3.2 Acyclic base-change: weaker versions for Banach modules

Let A be a (not necessarily unital) Banach algebra. Recall that the enveloping algebra of A is $A_{\mathsf{un}}\widehat{\otimes}A^{\mathsf{op}}_{\mathsf{un}}$, and that we denote it by A^e.

Suppose we have a unital Banach algebra T and a continuous unital homomorphism $\varphi : A^e \to T$. We may regard T as a unit-linked, *right* Banach A^e-module T_φ, and thence as a Banach A-bimodule for the action

$$atb := t \cdot \varphi(b \otimes a) \qquad (a, b \in A; t \in T).$$

Recall that

$$\mathcal{C}_n(A, T_\varphi) = T_\varphi \underset{A^e}{\widehat{\otimes}} \beta_n(A)$$

where $\beta_n(A) := A_{\mathsf{un}} \widehat{\otimes} A^{\widehat{\otimes} n} \widehat{\otimes} A_{\mathsf{un}}$ with the following A^e-module action:

$$(b \otimes c) \cdot (a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) := b a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1} c$$

$$(a_1, \ldots, a_n \in A;\ a_0, a_{n+1}, b, c \in A_{\mathsf{un}})$$

and $\beta_*(A)$ is a complex of A^e-modules with differential d given by

$$d(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) := \sum_{j=0}^{n} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{n+1}$$

We need to be precise about the formula for the differential on $\mathcal{C}_*(A, T_\varphi)$, which will be denoted d_T. We have $d_T = \mathsf{id}_T \underset{A^e}{\otimes} d$, so that

$$\begin{aligned} d_T(t \underset{A^e}{\otimes} (x \otimes a_1 \otimes \ldots \otimes a_n \otimes y)) &= t \underset{A^e}{\otimes} d(x \otimes a_1 \otimes \ldots \otimes a_n \otimes y) \\ &= \begin{cases} t \underset{A^e}{\otimes} (x a_1 \otimes a_2 \ldots \otimes a_n \otimes y) \\ + \displaystyle\sum_{j=1}^{n-1} (-1)^j t \underset{A^e}{\otimes} (x \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes y) \\ +(-1)^n t \underset{A^e}{\otimes} (x \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n y) \end{cases} \\ &= \begin{cases} t \underset{A^e}{\otimes} (x a_1 \otimes a_2 \ldots \otimes a_n \otimes y) \\ + \displaystyle\sum_{j=1}^{n-1} (-1)^j t \underset{A^e}{\otimes} (x \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes y) \\ +(-1)^n t \underset{A^e}{\otimes} (x \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n y) \end{cases} \end{aligned}$$

It is clear from the above that each $\mathcal{C}^n(A, T_\varphi)$ is a left Banach T-module, and that d_T is a left T-module map. This becomes even clearer if we make the usual identification of $\mathcal{C}^n(A, T_\varphi)$ with $T \widehat{\otimes} A^{\widehat{\otimes} n}$. To be more precise, we identify

$$t \underset{A^e}{\otimes} (1 \otimes a_1 \otimes \ldots \otimes a_n \otimes 1) \quad \text{with} \quad t \otimes a_1 \otimes \ldots \otimes a_n$$

The differential d_T now takes the following form:

$$d_T(t \otimes a_1 \otimes \ldots \otimes a_n) = \begin{cases} t \cdot \varphi(a_1 \otimes 1) \otimes a_2 \ldots \otimes a_n \\ + \displaystyle\sum_{j=1}^{n-1} (-1)^j t \otimes a_1 \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n \\ + (-1)^n t \cdot \varphi(1 \otimes a_n) \otimes a_0 \otimes \ldots \otimes a_{n-1} \end{cases}$$

Suppose now that M is a unit-linked, left Banach T-module. Just as in the algebraic case, the unital homomorphism $A^e \xrightarrow{\varphi} T$ gives M the structure of a left Banach A^e-module ${}_\varphi M$. We may then regard ${}_\varphi M$ as a Banach A-bimodule, with the action given by

$$amb := \varphi(a\otimes b)m \qquad (a, b \in A; m \in M) \ .$$

Although the following observation is then trivial, it seems worth highlighting.

Lemma 2.3.3. *Let A, T be as above, and let M be a unit-linked, left Banach T-module. Then the cochain complex ${}_T\text{Hom}\big(\mathcal{C}_*(A, T_\varphi), {}_\varphi M\big)$ is isomorphic (as a chain complex of Banach spaces) to the Hochschild cochain complex $\mathcal{C}^*(A, {}_\varphi M)$.*

Proof sketch. Define comparison maps $\theta_n : {}_T\text{Hom}\big(\mathcal{C}_n(A, T_\varphi), {}_\varphi M\big) \to \mathcal{C}^n(A, {}_\varphi M)$ by

$$(\theta_n F)(a_1, \ldots, a_n) := F(1_T\otimes a_1\otimes \ldots \otimes a_n) \ ;$$

clearly each θ_n is an isomorphism of Banach spaces, and a tedious check shows that $\delta\theta_n F = \theta_{n+1} d_T^* F$ for all n. □

We can now give a weaker version of Corollary 2.3.2 for Banach modules.

Proposition 2.3.4 (Base change for Banach Hochschild cohomology). *Let A, T be Banach algebras with T unital; let $A^e \xrightarrow{\varphi} T$ be a continuous unital homomorphism. Suppose that the Hochschild chain complex*

$$\mathcal{C}_0(A, T_\varphi) \xleftarrow{d_0^T} \mathcal{C}_1(A, T_\varphi) \xleftarrow{d_1^T} \mathcal{C}_2(A, T_\varphi) \xleftarrow{d_2^T} \ \ldots$$

splits as a complex of Banach spaces in degrees ≥ 1, i.e. suppose there exist bounded linear maps $\sigma_j : \mathcal{C}_j(A, T_\varphi) \to \mathcal{C}_{j+1}(A, T_\varphi)$ for $j \geq 0$ which satisfy

$$\sigma_{n-1} d_{n-1}^T + d_n^T \sigma_n = \mathsf{id} \qquad \forall\, n \geq 1 \quad .$$

Then $\mathcal{H}_0(A, T_\varphi)$ is a Banach left T-module, linearly complemented in $\mathcal{C}_0(A, T_\varphi)$.

Moreover, if M is any unit-linked Banach left T-module, and thence an A-bimodule ${}_\varphi M$ via the action $amb := \varphi(a\otimes b)m$, then for all n there are isomorphisms of seminormed spaces

$$\text{Ext}_T^n\left(\mathcal{H}_0(A, T_\varphi), {}_\varphi M\right) \cong \mathcal{H}^n(A, {}_\varphi M) \tag{2.6}$$

Proof. We must first prove that $\mathcal{H}_0(A, T_\varphi)$ is Banach – otherwise the LHS of Equation (2.6) isn't even defined. This however follows easily from the hypotheses of the proposition: for since $\sigma_0 d_0^T + d_1^T \sigma_1 = \mathsf{id}$ we have

$$d_0^T = d_0^T(\sigma_0 d_0^T + d_1^T \sigma_1) = d_0^T \sigma_0 d_0^T$$

and so d_0^T has closed range. Since the range of d_0^T is also a unit-linked left T-module,

$$\mathcal{H}_0(A, T_\varphi) = \frac{T}{d_0^T\left(\mathcal{C}_1(A, T_\varphi)\right)}$$

is the quotient of a unit-linked Banach left T-module by a *unit-linked, closed* submodule, and is therefore itself a unit-linked Banach left T-module.

Let $q : \mathcal{C}_0(A, T_\varphi) \to \mathcal{H}_0(A, T_\varphi)$ be the canonical quotient map. Note that $d_0^T \sigma_0$ is a linear projection onto $\ker(q)$. Hence the augmented complex

$$0 \leftarrow \mathcal{H}_0(A, T_\varphi) \xleftarrow{q} \mathcal{C}_0(A, T_\varphi) \xleftarrow{d_0^T} \mathcal{C}_1(A, T_\varphi) \xleftarrow{d_1^T} \mathcal{C}_2(A, T_\varphi) \xleftarrow{d_2^T} \cdots$$

is an admissible complex in ${}_T\mathsf{unmod}$. Moreover, since T is unital, each $\mathcal{C}^n(A, T_\varphi)$ is free as a left T-module, so this augmented complex is an *admissible T-projective resolution* of $\mathcal{H}_0(A, T_\varphi)$. Therefore we have an isomorphism of seminormed spaces

$$\mathrm{Ext}_T^n\left(\mathcal{H}_0(A, T_\varphi), {}_\varphi M\right) \cong H^n\left\{{}_T\mathrm{Hom}\big(\mathcal{C}_*(A, T_\varphi), {}_\varphi M\big)\right\}$$

and by Lemma 2.3.3 the RHS is isomorphic to the nth Hochschild cohomology group $\mathcal{H}^n(A, {}_\varphi M)$ as required. □

2.3.3 An application of base change

Proposition 2.3.4 has the following corollary.

Corollary 2.3.5. *Let A be a commutative Banach algebra and suppose that the Hochschild chain complex*

$$\mathcal{C}_0(A, A_{\mathsf{un}}) \xleftarrow{d_0} \mathcal{C}_1(A, A_{\mathsf{un}}) \xleftarrow{d_1} \mathcal{C}_2(A, A_{\mathsf{un}}) \xleftarrow{d_2} \cdots$$

is split in Ban *in degrees ≥ 1. Then for any symmetric Banach A-bimodule M, $\mathcal{H}^n(A, M) = 0$ for all $n \geq 1$.*

Proof. Take $T = A_{\mathsf{un}}$ and $\varphi : A^e \to A_{\mathsf{un}}$ to be the product map. By Equation (2.6)

$$\mathcal{H}^n(A, {}_\varphi M) \cong \mathrm{Ext}^n_{A_{\mathsf{un}}}\left(\mathcal{H}_0(A, A_{\mathsf{un}}), {}_\varphi M\right)$$

but since M is a symmetric A-bimodule, $\mathcal{H}_0(A, A_{\mathsf{un}}) = A_{\mathsf{un}}$ is left A_{un}-projective, and hence the Ext vanishes in all degrees ≥ 1. □

By the main results of the author's article [5], which is based on work predating the results of this chapter, the conditions of this corollary hold when A is the ℓ^1-convolution algebra of a semilattice. Hence for such A we are able to deduce the vanishing of $\mathcal{H}^*(A, M)$ in degrees 1 and above for any symmetric, Banach A-bimodule M.

More generally we shall apply Corollary 2.3.5 later on, in Section 5.7, as a tool for showing that certain cohomology groups of convolution algebras of Clifford semigroups vanish. For the moment let us make some comments on the corollary and some possible implications.

Remark. A slightly more direct proof of the corollary can be obtained by specialising the *proof* of Proposition 2.3.4. This is the approach taken in [5].

Remark. It is possible to formally weaken the hypotheses in Corollary 2.3.5 if we put some extra assumptions on the Banach space structure of A. Specifically: if we furthermore assume that A is isomorphic as a Banach *space* to $\ell^1(\Gamma)$ for some index set Γ, then we need only assume that $\mathcal{H}_*(A, A_{\mathsf{un}}) = 0$ vanishes in degrees 1 and higher. (It is interesting to compare this extra hypothesis, which essentially says that the Banach algebra A is "projective in Ban", with the requirement in Corollary 2.3.2 that the k-algebra A should be projective as a k-module.)

Remark. Corollary 2.3.5 might suggest on a first reading that when A is a unital commutative Banach algebra, then simplicial triviality of A implies vanishing of cohomology with all symmetric coefficients. However, simplicial triviality merely says that the Hochschild chain complex $\mathcal{C}_*(A, A)$ is exact; we do not know *a priori* that it splits in Ban, and so the corollary cannot be applied. Moreover, while the unital, commutative Banach algebra $C([0,1])$ is amenable (hence *a fortiori* simplicially trivial), it is to the author's knowledge still unknown if $\mathcal{H}^3(C([0,1]), X) \neq 0$ for some symmetric Banach bimodule X.

This concludes our discussion of simplicial triviality for the moment. In the next chapter we shall investigate what can be said for symmetric coefficients under weaker conditions on simplicial cohomology.

Chapter 3

On the first order simplicial (co)homology

It was shown recently in [12] that the simplicial cohomology of the convolution algebra $\ell^1(\mathbb{Z}_+)$ vanishes in degrees 2 and above. In analogy with the result just proved for simplicial triviality, one might hope that we could deduce the vanishing of $\mathcal{H}^2(\ell^1(\mathbb{Z}_+), X)$ in degrees 2 and above for a significant class of symmetric bimodules X.

Unfortunately matters are not so straightforward, as we shall see in Chapter 4. In this chapter we shall establish some preliminary results for use in the next chapter.

3.1 Harrison (co)homology as a derived functor

The following computations are analogous to those in Section 2.3.2 – indeed, they are motivated by the spectral sequence discussed at the start of Section 2.3. We shall use them later in Chapter 4.

Proposition 3.1.1. *Let B be a unital, commutative Banach algebra such that the chain complex*

$$\mathcal{C}_1(B,B) \xleftarrow{\mathsf{d}_1} \mathcal{H}ar\mathcal{C}_2(B,B) \xleftarrow{\mathsf{d}_2} \mathcal{H}ar\mathcal{C}_3(B,B) \xleftarrow{\mathsf{d}_3} \dots \tag{3.1}$$

is split exact in Ban*. Then $\mathcal{H}_1(B,B)$ is a unit-linked, left Banach B-bimodule.*

Moreover, for each $n \geq 1$ and any symmetric, unit-linked Banach B-bimodule X, we have isomorphisms of seminormed spaces

$$\mathcal{H}ar\mathcal{H}^n(B,X) \cong \operatorname{Ext}_B^{n-1}(\mathcal{H}_1(B,B), X_L)$$
$$\mathcal{H}ar\mathcal{H}_n(B,X) \cong \operatorname{Tor}_{n-1}^B(X_R, \mathcal{H}_1(B,B))$$

where X_L and X_R denote the B-modules obtained by restricting the 2-sided action on B to a left and right action respectively.

Proof. Recall from Proposition 1.4.7 and the remark after it that

$$\mathcal{C}_1(B,B) \xleftarrow{\mathsf{d}_1} \mathcal{H}ar\mathcal{C}_2(B,B) \xleftarrow{\mathsf{d}_2} \mathcal{H}ar\mathcal{C}_3(B,B) \xleftarrow{\mathsf{d}_3} \dots$$

is a complex in ${}_B\mathsf{unmod}$.

The hypothesis (3.1) says that there exist bounded linear maps

$$\sigma_n : \mathcal{H}ar\mathcal{C}_n(B,B) \to \mathcal{H}ar\mathcal{C}_{n+1}(B,B)$$

such that

$$\sigma_n \mathsf{d}_n + \mathsf{d}_{n+1}\sigma_{n+1} = \mathsf{id} \qquad \text{for } n = 1,2,\dots.$$

In particular $\sigma_1\mathsf{d}_1 = \mathsf{id} - \mathsf{d}_2\sigma_2$; hence $\mathsf{d}_1\sigma_1\mathsf{d}_1 = \mathsf{d}_1(\mathsf{id} - \mathsf{d}_2\sigma_2) = \mathsf{d}_1$ and this implies that d_1 has closed range. Thus $\mathcal{H}_1(B,B) = \operatorname{coker}(\mathsf{d}_1)$ is the quotient of a unit-linked Banach B-module by a closed submodule, and is therefore itself a unit-linked Banach B-module.

Since B is unital, each $\mathcal{C}_n(B,B)$ is B-projective as a Banach B-module; therefore, since the BGS projections are B-module maps, each $\mathcal{H}ar\mathcal{C}_n(B,B)$ is a B-module summand of a B-projective module and is thus B-projective. Hence by the hypothesis (3.1) the complex

$$0 \leftarrow \mathcal{H}_1(B,B) \xleftarrow{q} \mathcal{C}_1(B,B) \xleftarrow{\mathsf{d}_1} \mathcal{H}ar\mathcal{C}_2(B,B) \xleftarrow{\mathsf{d}_2} \dots$$

is an admissible B-projective resolution of $\mathcal{H}_1(B,B)$, and by the definitions of Tor and Ext we have

$$\operatorname{Ext}^{n-1}_B\left[\mathcal{H}_1(B,B), X_L\right] \cong H^n\left[{}_B\operatorname{Hom}\left(\mathcal{H}ar\mathcal{C}_*(B,B), X_L\right)\right]$$

and

$$\operatorname{Tor}^B_{n-1}\left[X_R, \mathcal{H}_1(B,B)\right] \cong H_n\left[X_R \widehat{\underset{B}{\otimes}} \mathcal{H}ar\mathcal{C}_*(B,B)\right]$$

for every $n \geq 1$.

To finish, we recall from (the remarks in) Definition 1.5.4 that the cochain complex ${}_B\operatorname{Hom}\left(\mathcal{H}ar\mathcal{C}_*(B,B), X_L\right)$ is isomorphic to $\mathcal{H}ar\mathcal{C}^*(A,X)$, and that the chain complex $X_R \widehat{\underset{B}{\otimes}} \mathcal{H}ar\mathcal{C}_*(B,B)$ is isomorphic to $\mathcal{H}ar\mathcal{C}_*(A,X)$. ☐

3.2 A "baby Künneth formula"

The Künneth formula of [13] is applied in that article to calculate the simplicial homology groups of $\ell^1(\mathbb{Z}_+^k)$ up to isomorphism of seminormed spaces; in particular one sees that $\mathcal{H}_n(\ell^1(\mathbb{Z}_+^k), \ell^1(\mathbb{Z}_+^k))$ is Banach for all n and all k. For later reference, we would like to determine the first simplicial homology group of $\ell^1(\mathbb{Z}_+^k)$ up to *isomorphism of Banach* $\ell^1(\mathbb{Z}_+^k)$*-modules*.

It seems likely that by chasing the relevant maps through the proof in [13], one could show that the Banach-space isomorphism calculated there is in fact an $\ell^1(\mathbb{Z}_+^k)$-module map. However, we have chosen a more abstract approach which holds for any unital commutative Banach algebra A: more precisely, we identify $\mathcal{H}_1(A, A)$ with a certain "module of differentials" $\widetilde{\Omega}_A$, and then give a decomposition theorem for $\widetilde{\Omega}_{\left(A^{\widehat{\otimes} k}\right)}$ under certain conditions on A. Since we are only dealing with the first simplicial homology group (rather than all the simplicial homology groups, as in [13]) we are able to carry out the computations quite explicitly.

Notation and other preliminaries

Let A be a unital commutative Banach algebra. Let I_A denote the kernel of the product map $A\widehat{\otimes}A \to A$, equipped with the A-bimodule structure it inherits from $A\widehat{\otimes}A$.

We let σ_A denote the projection from $A\widehat{\otimes}A$ onto I_A which is defined by

$$\sigma_A(x\otimes y) = x\otimes y - xy\otimes 1_A$$

and note that $\ker(\sigma_A) = A\widehat{\otimes}\mathbb{C}1_A$. Note also that σ_A is a left A-module map.

Let $\tau_A : A\widehat{\otimes}A\widehat{\otimes}A \to I_A$ be the bounded linear map defined by

$$\begin{aligned}\tau_A(x\otimes y\otimes a) &= \sigma(x\otimes y)\cdot a - a\cdot\sigma(x\otimes y)\\ &= x\otimes ay - xy\otimes a - ax\otimes y + axy\otimes 1_A\end{aligned}$$

Although τ_A need not have closed range, note that it is a left A-submodule of I_A, since $u\tau_A(x\otimes y\otimes a) = \tau_A(ux\otimes y\otimes a)$. We may therefore form the quotient seminormed space

$$\widetilde{\Omega}_A := I_A/\operatorname{im}(\tau_A)$$

which inherits the structure of a left A-module.

The following result is somehow implicit in the setup of [28], but the precise formulation here is new as far as I know. It is in any case a straightforward if fiddly modification of standard ideas from commutative algebra (see [30, 9.2.4] for instance).

Proposition 3.2.1. *There is an isomorphism of seminormed spaces $\widetilde{\Omega}_A \to \mathcal{H}_1(A, A)$, which is also an isomorphism of left A-modules.*

Note that in particular $\widetilde{\Omega}_A$ is a Banach space if and only if $\mathcal{H}_1(A, A)$ is.

Proof. Let $\mathsf{d}_1 : \mathcal{C}_2(A, A) \to \mathcal{C}_1(A, A)$ be the Hochschild boundary map, given by the formula

$$\mathsf{d}_1(x \otimes a_1 \otimes a_2) = xa_1 \otimes a_2 - x \otimes a_1a_2 + a_2x \otimes a_1 \ .$$

A quick calculation shows that $\sigma_A \mathsf{d}_1 = -\tau_A$, and so in particular the composite map

$$\mathcal{C}_1(A, A) = A \widehat{\otimes} A \xrightarrow{\sigma_A} I_A \longrightarrow I_A / \operatorname{im}(\tau_A)$$

vanishes on $\operatorname{im}(\mathsf{d}_1) = \mathcal{B}_1(A, A)$, hence descends to a well-defined and bounded linear A-module map $\widetilde{\sigma_A}$ as shown in the diagram below:

$$\begin{array}{ccc} \mathcal{C}_1(A, A) & \xrightarrow{\sigma_A} & I_A \\ {\scriptstyle q}\downarrow & & \downarrow \\ \mathcal{H}_1(A, A) & \underset{\widetilde{\sigma_A}}{\cdots\cdots\to} & \widetilde{\Omega}_A \end{array}$$

It suffices to construct a bounded linear 2-sided inverse to $\widetilde{\sigma_A}$, which we do as follows. Let $J : I_A \to A \widehat{\otimes} A = \mathcal{C}_1(A, A)$ be the inclusion map: then $\sigma_A J = \mathsf{id}$. Moreover, for any $x, y, a \in A$

$$\begin{aligned} J\tau_A(x \otimes y \otimes a) &= x \otimes ay - xy \otimes a - ax \otimes y + axy \otimes 1_A \\ &= -\mathsf{d}_1(x \otimes y \otimes a) + \mathsf{d}_1(axy \otimes 1_A \otimes 1_A) \end{aligned}$$

and so qJ vanishes on $\operatorname{im}(\tau)$, inducing a bounded linear map $\widetilde{J} : \widetilde{\Omega}_A \to \mathcal{H}_1(A, A)$. Since $\sigma_A J = \mathsf{id}$, $\widetilde{\sigma_A}\widetilde{J}$ is the identity map, and it remains only to show that $\mathsf{id} - J\sigma_A$ takes values in $\ker(q) = \operatorname{im}(\mathsf{d}_1)$. But this is immediate from the following computation:

$$(\mathsf{id} - J\sigma_A)(x \otimes y) = xy \otimes 1_A = \mathsf{d}_1(xy \otimes 1_A \otimes 1_A)$$

□

The main calculation

Let A and B be unital commutative Banach algebras; then their projective tensor product $A\widehat{\otimes}B$ is also a unital commutative Banach algebra, which we denote by C.

Theorem 3.2.2 (Differentials of tensor products). *There exist mutually inverse, bounded linear C-module maps*

$$\widetilde{\Omega}_C \underset{\mathsf{Ass}}{\overset{\mathsf{Ex}}{\rightleftarrows}} \frac{I_A\widehat{\otimes}B}{\operatorname{im}(\tau_A\widehat{\otimes}\mathsf{id}_B)} \oplus \frac{A\widehat{\otimes}I_B}{\operatorname{im}(\mathsf{id}_A\widehat{\otimes}\tau_B)}$$

In particular, if both τ_A and τ_B have closed range, then so does τ_C, and we then have an isomorphism of Banach C-modules

$$\widetilde{\Omega}_C \cong \widetilde{\Omega}_A\widehat{\otimes}B \oplus A\widehat{\otimes}\widetilde{\Omega}_B\ .$$

Clearly this generalises to k-fold tensor products of unital commutative Banach algebras, just by induction on k:

Corollary 3.2.3. *Let $A_1,\dots,A_k$ be unital, commutative Banach algebras, and let $\mathfrak{A} = \widehat{\bigotimes}_{i=1}^k A_i$. Suppose that $\mathcal{H}_1(A_i,A_i)$ is a Banach space for each i. Then $\mathcal{H}_1(\mathfrak{A},\mathfrak{A})$ is Banach, and there is an isomorphism of Banach $\mathfrak{A}$-modules*

$$\mathcal{H}_1(\mathfrak{A},\mathfrak{A}) \cong \bigoplus_{i=1}^k A_1\widehat{\otimes}\dots\widehat{\otimes}\mathcal{H}_1(A_i,A_i)\widehat{\otimes}\dots\widehat{\otimes}A_k$$

Proof of Theorem 3.2.2. Our proof is quite space-consuming, but at heart very simple. The strategy is to "induce" the maps Ex and Ass from bounded, Banach C-module maps

$$\widetilde{\mathsf{Ex}} : I_C \to I_A\widehat{\otimes}B \oplus A\widehat{\otimes}I_B \quad \text{and} \quad \widetilde{\mathsf{Ass}} : I_A\widehat{\otimes}B \oplus A\widehat{\otimes}I_B \to I_C$$

and then check that these descend to mutually inverse maps at the level of quotient spaces.

We define $\widetilde{\mathsf{Ex}}$ by the formula

$$\widetilde{\mathsf{Ex}}\,(\sigma_C(a\otimes b\otimes x\otimes y)) := \begin{pmatrix} \sigma_A(a\otimes x)\otimes by \\ ax\otimes\sigma_B(b\otimes y)\end{pmatrix} \qquad (a,x\in A; b,y\in B) \tag{3.2}$$

(this is well-defined, since $\ker(\sigma_C) = C\widehat{\otimes}\mathbb{C}1_C = A\widehat{\otimes}B\widehat{\otimes}\mathbb{C}(1_A\otimes 1_B)$ and the RHS of equation (3.2) vanishes if $x\in\mathbb{C}1_A$ and $y\in\mathbb{C}1_B$). One easily checks that $\widetilde{\mathsf{Ex}}$ is a Banach C-module map.

Now we must check that $\widetilde{\mathsf{Ex}}$ descends to a bounded linear C-module map from $I_C/\operatorname{im}(\tau_c)$ to $[(I_A\widehat{\otimes}B)/\operatorname{im}(\tau_A\widehat{\otimes}\mathsf{id}_B)]\oplus[(A\widehat{\otimes}I_B)/\operatorname{im}(\mathsf{id}_A\widehat{\otimes}\tau_B)]$, i.e. that there is a well-defined, bounded linear C-module map Ex which makes the following diagram commute:

$$\begin{array}{ccc} I_C & \xrightarrow{\quad\widetilde{\mathsf{Ex}}\quad} & I_A\widehat{\otimes}B\oplus A\widehat{\otimes}I_B \\ \downarrow & & \downarrow \\ \operatorname{coker}(\tau_C) & \overset{\mathsf{Ex}}{\dashrightarrow} & \operatorname{coker}(\tau_A\widehat{\otimes}\mathsf{id}_B)\oplus\operatorname{coker}(\mathsf{id}_A\widehat{\otimes}\tau_B) \end{array}$$

The existence of such a map Ex is guaranteed by standard diagram chasing, once we establish that

$$\operatorname{im}(\widetilde{\mathsf{Ex}}\circ\tau_C)\subseteq\operatorname{im}((\tau_A\widehat{\otimes}\mathsf{id}_B,\mathsf{id}_A\widehat{\otimes}\tau_B)) \quad ;$$

this inclusion in turn follows from the following claim:

Claim #1. There exists a bounded linear map θ making the following diagram commute:

$$\begin{array}{ccc} C^{\widehat{\otimes}3} & \xrightarrow{\quad\tau_C\quad} & I_C \\ {\scriptstyle\theta}\,\vdots & & \downarrow{\scriptstyle\widetilde{\mathsf{Ex}}} \\ A^{\widehat{\otimes}3}\widehat{\otimes}B\oplus A\widehat{\otimes}B^{\widehat{\otimes}3} & \xrightarrow[(\tau_A\widehat{\otimes}\mathsf{id}_B,\,\mathsf{id}_A\widehat{\otimes}\tau_B)]{} & I_A\widehat{\otimes}B\oplus A\widehat{\otimes}I_B \end{array}$$

(For if we assume the claim holds, then

$$\operatorname{im}(\widetilde{\mathsf{Ex}}\circ\tau_C)=\operatorname{im}((\tau_A\widehat{\otimes}\mathsf{id}_B,\mathsf{id}_A\widehat{\otimes}\tau_B)\circ\theta)\subseteq\operatorname{im}((\tau_A\widehat{\otimes}\mathsf{id}_B,\mathsf{id}_A\widehat{\otimes}\tau_B))$$

as required.)

Proof of Claim #1. Let $x_1, x_2, a \in A$ and $y_1, y_2, b \in B$. Direct computation yields

$$\begin{aligned}
&\widetilde{\mathsf{Ex}}\,\tau_C(x_1\otimes y_1\otimes x_2\otimes y_2\otimes a\otimes b)\\
&= \widetilde{\mathsf{Ex}}[x_1\otimes y_1\otimes ax_2\otimes by_2 - x_1x_2\otimes y_1y_2\otimes a\otimes b - ax_1\otimes by_1\otimes x_2\otimes y_2 + ax_1x_2\otimes by_1y_2\otimes 1_A\otimes 1_B]\\
&= \widetilde{\mathsf{Ex}}\,[\sigma_C(x_1\otimes y_1\otimes ax_2\otimes by_2) - \sigma_C(x_1x_2\otimes y_1y_2\otimes a\otimes b) - \sigma_C(ax_1\otimes by_1\otimes x_2\otimes y_2)]\\
&= \begin{pmatrix}\sigma_A(x_1x_2\otimes a)\otimes by_1y_2\\ ax_1x_2\otimes\sigma_B(y_1\otimes by_2)\end{pmatrix} - \begin{pmatrix}\sigma_A(x_1x_2\otimes a)\otimes by_1y_2\\ ax_1x_2\otimes\sigma_B(y_1y_2\otimes b)\end{pmatrix} - \begin{pmatrix}\sigma_A(ax_1x_2)\otimes by_1y_2\\ ax_1x_2\otimes\sigma_B(by_1\otimes y_2)\end{pmatrix}\\
&= \begin{pmatrix}(x_1\otimes ax_2 - x_1x_2\otimes a - ax_1\otimes x_2 + ax_1x_2\otimes 1_A)\otimes by_1y_2\\ ax_1x_2\otimes(y_1\otimes by_2 - y_1y_2\otimes b + by_1\otimes y_2 - by_1t_2\otimes 1_B)\end{pmatrix}\\
&= \begin{pmatrix}\tau_A(x_1\otimes x_2\otimes a)\otimes by_1y_2\\ ax_1x_2\otimes\tau_B(y_1\otimes y_2\otimes b)\end{pmatrix}\,.
\end{aligned}$$

We therefore define θ by the formula

$$\theta(x_1\otimes y_1\otimes x_2\otimes y_2\otimes a\otimes b) := \begin{pmatrix}(x_1\otimes x_2\otimes a)\otimes by_1y_2\\ ax_1x_2\otimes(y_1\otimes y_2\otimes b)\end{pmatrix}$$

and observe that θ is bounded linear; by linearity and continuity the calculation above implies that $\widetilde{\mathsf{Ex}}\,\tau_C = \theta(\tau_A\widehat{\otimes}\mathsf{id}_B, \mathsf{id}_A\widehat{\otimes}\tau_B)$ as claimed. ☐

Now let us turn to the definition of the map $\widetilde{\mathsf{Ass}}$. It is convenient to introduce auxiliary maps $\widetilde{\mathsf{Ass}}_A : I_A\widehat{\otimes}B \to I_C$ and $\widetilde{\mathsf{Ass}}_B : A\widehat{\otimes}I_B \to I_C$, defined by

$$\widetilde{\mathsf{Ass}}_A(\sigma_A(u\otimes x)\otimes b) = \sigma_C(u\otimes b\otimes x\otimes 1_B)$$ (well-defined, since RHS vanishes if $x \in \mathbb{C}1_A$)

$$\widetilde{\mathsf{Ass}}_B(a\otimes\sigma_B(v\otimes y)) = \sigma_C(a\otimes v\otimes 1_A\otimes y)$$ (well-defined, since RHS vanishes if $y \in \mathbb{C}1_B$)

and to then let $\widetilde{\mathsf{Ass}} := \mathsf{Ass}_A \oplus \mathsf{Ass}_B : \begin{pmatrix}I_A\widehat{\otimes}B\\ A\widehat{\otimes}I_B\end{pmatrix} \to I_C$. One checks easily that $\widetilde{\mathsf{Ass}}_A$ and $\widetilde{\mathsf{Ass}}_B$ are Banach C-module maps, so that their coproduct $\widetilde{\mathsf{Ass}}$ is also a Banach C-module map.

Now we must check that $\widetilde{\mathsf{Ass}}$ descends to a bounded linear C-module map from $[(I_A\widehat{\otimes}B)/\operatorname{im}(\tau_A\widehat{\otimes}\mathsf{id}_B)] \oplus [(A\widehat{\otimes}I_B)/\operatorname{im}(\mathsf{id}_A\widehat{\otimes}\tau_B)]$ to $I_C/\operatorname{im}(\tau_c)$, i.e. that there is a well-defined, bounded linear C-module map Ass which makes the following diagram commute:

$$\begin{array}{ccc}
I_A \widehat{\otimes} B \oplus A \widehat{\otimes} I_B & \xrightarrow{\widetilde{\mathsf{Ass}}} & I_C \\
\downarrow & & \downarrow \\
\operatorname{coker}(\tau_A \widehat{\otimes} \mathsf{id}_B) \oplus \operatorname{coker}(\mathsf{id}_A \widehat{\otimes} \tau_B) & \underset{\mathsf{Ass}}{\cdots\cdots\rightarrow} & \operatorname{coker}(\tau_C)
\end{array}$$

The existence of such a map Ass is guaranteed by standard diagram chasing, once we establish that

$$\operatorname{im}(\widetilde{\mathsf{Ass}} \circ (\tau_A \widehat{\otimes} \mathsf{id}_B, \mathsf{id}_A \widehat{\otimes} \tau_B)) \subseteq \operatorname{im}(\tau_C) \quad ;$$

this inclusion in turn follows from the following claim:

Claim #2. There exists a bounded linear map γ making the following diagram commute:

$$\begin{array}{ccc}
A^{\widehat{\otimes} 3} \widehat{\otimes} B \oplus A \widehat{\otimes} B^{\widehat{\otimes} 3} & \xrightarrow{(\tau_A \widehat{\otimes} \mathsf{id}_B, \mathsf{id}_A \widehat{\otimes} \tau_B)} & I_A \widehat{\otimes} B \oplus A \widehat{\otimes} I_B \\
\gamma \,\vdots & & \downarrow \widetilde{\mathsf{Ex}} \\
C^{\widehat{\otimes} 3} & \xrightarrow[\tau_C]{} & I_C
\end{array}$$

(For if we assume the claim holds, then

$$\operatorname{im}(\widetilde{\mathsf{Ass}} \circ (\tau_A \widehat{\otimes} \mathsf{id}_B, \mathsf{id}_A \widehat{\otimes} \tau_B)) = \operatorname{im}(\tau_C \circ \gamma) \subseteq \operatorname{im}(\tau_C)$$

as required.)

Proof of Claim #2. Let $x_1, x_2, u \in A$ and $b \in B$. Direct computation yields

$$\begin{aligned}
& \widetilde{\mathsf{Ass}}_A (\tau_A \widehat{\otimes} \mathsf{id}_B)(x_1 \otimes x_2 \otimes u \otimes b) \\
&= \widetilde{\mathsf{Ass}}_A [(x_1 \otimes u x_2 - x_1 x_2 \otimes u - u x_1 \otimes x_2 + u_2 x_1 x_2 \otimes 1_A) \otimes b] \\
&= \widetilde{\mathsf{Ass}}_A \left[\sigma_A(x_1 \otimes u x_2) \otimes b - \sigma_A(x_1 x_2 \otimes u) \otimes b - \sigma_A(u x_1 \otimes x_2) \otimes b\right] \\
&= \sigma_C(x_1 \otimes b \otimes u x_2 \otimes 1_B - x_1 x_2 \otimes b \otimes u \otimes 1_B - u x_1 \otimes b \otimes x_2 \otimes 1_B) \\
&= x \otimes b \otimes \otimes u x_2 \otimes 1_B - x_1 x_2 \otimes b \otimes u \otimes 1_B - u x_1 \otimes b \otimes x_2 \otimes 1_B \\
&= \tau_C(x_1 \otimes b \otimes x_2 \otimes 1_B \otimes u \otimes 1_B) \quad ;
\end{aligned}$$

by symmetry, if $a \in A$ and $y_1, y_2, v \in B$, we have

$$\widetilde{\mathsf{Ass}}_B (\mathsf{id}_A \widehat{\otimes} \tau_B)(a \otimes y_1 \otimes y_2 \otimes v) = \tau_C(a \otimes y_1 \otimes 1_A \otimes y_2 \otimes 1_A \otimes v) \, .$$

We therefore define γ by the formula

$$\gamma \begin{pmatrix} (x_1 \otimes x_2 \otimes u) \otimes b \\ a \otimes (y_1 \otimes y_2 \otimes v) \end{pmatrix} := x_1 \otimes b \otimes x_2 \otimes 1_B \otimes u \otimes 1_B + a \otimes y_1 \otimes 1_A \otimes y_2 \otimes 1_A \otimes v$$

and observe that γ is bounded linear; by linearity and continuity the calculations above imply that

$$\widetilde{\mathsf{Ass}}_A \left(\tau_A \widehat{\otimes} \mathsf{id}_B, \mathsf{id}_A \widehat{\otimes} \tau_B \right) = \tau_C \gamma$$

as claimed.

□

Proof that Ass *and* Ex *are mutually inverse.* Consider the map

$$\widetilde{\mathsf{Ex}}\, \widetilde{\mathsf{Ass}} : I_A \widehat{\otimes} B \oplus A \widehat{\otimes} I_B \to I_A \widehat{\otimes} B \oplus A \widehat{\otimes} I_B \ .$$

Evaluating on elementary tensors, we find that

$$\begin{aligned} \widetilde{\mathsf{Ex}}\, \widetilde{\mathsf{Ass}} \begin{pmatrix} \sigma_A(u \otimes x) \otimes b \\ a \otimes \sigma_B(v \otimes y) \end{pmatrix} &= \widetilde{\mathsf{Ex}}\, \sigma_C(u \otimes b \otimes x \otimes 1_B) + \widetilde{\mathsf{Ex}}\, \sigma_C(a \otimes v \otimes 1_A \otimes y) \\ &= \begin{pmatrix} \sigma_A(u \otimes x) \otimes b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a \otimes \sigma_B(v \otimes y) \end{pmatrix} = \begin{pmatrix} \sigma_A(u \otimes x) \otimes b \\ a \otimes \sigma_B(v \otimes y) \end{pmatrix} \end{aligned}$$

and so by continuity and linearity, $\widetilde{\mathsf{Ex}}\, \widetilde{\mathsf{Ass}}$ is the identity map on $I_A \widehat{\otimes} B \oplus A \widehat{\otimes} I_B$; in particular, $\mathsf{Ex}\, \mathsf{Ass}$ is the identity map on $\operatorname{coker}(\tau_A \widehat{\otimes} \mathsf{id}_B) \oplus \operatorname{coker}(\mathsf{id}_A \widehat{\otimes} \tau_B)$.

It remains only to show that the map $\widetilde{\mathsf{Ass}}\, \widetilde{\mathsf{Ex}} - \mathsf{id}$ takes values in $\operatorname{im}(\tau_C)$. For this it suffices to construct a bounded linear map $\rho : C \widehat{\otimes} C \to C \widehat{\otimes} C \widehat{\otimes} C$ such that

$$\widetilde{\mathsf{Ass}}\, \widetilde{\mathsf{Ex}}\, \sigma_C = \sigma_C + \tau_C \rho$$

which we do as follows. For any $a, x \in A$ and $b, y \in B$, direct computation yields

$$\begin{aligned} &(\widetilde{\mathsf{Ass}}\, \widetilde{\mathsf{Ex}}\, \sigma_C - \sigma_C)(a \otimes b \otimes x \otimes y)) \\ &= \widetilde{\mathsf{Ass}}_A(\sigma_A(a \otimes x) \otimes by) + \widetilde{\mathsf{Ass}}_B(ax \otimes \sigma_B(b \otimes y)) - \sigma_C(a \otimes b \otimes x \otimes y) \\ &= \sigma_C(a \otimes by \otimes x \otimes 1_B + ax \otimes b \otimes 1_A \otimes y - a \otimes b \otimes x \otimes y) \\ &= a \otimes by \otimes x 1_B - ax \otimes by \otimes 1_A \otimes 1_B + ax \otimes b \otimes 1_A \otimes y - a \otimes b \otimes x \otimes y \\ &= -\tau_C(a \otimes b \otimes x \otimes 1_B \otimes 1_A \otimes y) \quad ; \end{aligned}$$

therefore, if we define ρ by the formula $\rho(a \otimes b \otimes x \otimes y) = -a \otimes b \otimes x \otimes 1_B \otimes 1_A \otimes y$, we see that ρ is indeed bounded linear and that by linearity and continuity

$$\widetilde{\mathsf{Ass}}\, \widetilde{\mathsf{Ex}}\, \sigma_C - \sigma_C = \tau_C \rho$$

as required. □

This concludes the proof of the theorem. □

Relation to the "Banach Kähler module"

We let $I_A^{[2]}$ denote the image of the product map $I_A \widehat{\otimes} I_A \to I_A$; note that this is *a priori* strictly larger than $I_A^2 = \operatorname{lin}\{vw \ : \ v, w \in I_A\}$, and is in general strictly smaller than $\overline{I_A^2}$.

Lemma 3.2.4. $I_A^{[2]} = \operatorname{im}(\tau_A)$.

I would like to thank Prof. Niels Grønbæk for suggestions which have shortened the following proof.

Proof. Given $x_1, x_2, y_1, y_2 \in A$ observe that

$$\begin{aligned} \sigma_A(x_1 \otimes x_2)\sigma_A(y_1 \otimes y_2) &= (x_1 \otimes x_2 - x_1x_2 \otimes 1_A)(y_1 \otimes y_2 - y_1y_2 \otimes 1_A) \\ &= x_1y_1 \otimes x_2y_2 - x_1x_2y_1 \otimes y_2 - x_1y_1y_2 \otimes x_2 + x_1x_2y_1y_2 \otimes 1_A \\ &= \tau_A(x_1y_1 \otimes x_2 \otimes y_2) \ . \end{aligned}$$

Let $\alpha : A^{\otimes 4} \to A^{\widehat{\otimes} 3}$ be given by $\alpha(x_1 \otimes x_2 \otimes y_1 \otimes y_2) := x_1y_1 \otimes x_2 \otimes y_2$; then the preceding calculation shows that $\tau_A \alpha = \pi_{I_A}(\sigma_A \widehat{\otimes} \sigma_A)$. Since A is unital α is surjective, and we conclude that

$$I_A^{[2]} = \operatorname{im}(\pi_{I_A}(\sigma_A \widehat{\otimes} \sigma_A)) = \operatorname{im}(\tau_A \alpha) = \operatorname{im}(\tau_A)$$

as required. □

From this the following corollary is immediate.

Corollary 3.2.5. *The Hausdorffification of $\widetilde{\Omega}_A$ is isomorphic, as a Banach A-module, to $I_A/\overline{I_A^2}$.*

The point of this corollary is that the Banach A-module $I_A/\overline{I_A^2}$ has already been studied, in Runde's paper [28]: it is the natural Banach analogue of the Kähler module of differentials for a commutative ring. For the purposes of this thesis, it is $\mathcal{H}_1(A, A)$ we are interested in and not $I_A/\overline{I_A^2}$, and we have arranged our proofs accordingly. The reader should nevertheless note that the decomposition theorem for $\widetilde{\Omega}_{A \widehat{\otimes} B}$, and its proof, are modelled on the corresponding result and proof for the Kähler module of a tensor product of rings. For example, the idea behind Theorem 3.2.2 is based on the formula

$$d_C(x \otimes y) = (1_A \otimes y) \cdot d_C(x \otimes 1_B) + (x \otimes 1_B) \cdot d_C(1_A \otimes y)$$

and the heuristic idea that we "identify" $d_C(x \otimes 1_B)$, $d_C(1_A \otimes y)$ with $d_A(x)$ and $d_B(y)$ respectively.

3.3 The norms on $\mathcal{Z}^1(\mathfrak{A}_\omega, \mathfrak{A}'_\omega)$ and $\mathcal{Z}^1(\ell^1(\mathbb{Z}_+^\infty), \ell^1(\mathbb{Z}_+^\infty)')$

The results of this section will not be needed in later parts of this thesis, but have been placed here since they follow on from the idea of trying to determine the first simplicial homology group of a tensor product of Banach algebras.

It was shown in the proof of [13, Propn 7.3] that $\mathcal{H}_1(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+))$ is isomorphic as a Banach space to ℓ^1. It follows by the Künneth formula of that paper (or by our Corollary 3.2.3) that for each $n \in \mathbb{N}$ the simplicial homology group $\mathcal{H}_1(\ell^1(\mathbb{Z}_+^n), \ell^1(\mathbb{Z}_+^n))$ is also isomorphic to $\ell^1(\mathbb{N})$, and it is natural to then ask if the distortion constant of this isomorphism is bounded as $n \to \infty$.

We shall show, somewhat indirectly, that the answer is negative. More precisely, if the Banach-Mazur distance from $\mathcal{H}_1(\ell^1(\mathbb{Z}_+^n), \ell^1(\mathbb{Z}_+^n))$ to $\ell^1(\mathbb{N})$ were bounded independently of n, then by duality the Banach-Mazur distance from $\mathcal{H}^1((\ell^1(\mathbb{Z}_+^n), \ell^1(\mathbb{Z}_+^n))$ to ℓ^∞ would also be bounded independently of n; by computing the norms on the Banach spaces $\mathcal{H}^1((\ell^1(\mathbb{Z}_+^n), \ell^1(\mathbb{Z}_+^n))$ we shall show that this is not the case.

In fact we can carry out the main computations in the slightly more general setting of weighted convolution algebras on $\mathbb{Z}_+^n$. We first need to set up some notation.

Notation. If ω is a weight on the semigroup $\mathbb{Z}_+^n$ then we write $\mathfrak{A}_\omega$ for the convolution algebra $\ell^1(\mathbb{Z}_+^n, \omega)$. We shall use bold letters to denote multi-indices in $\mathbb{Z}_+^n$, writing $\boldsymbol{\alpha}$ for the n-tuple $(\alpha_1, \dots, \alpha_n)$ for example. If $j \in \{1, \dots, n\}$ then e_j denotes the element of $\mathbb{Z}_+^n$ which is 1 in position j and 0 everywhere else.

We shall denote the basis element in $\mathfrak{A}_\omega$ that corresponds to a multi-index $\boldsymbol{\alpha}$ by $[\boldsymbol{\alpha}]$. This is purely to try and make some of the formulas to follow more readable, *and in later chapters we shall revert to the more usual "monomial" notation* z^α.

The following lemma is trivial but worth isolating as a technical step.

Lemma 3.3.1. *Let $F : \mathfrak{A}_\omega \to \mathfrak{A}'_\omega$ be a bounded linear map. Then F is a derivation if and only if*

$$F([\boldsymbol{\alpha}])([\mathbf{N} - \boldsymbol{\alpha}]) = \sum_{j=1}^{n} \frac{\alpha_j}{N_j} F([N_j e_j])([\mathbf{N} - N_j e_j]) \qquad \textit{for all } \mathbf{0} \le \boldsymbol{\alpha} \le \mathbf{N} \tag{3.3}$$

(We use the notational convention that if $\alpha_j = N_j = 0$ then $\frac{\alpha_j}{N_j} = 0$.)

Proof. Suppose F is a derivation. A simple induction using the Leibnitz rule shows that whenever $0 \leq \boldsymbol{\alpha} \leq \mathbf{N}$

$$F([\boldsymbol{\alpha}])([\mathbf{N}-\boldsymbol{\alpha}]) = \sum_{j=1}^{n} \alpha_j F([e_j])([\mathbf{N}-\alpha_j e_j])$$

In particular, for each $k = 1, 2, \ldots n$ we set $\boldsymbol{\alpha} = \mathbf{N}_k e_k$ and get

$$F([\mathbf{N}_k e_k])([\mathbf{N}-N_k e_k]) = N_k F([e_k])([\mathbf{N}-e_k])$$

Substituting this back in we find that

$$F([\boldsymbol{\alpha}])([\mathbf{N}-\boldsymbol{\alpha}]) = \sum_{j=1}^{n} \frac{\alpha_j}{N_j} F([N_j e_j])([\mathbf{N}-N_j e_j])$$

and thus F satisfies the identity (3.3).

Conversely, suppose F is a bounded linear map from $\mathfrak{A}_\omega$ to $\mathfrak{A}'_\omega$ that satisfies the identity (3.3). Then for any $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{Z}_+^n$, we let $\mathbf{N} := \boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}$ and observe that

$$\begin{aligned}
F([\boldsymbol{\alpha}]\cdot[\boldsymbol{\beta}])([\boldsymbol{\gamma}]) &= F([\boldsymbol{\alpha}+\boldsymbol{\beta}])([\mathbf{N}-\boldsymbol{\alpha}-\boldsymbol{\beta}]) \\
&= \sum_{j=1}^{n} \frac{\alpha_j+\beta_j}{N_j} F([N_j e_j])([\mathbf{N}-N_j e_j]) && \text{by (3.3)} \\
&= \begin{cases} \displaystyle\sum_{j=1}^{n} \frac{\alpha_j}{N_j} F([N_j e_j])([\mathbf{N}-N_j e_j]) \\ \displaystyle +\sum_{k=1}^{n} \frac{\beta_k}{N_k} F([N_k e_k])([\mathbf{N}-N_k e_k]) \end{cases} \\
&= F([\boldsymbol{\alpha}])([\mathbf{N}-\boldsymbol{\alpha}]) + F([\boldsymbol{\beta}])([\mathbf{N}-\boldsymbol{\beta}]) && \text{by (3.3)} \\
&= F([\boldsymbol{\alpha}])([\boldsymbol{\beta}]\cdot[\boldsymbol{\gamma}]) + F([\boldsymbol{\beta}])([\boldsymbol{\gamma}]\cdot[\boldsymbol{\alpha}])
\end{aligned}$$

It follows, by continuity and linearity, that $F(fg)(h) = F(f)(gh) + F(g)(hf)$ for all $f, g, h \in \mathfrak{A}_\omega$, and so F is a derivation. □

The norm of any bounded linear map $T : \mathfrak{A}_\omega \to \mathfrak{A}'_\omega$ is given by

$$\begin{aligned}
\|T\| &= \sup_{\alpha,\beta\in\mathbb{Z}_+^n} \frac{|T([\boldsymbol{\alpha}])([\boldsymbol{\beta}])|}{\omega(\boldsymbol{\alpha})\omega(\boldsymbol{\beta})} \\
&= \sup_{\mathbf{N}\in\mathbb{Z}_+^n} \max_{\mathbf{0}\leq\boldsymbol{\alpha}\leq\mathbf{N}} \frac{|T([\boldsymbol{\alpha}])([\mathbf{N}-\boldsymbol{\alpha}])|}{\omega(\boldsymbol{\alpha})\omega(\mathbf{N}-\boldsymbol{\alpha})}
\end{aligned}$$

Therefore, if $D : \mathfrak{A}_\omega \to \mathfrak{A}'_\omega$ is a derivation,

$$\begin{aligned}
\|D\| &= \sup_{\mathbf{N}\in\mathbb{Z}_+^n} \max_{\mathbf{0}\leq\boldsymbol{\alpha}\leq\mathbf{N}} \frac{|D([\boldsymbol{\alpha}])([\mathbf{N}-\boldsymbol{\alpha}])|}{\omega(\boldsymbol{\alpha})\omega(\mathbf{N}-\boldsymbol{\alpha})} \\
&= \sup_{\mathbf{N}\in\mathbb{Z}_+^n} \max_{\mathbf{0}\leq\boldsymbol{\alpha}\leq\mathbf{N}} \left| \sum_{j=1}^{n} \frac{\alpha_j}{N_j} \frac{D([N_j e_j])([\mathbf{N}-N_j e_j])}{\omega(\boldsymbol{\alpha})\omega(\mathbf{N}-\boldsymbol{\alpha})} \right|
\end{aligned}$$

We take a momentary detour at this point. Let $g_n : \mathbb{C}^n \to \mathbb{R}_+$ be the function defined as follows:

$$g_n(x) := \max_{S \subseteq \{1,\dots,n\}} \left| \sum_{j \in S} x_j \right| = \max_{\varepsilon \in \{0,1\}^n} \left| \sum_{j=1}^n \varepsilon_j x_j \right| \tag{3.4}$$

Lemma 3.3.2. *Let $x \in \mathbb{C}^n$ and let $t_1, \dots, t_n \in [0,1]$. Then $|\sum_{i=1}^n t_i x_i| \leq g_n(x)$.*

Proof. We use the familiar fact that the n-cube $[0,1]^n \subset \mathbb{R}^n$ is the convex hull of the set $\{0,1\}^n$.

The function

$$h : \begin{cases} \mathbb{R}^n \longrightarrow & \mathbb{R}_+ \\ (s_1, \dots, s_n) \mapsto & \left| \sum_{i=1}^n s_i x_i \right| \end{cases}$$

is continuous and convex: hence for each $\mathbf{s} \in [0,1]^n$ $g(\mathbf{s})$ lies in the convex hull of the set $\{h(\mathbf{d}) \ : \ d_i \in \{0.1\} \ \forall\, i\}$. In particular

$$|h(\mathbf{s})| \leq \max_{\mathbf{d} \in \{0,1\}^n} |h(\mathbf{d})| = g_n(x)$$

as claimed. ∎

Lemma 3.3.3. *g is a norm on $\mathbb{C}^n$ and*

$$\frac{1}{4}\|x\|_1 \leq g_n(x) \leq \|x\|_1$$

for all $x \in \mathbb{C}^n$.

Proof. We first establish the claimed estimates on $g_n(x)$: the upper bound is trivial, and the lower one can be proved as follows. Splitting each x_j into real and imaginary parts and then into positive and negative parts, we find that

$$\|x\|_1 = \sum_j |x_j| \leq \sum_j (\operatorname{Re} x_j)_+ + \sum_j (\operatorname{Re} x_j)_- + \sum_j (\operatorname{Im} x_j)_+ + \sum_j (\operatorname{Im} x_j)_- \ ;$$

a standard trick then shows that $g_n(x)$ is an upper bound for each of the four summands on the RHS, giving us $\|x\|_1 \leq 4g_n(x)$ as claimed.

[To see how this works for the last summand, let I denote the set $\{j \ : \ \operatorname{Im} x_j < 0\} \subseteq \{1, \dots n\}$ and observe that

$$\sum_j (\operatorname{Im} x_j)_- = -\sum_{j \in I} \operatorname{Im} x_j = -\operatorname{Im} \left(\sum_{j \in I} x_j \right) \leq \left| \sum_{j \in I} x_j \right| \leq g_n(x)$$

as required. The other three summands follow similarly.]

In particular, the lower bound on the function g shows that $g_n(x) = 0 \iff x = 0$. Moreover, if we pick $x, y \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ we see that

$$\begin{aligned} g_n(x+y) &\leq \sup_{S \subseteq \{1,\dots,n\}} \left(\left| \sum_{j \in S} x_j \right| + \left| \sum_{k \in S} y_k \right| \right) \\ &\leq \sup_{S \subseteq \{1,\dots,n\}} \left| \sum_{j \in S} x_j \right| + \sup_{T \subseteq \{1,\dots,n\}} \left| \sum_{k \in T} y_k \right| \quad = g_n(x) + g_n(y) \end{aligned}$$

while

$$g_n(\lambda x) = \sup_{S \subseteq \{1,\dots,n\}} \left| \lambda \sum_{j \in S} x_j \right| = |\lambda| g_n(x) \ .$$

Thus g is indeed a norm on $\mathbb{C}^n$. □

Remark. The constant $\frac{1}{4}$ in the lower bound is probably not optimal. On the other hand, for $n \geq 3$ the Banach-Mazur distance between $(\mathbb{C}^n, g)$ and $(\mathbb{C}^n, \|\_\|_1)$ is at least 3, as can be seen by considering the vector $(e^{4\pi i/3}, e^{2\pi i/3}, 1, \dots, 1) \in \mathbb{C}^n$.

Let us return to $\mathfrak{A}_\omega$ and derivations. Recall that for every bounded derivation $D : \mathfrak{A}_\omega \to \mathfrak{A}'_\omega$,

$$\|D\| = \sup_{\mathbf{N} \in \mathbb{Z}_+^n} \omega(\mathbf{N})^{-1} \max_{\mathbf{0} \leq \boldsymbol{\alpha} \leq \mathbf{N}} \left| \sum_{j=1}^n \frac{\alpha_j}{N_j} \frac{\omega(\mathbf{N})}{\omega(\boldsymbol{\alpha})\omega(\mathbf{N}-\boldsymbol{\alpha})} D([N_j e_j])([\mathbf{N} - N_j e_j]) \right| \tag{3.5}$$

Let $\widetilde{D}_{\mathbf{N};j} := D([N_j e_j])([\mathbf{N} - N_j e_j])$.

Proposition 3.3.4 (Norm on $\mathcal{Z}^1(\mathfrak{A}_\omega, \mathfrak{A}'_\omega)$). *There is a contractive linear map*

$$\theta : \bigoplus_{\mathbf{N} \in \mathbb{Z}_+}^{(\infty)} (\mathbb{C}^n; \omega(\mathbf{N})^{-1} g_n) \to \mathcal{Z}^1(\mathfrak{A}_\omega, \mathfrak{A}'_\omega)$$

given by $\theta\psi([\boldsymbol{\alpha}])([\mathbf{N} - \boldsymbol{\alpha}]) := \sum_{j=1}^n \frac{\alpha_j}{N_j} \psi_{\mathbf{N};j}$.

If ω is a product weight, i.e. there exist weights $\omega_1, \dots, \omega_n$ on $\mathbb{Z}_+^n$ such that

$$\omega(\boldsymbol{\alpha}) = \omega_1(\alpha_1) \cdots \omega_n(\alpha_n) \qquad \text{for all } \boldsymbol{\alpha} \in \mathbb{Z}_+^n,$$

then θ is surjective and an isometry.

Proof. Fix $\mathbf{N} \in \mathbb{Z}_+^n$. For each $\boldsymbol{\alpha} \in \mathbb{Z}_+^n$ such that $\boldsymbol{\alpha} \leq \mathbf{N}$,

$$\frac{\alpha_j}{N_j} \frac{\omega(\mathbf{N})}{\omega(\boldsymbol{\alpha})\omega(\mathbf{N}-\boldsymbol{\alpha})} \in [0,1] \qquad \text{for } j = 1, \dots, n$$

and so by Lemma 3.3.2

$$\max_{\mathbf{0}\leq\boldsymbol{\alpha}\leq\mathbf{N}}\left|\sum_{j=1}^n \frac{\alpha_j}{N_j}\frac{\omega(\mathbf{N})}{\omega(\boldsymbol{\alpha})\omega(\mathbf{N}-\boldsymbol{\alpha})}D([N_je_j])([\mathbf{N}-N_je_j])\right| \leq g_n(\widetilde{D}_{\mathbf{N};\bullet})$$

Suppose now that ω is a product weight. Let $S\subseteq\{1,\dots,n\}$ and let $\boldsymbol{\gamma}=\sum_{i\in S}N_ie_i$: then since $\omega=\omega_1\times\ldots\times\omega_n$,

$$\omega(\boldsymbol{\gamma})\omega(\mathbf{N}-\boldsymbol{\gamma})=\prod_{i\in S}\omega_i(N_i)\prod_{j\notin S}\omega_j(N_j)=\prod_{k=1}^n\omega_k(N_k)=\omega(\mathbf{N})$$

and therefore

$$\begin{aligned}\|D\|\geq|D(\boldsymbol{\gamma})(\mathbf{N}-\boldsymbol{\gamma})| &= \left|\sum_{j=1}^n\frac{\gamma_j}{N_j}\frac{D([N_je_j])([\mathbf{N}-N_je_j])}{\omega(\boldsymbol{\gamma})\omega(\mathbf{N}-\boldsymbol{\gamma})}\right| \\ &= \left|\sum_{j\in S}\frac{D([N_je_j])([\mathbf{N}-N_je_j])}{\omega(\mathbf{N})}\right|\end{aligned}$$

Taking the supremum on the RHS over all subsets S yields

$$\|D\|\geq\omega(\mathbf{N})^{-1}g_n(\widetilde{D}_{\mathbf{N};\bullet})$$

as required. □

Corollary 3.3.5. *Let ω be a product weight on $\mathbb{Z}_+^n$. Then there is an isomorphism of Banach spaces*

$$\mathcal{Z}^1(\mathfrak{A}_\omega,\mathfrak{A}'_\omega)\cong\bigoplus_{\mathbf{N}\in\mathbb{Z}_+^n}^{(\infty)}(\mathbb{C}^n;\|\underline{\quad}\|_1)$$

where the isomorphism has distortion constant ≤ 4.

Remark. Note that the identity map gives an isomorphism

$$\bigoplus_{\mathbf{N}\in\mathbb{Z}_+^n}^{(\infty)}(\mathbb{C}^n;\|\underline{\quad}\|_1)\cong\bigoplus_{1\leq i\leq n}^{(1)}\ell^\infty(\mathbb{Z}_+^n)$$

but the distortion constant of this isomorphism grows as $n\to\infty$.

In the case where $\omega\equiv 1$, Proposition 3.3.4 gives us the norm on $\mathcal{Z}^1(\ell^1(\mathbb{Z}_+^n),\ell^1(\mathbb{Z}_+^n)')$. A routine exhaustion argument the yields the following.

Corollary 3.3.6. *Let E denote the completion of c_{00} with respect to the norm g_∞. Then there is an isometric isomorphism of Banach spaces*

$$\theta:\bigoplus_{\mathbf{N}\in\mathbb{Z}_+^\infty}^{(\infty)}E\to\mathcal{Z}^1(\ell^1(\mathbb{Z}_+^\infty),\ell^1(\mathbb{Z}_+^\infty)')$$

Remark. The proof of Lemma 3.3.3 shows that E is isomorphic to ℓ^1 (indeed, that the Banach-Mazur distance between them is at most 4). It follows that $\bigoplus^{(\infty)}_{\mathbf{N}\in\mathbb{Z}_+^\infty} E$ contains a complemented subspace isomorphic to ℓ^1, and hence cannot be isomorphic to ℓ^∞ (since by a classical result of Lindenstrauss all complemented subspaces of ℓ^∞ are isomorphic to ℓ^∞).

In particular, the Hausdorffification of $\mathcal{H}_1(\ell^1(\mathbb{Z}_+^\infty), \ell^1(\mathbb{Z}_+^\infty))$ cannot be isomorphic as a Banach space to ℓ^1 (nor, indeed, to any Banach space predual of ℓ^∞).

Chapter 4

Hochschild (co)homology of $\ell^1(\mathbb{Z}_+^k)$ with symmetric coefficients

Throughout this chapter we let A denote the convolution algebra $\ell^1(\mathbb{Z}_+)$, and write $\mathfrak{A}_k$ for the algebra $A^{\widehat{\otimes} k} \equiv \ell^1(\mathbb{Z}_+^k)$. Our aim is to investigate the Hochschild homology groups $\mathcal{H}_n(A, \ell^1(X))$, where X is a set equipped with an action of the monoid $\mathbb{Z}_+$ and $\ell^1(X)$ is the induced symmetric A-bimodule. We develop methods to calculate these homology groups for certain X, by reducing them to the study of the homological properties of $\ell^1(X)$ as a *one-sided* A-module.

First, we show how one can deduce partial results on the cohomology of $\mathfrak{A}_k$ with symmetric coefficients from knowledge of cohomology of A with symmetric coefficients. These results rely crucially on results from [12], [13]: our approach is to build on the results rather than try to generalise their proofs, and it is here that the technical results of the previous chapter are brought into play.

4.1 Hochschild homology via Tor^A

The following lemma is taken from the proof of [13, Propn 7.3].

Lemma 4.1.1. *Let $q : \mathcal{C}_1(A, A) \to \ell^1(\mathbb{N})$ be the bounded linear map defined by $q(1 \otimes 1) = 0$ and*

$$q(z^k \otimes z^l) = \frac{l}{k+l} z^{k+l} \quad (k, l \in \mathbb{Z}_+; k + l \geq 1).$$

Then q *is surjective and* $\ker(q) = \mathcal{B}_1(A, A)$.

We note that the proof of this in [13] can be shortened slightly: see Appendix C for the details.

Corollary 4.1.2. $\mathcal{H}_1(A, A)$ *is a unit-linked, Banach* A*-module, whose underlying Banach space is isomorphic to* ℓ^1.

Proof. First note that $\mathcal{C}_1(A, A) = \mathcal{Z}_1(A, A)$ (since A is commutative).

By Lemma 4.1.1, $\mathcal{B}_1(A, A)$ is a closed linear subspace of $\mathcal{C}_1(A, A)$ and the quotient space $\mathcal{C}_1(A, A)/\mathcal{B}_1(A, A)$ is isomorphic as a Banach space to $\mathcal{C}_1(A, A)/\ker(q) \cong \ell^1(\mathbb{N})$.

Moreover, $\mathcal{B}_1(A, A)$ is a submodule of the unit-linked A-module $\mathcal{C}_1(A, A)$: hence $\mathcal{H}_1(A, A) = \mathcal{Z}_1(A, A)/\mathcal{B}_1(A, A) = \mathcal{C}_1(A, A)/\mathcal{B}_1(A, A)$ is the quotient of a unit-linked Banach A-module by a closed submodule, and is thus itself a unit-linked Banach A-module as claimed. □

Proposition 4.1.3. *Let* $k \in \mathbb{N}$*; let* $A_1, \dots, A_k$ *denote copies of the Banach algebra* $A = \ell^1(\mathbb{Z}_+)$*, and identify the convolution algebra* $\mathfrak{A}_k = \ell^1(\mathbb{Z}_+^k)$ *with the tensor product* $A_1 \widehat{\otimes} \dots \widehat{\otimes} A_k$.

Then $\mathcal{H}_1(\mathfrak{A}_k, \mathfrak{A}_k)$ *is a symmetric, unit-linked, Banach* $\mathfrak{A}_k$*-bimodule, and we have an isomorphism of Banach* $\mathfrak{A}$*-modules*

$$\mathcal{H}_1(\mathfrak{A}_k, \mathfrak{A}_k) \cong \bigoplus_{i=1}^{k} A_1 \widehat{\otimes} \dots \widehat{\otimes} \mathcal{H}_1(A_i, A_i) \widehat{\otimes} \dots \widehat{\otimes} A_k$$

In particular, the underlying Banach space of $\mathcal{H}_1(\mathfrak{A}_k, \mathfrak{A}_k)$ *is isomorphic to* ℓ^1.

Proof. This is immediate from Corollaries 4.1.2 and 3.2.3. □

Theorem 4.1.4. *Let* N *be a unit-linked, symmetric* A*-bimodule and let* $n \geq 1$*. Then the canonical maps*

$$\mathcal{H}ar\mathcal{H}_n(A, N) \longrightarrow \mathcal{H}_n(A, N)$$

$$\mathcal{H}ar\mathcal{H}^n(A, N) \longrightarrow \mathcal{H}^n(A, N)$$

induce isomorphisms on homology and cohomology respectively. Moreover, there are isomorphisms of seminormed spaces

$$\mathcal{H}_n(A, N) \cong \mathrm{Tor}^A_{n-1}\left[N_R, \mathcal{H}_1(A, A)\right] \cong \mathcal{H}ar\mathcal{H}_n(A, N)$$

$$\mathcal{H}^n(A, N) \cong \mathrm{Ext}_A^{n-1}\left[\mathcal{H}_1(A, A), N_L\right] \cong \mathcal{H}ar\mathcal{H}^n(A, N)$$

Proof. By [13, Propn 7.3] the following facts hold:

- $\mathcal{B}_1(A, A)$ is a closed subspace of $\mathcal{C}_1(A, A)$;
- the Banach space $\mathcal{H}_1(A, A)$ is isomorphic to ℓ^1;
- the chain complex

$$\mathcal{H}_1(A, A) \xleftarrow{q} \mathcal{C}_1(A, A) \xleftarrow{\mathsf{d}_1} \mathcal{C}_2(A, A) \longleftarrow \ldots \tag{4.1}$$

  is an exact sequence of Banach spaces.

We can sharpen this last property slightly. Since $\mathcal{H}_1(A, A)$ is isomorphic as a Banach space to ℓ^1, the lifting property of ℓ^1-spaces with respect to open mappings allows us to find a bounded linear map $\rho_0 : \mathcal{H}_1(A, A) \to \mathcal{C}_1(A, A)$ such that $q\rho = \mathsf{id}$. Then since d_1 surjects onto $\ker(q)$, and since $\mathcal{C}_1(A, A)$ is isomorphic as a Banach space to ℓ^1, the aforementioned lifting property of ℓ^1-spaces allows us to find a bounded linear map $\rho_1 : \mathcal{C}_1(A, A) \to \mathcal{C}_2(A, A)$ such that $\mathsf{d}_1\rho_1 = \mathsf{id} - \rho_0 q$.

$$\begin{array}{ccc} & & \mathcal{C}_2 \\ & \overset{\rho_1}{\nearrow} & \downarrow \mathsf{d}_1 \\ \mathcal{C}_1 & \xrightarrow[\mathsf{id} - \rho_0 q]{} & \ker(q) \end{array}$$

In this way, at each stage using the fact that each $\mathcal{C}_n(A, A)$ is isomorphic to an ℓ^1-space, we may inductively construct bounded linear maps $\rho_n : \mathcal{C}_n(A, A) \to \mathcal{C}_{n+1}(A, A)$ such that $\mathsf{d}_n\rho_n + \rho_{n-1}\mathsf{d}_{n-1} = \mathsf{id}$. Thus the complex (4.1) is not merely exact, but is split exact in Ban.

Now let $\pi = \mathsf{id}\widehat{\otimes}e_\bullet(1) : \mathcal{C}_\bullet(A, A) \to \mathcal{H}ar\mathcal{C}_\bullet(A, A)$ be the BGS projection onto the Harrison summand. π is a chain map, so we have a commuting diagram in ${}_A\mathsf{mod}$:

$$\begin{array}{ccccccccc} 0 \leftarrow & \mathcal{H}_1(A, A) & \xleftarrow{q} & \mathcal{C}_1(A, A) & \xleftarrow{\mathsf{d}_1} & \mathcal{C}_2(A, A) & \xleftarrow{\mathsf{d}_2} & \mathcal{C}_3(A, A) & \leftarrow \ldots \\ & \| & & \| & & \downarrow \pi_2 & & \downarrow \pi_3 & \\ 0 \leftarrow & \mathcal{H}ar\mathcal{H}_1(A, A) & \xleftarrow[q]{} & \mathcal{H}ar\mathcal{C}_1(A, A) & \xleftarrow[\mathsf{d}_1]{} & \mathcal{H}ar\mathcal{C}_2(A, A) & \xleftarrow[\mathsf{d}_2]{} & \mathcal{H}ar\mathcal{C}_3(A, A) & \leftarrow \ldots \end{array}$$

We have already observed that the top row of (4.1) is split exact in Ban. Since π is a *chain projection*, the bottom row is a direct summand of the top row and therefore (by a standard diagram-chase) must itself be split exact in Ban.

Thus both rows are admissible resolutions of $\mathcal{H}_1(A, A)$ *by A-projective Banach modules.* Since π is left inverse to the inclusion chain map $\iota : \mathcal{H}ar\mathcal{C}_*(A, A) \to \mathcal{C}_*(A, A)$, the standard comparison theorem for projective resolutions tells us that $\iota\pi$

is chain homotopic to the identity. Therefore each of the induced chain maps

$$N\underset{A}{\widehat{\otimes}}\mathcal{C}_{*+1}(A,A) \xrightarrow{\mathsf{id}_N\underset{A}{\widehat{\otimes}}\pi} N\underset{A}{\widehat{\otimes}}\mathcal{H}ar\mathcal{C}_{*+1}(A,A)$$

$${}_A\mathrm{Hom}\left(\mathcal{C}_{*+1}(A,A),N\right) \xleftarrow[{}_A\mathrm{Hom}(\pi,N)]{} {}_A\mathrm{Hom}\left(\mathcal{H}ar\mathcal{C}_{*+1}(A,A),N\right)$$

is chain homotopic to the identity, hence induces isomorphism on (co)homology.

Moreover, since Tor and Ext may be calculated using A-projective resolutions in the first variable,

$$H_m\left[N\underset{A}{\widehat{\otimes}}\mathcal{C}_{*+1}(A,A)\right] \cong \operatorname{Tor}_m^A\left[N_R,\mathcal{H}_1(A,A)\right] \cong H_m\left[N\underset{A}{\widehat{\otimes}}\mathcal{H}ar\mathcal{C}_{*+1}(A,A)\right]$$

and

$$H^m\left[{}_A\mathrm{Hom}\left(\mathcal{C}_{*+1}(A,A),N\right)\right] \cong \operatorname{Ext}_A^m\left[\mathcal{H}_1(A,A),N_L\right] \cong H^m\left[{}_A\mathrm{Hom}\left(\mathcal{H}ar\mathcal{C}_{*+1}(A,A),N\right)\right] .$$

Recall from Proposition 1.4.7 and the comments made in Definition 1.5.4 that there are chain isomorphisms

$$\mathcal{C}_*(A,M) \underset{1}{\cong} M\widehat{\otimes}_{A_{\mathsf{un}}}\mathcal{C}_*(A,A_{\mathsf{un}})$$

$$\mathcal{C}^*(A,M) \underset{1}{\cong} {}_{(A_{\mathsf{un}})}\mathrm{Hom}\left(\mathcal{C}_*(A,A_{\mathsf{un}}),M\right)$$

and

$$N\underset{A}{\widehat{\otimes}}\mathcal{H}ar\mathcal{C}_*(A,A) \cong \mathcal{H}ar\mathcal{C}_*(A,N)$$

$${}_A\mathrm{Hom}\left(\mathcal{H}ar\mathcal{C}^*(A,A),N\right) \cong \mathcal{H}ar\mathcal{C}^*(A,N) .$$

Under these chain isomorphisms we identify $\mathsf{id}_N\underset{A}{\widehat{\otimes}}\pi$ with the BGS projection of $\mathcal{C}_*(A,N)$ onto $\mathcal{H}ar\mathcal{C}_*(A,N)$ and identify ${}_A\mathrm{Hom}(\pi,N)$ with the inclusion of $\mathcal{H}ar\mathcal{C}^*(A,N)$ into $\mathcal{C}^*(A,N)$. By the previous remarks both these maps induce isomorphism on (co)homology, and we are done. □

We shall build on this idea slightly to obtain partial results for cohomology of $\mathfrak{A}_k$. Our approach requires some results from the purely algebraic Hochschild homology groups $\mathcal{H}_*^{\mathsf{alg}}(\mathsf{R}_k,\mathsf{R}_k)$, where R_k denotes the polynomial algebra $\mathbb{C}[z_1,\dots,z_k]$.

Theorem 4.1.5 (see [30, 8.8.8]). *Let* $n\geq 2$*. Then* $\mathcal{H}_{i,n-i}^{\mathsf{alg}}(\mathsf{R}_k,\mathsf{R}_k)=0$ *for* $1\leq i\leq n-1$*.*

Informally, the theorem tells us that the simplicial homology of a polynomial algebra is confined to the Lie component. (The proof goes as follows: recall that the Lie component consists precisely of those chains which are alternating; deduce that the Lie component in degree d is a free R_k-module of rank $\binom{k}{d}$; then use the standard Koszul resolution of R_k to compute the 'full' Hochschild homology to find that $\mathcal{H}_d(\mathsf{R}_k, \mathsf{R}_k)$ is also a free R_k-module of rank $\binom{k}{d}$, so that there can be no non-zero homology outside the Lie component.)

We shall use Theorem 4.1.5, combined with analytic results from [12] and [13], to derive the analogous result for the simplicial homology of $\mathfrak{A}_k \cong \ell^1(\mathbb{Z}_+^k)$. To pass between the algebraic and analytic settings we need a good way to approximate simplicial cycles on $\mathfrak{A}_k$ by simplicial cycles on R_k; this is done by establishing a suitable "density lemma" (Lemma 4.1.7 below).

Remark. Note that such a "density lemma" does not follow automatically from the density of R_k in $\mathfrak{A}_k$: recall that if E is a Banach space, V a dense subspace of E and F a closed subspace of E, $V \cap F$ need not be dense in F (for example, take $E = \mathbb{R}^2$, $V = \mathbb{Q}^2$ and F to be a line through the origin with irrational gradient.)

Identify R_k with the dense subalgebra of $\mathfrak{A}_k$ spanned by polynomials. The inclusion homomorphism $\mathsf{R}_k \hookrightarrow \mathfrak{A}_k$ yields an inclusion of chain complexes

$$\begin{array}{ccccc}
\ldots \longleftarrow & \mathcal{C}_{n-1}(\mathfrak{A}_k, \mathfrak{A}_k) & \leftarrow & \mathcal{C}_n(\mathfrak{A}_k, \mathfrak{A}_k) & \leftarrow \ldots \\
& \uparrow & & \uparrow & \\
\ldots \longleftarrow & \mathcal{C}^{\mathsf{alg}}_{n-1}(\mathsf{R}_k, \mathsf{R}_k) & \leftarrow & \mathcal{C}^{\mathsf{alg}}_n(\mathsf{R}_k, \mathsf{R}_k) & \leftarrow \ldots
\end{array}$$

and, identifying $\mathcal{C}_n(A_k, A_k)$ with $\ell^1(\mathbb{Z}_+^k \times \ldots \times \mathbb{Z}_+^k)$, we see that $\mathcal{C}^{\mathsf{alg}}_n(\mathsf{R}_k, \mathsf{R}_k)$ is dense in $\mathcal{C}_n(\mathfrak{A}_k, \mathfrak{A}_k)$ for each n.

We use multi-index notation, so that monomials in R_k are written as z^α rather than $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

Definition 4.1.6. A monomial chain in $\mathcal{C}^{\mathsf{alg}}_n(\mathsf{R}_k.\mathsf{R}_k)$ is just a tensor of the form

$$x = z^{\alpha(0)} \otimes z^{\alpha(1)} \otimes \ldots \otimes z^{\alpha(n)}$$

where $\alpha(0), \alpha(1), \ldots, \alpha(n) \in \mathbb{Z}_+^k$. The total degree of x is the k-tuple $\alpha(0) + \alpha(1) + \ldots + \alpha(n)$, and is denoted by $\deg(x)$.

Given $N \in \mathbb{Z}_+^k$, we let $\pi_n^N : \mathcal{C}_n(\mathfrak{A}_k, \mathfrak{A}_k) \to \mathcal{C}_n(\mathfrak{A}_k, \mathfrak{A}_k)$ denote the norm-1 projection onto the closed linear span of the monomial chains with total degree N. More precisely, we define π_n^N on monomial chains by

$$\pi_n^N(x) := \left\{ \begin{array}{ll} x & \text{if } \deg(x) = N \\ 0 & \text{otherwise} \end{array} \right\}$$

and extend by linearity and continuity.

It is clear from this explicit definition that π_n^N commutes with the action of $\mathsf{id}\otimes S_n$ on $\mathcal{C}_n(\mathfrak{A}_k, \mathfrak{A}_k)$, and hence commutes with each of the BGS idempotents $(e_n(i))_{i=1}^n$.

Claim: π_*^N is a chain map.

Proof. We need to show that $\pi_n^N \mathsf{d}_n = \mathsf{d}_n \pi_{n+1}^N$ where $\mathsf{d}_n : \mathcal{C}_{n+1}(\mathfrak{A}_k, \mathfrak{A}_k) \to \mathcal{C}_n(\mathfrak{A}_k, \mathfrak{A}_k)$ is the Hochschild boundary map. Since $\mathsf{d}_n = \sum_{i\geq 0}(-1)^i \partial_i^n$ is the alternating sum of face maps, it suffices to show that $\pi_n^N \partial_i^n = \partial_i^n \pi_{n+1}^N$ for each i. But this is immediate once we observe that each face map $\partial_i^n : \mathcal{C}_{n+1}(\mathfrak{A}_k, \mathfrak{A}_k) \to \mathcal{C}_n(\mathfrak{A}_k, \mathfrak{A}_k)$ preserves the total degree of monomial chains. □

Observe also that for given $N \in \mathbb{Z}_+^k$ and $n \in \mathbb{N}$, there are only finitely many monomial n-chains of degree N; hence the range of π_n^N is contained in $\mathcal{C}_n^{\mathsf{alg}}(\mathfrak{A}_k, \mathfrak{A}_k)$.

For $m \in \mathbb{N}$, we may define a chain projection $P_*^m : \mathcal{C}_*(\mathfrak{A}_k, \mathfrak{A}_k) \to \mathcal{C}_*(\mathfrak{A}_k, \mathfrak{A}_k)$ by

$$P_n^m := \sum_{N\in\mathbb{Z}_+^k\,:\,|N|\leq m} \pi_n^N$$

From the comments above, we see that P_n^m takes values in $\mathcal{C}_n^{\mathsf{alg}}(\mathsf{R}_k, \mathsf{R}_k)$ and that for every n-chain x

$$P_n^m x \to x \quad \text{as } m \to \infty\ ;$$

moreover, P^m commutes with the BGS projections.

We now have everything in place for the following technical lemma.

Lemma 4.1.7 (Density lemma). *Let $1 \leq i \leq n$ and let $x \in \mathcal{Z}_{i,n-i}(\mathfrak{A}_k, \mathfrak{A}_k)$. Then for every $\varepsilon > 0$ there exists $y \in \mathcal{Z}_{i,n-i}^{\mathsf{alg}}(\mathsf{R}_k, \mathsf{R}_k)$ with $\|x - y\| \leq \varepsilon$.*

Proof. We know that $P_n^m(x) \to x$ as $m \to \infty$. Choose M such that $\|P_n^M(x) - x\| \leq \varepsilon$ and let $y := P_n^M(x) \in \mathcal{C}_n^{\mathsf{alg}}(\mathsf{R}_k, \mathsf{R}_k)$. Since P^M is a chain map,

$$\mathsf{d}y = \mathsf{d}P_n^M(x) = P_{n-1}^M \mathsf{d}(x) = 0$$

and thus $y \in \mathcal{Z}_n^{\mathsf{alg}}(\mathsf{R}_k, \mathsf{R}_k)$. Finally,

$$e_n(i)y = e_n(i)P_n^M(x) = P_n^M e_n(i)(x) = P_n^M(x) = y$$

and thus y has BGS type $(i, n-i)$ as required. □

Proposition 4.1.8 (Simplicial homology confined to Lie component). *Let $n \geq 2$ and let $1 \leq i \leq n-1$. Then $\mathcal{H}_{i,n-i}(\mathfrak{A}_k, \mathfrak{A}_k) = 0$.*

Proof. By [13, Thm 7.5] we know that the boundary maps on the Hochschild chain complex $\mathcal{C}_*(\mathfrak{A}_k, \mathfrak{A}_k)$ are open mappings. Let C be the constant of openness of the boundary map $\mathsf{d}_n : \mathcal{C}_{n+1}(\mathfrak{A}_k, \mathfrak{A}_k) \to \mathcal{C}_n(\mathfrak{A}_k, \mathfrak{A}_k)$.

Fix $\varepsilon \in (0, 1)$ and let $x \in \mathcal{Z}_{i,n-i}(\mathfrak{A}_k, \mathfrak{A}_k)$.

Claim: there exists $\gamma \in \mathcal{C}^{\mathsf{alg}}_{i,n+1-i}(\mathsf{R}_k, \mathsf{R}_k)$ with $\|\gamma\| \leq C(1 + \varepsilon)^2\|x\|$ and $\|x - \mathsf{d}\gamma\| \leq \varepsilon\|x\|$.

Assuming that the claim holds, a standard inductive approximation argument may be used to produce $u \in \mathcal{C}^{\mathsf{alg}}_{i,n+1-i}(\mathsf{R}_k, \mathsf{R}_k)$ with $\|u\| \leq (1 - \varepsilon)^{-1}(1 + \varepsilon)^2C\|x\|$ and $\mathsf{d}u = x$; in particular $x \in \mathcal{B}_{i,n-i}(\mathfrak{A}_k, \mathfrak{A}_k)$. Since x was an arbitrary cycle of type $(i, n - i)$, this shows that $\mathcal{Z}_{i,n-i}(\mathfrak{A}_k, \mathfrak{A}_k) = \mathcal{B}_{i,n-i}(\mathfrak{A}_k, \mathfrak{A}_k)$ as required.

It therefore suffices to prove that we can find such a γ, which we do as follows. By our density lemma 4.1.7 we know there exists $y \in \mathcal{Z}^{\mathsf{alg}}_{i,n-i}(\mathsf{R}_k, \mathsf{R}_k)$ with $\|x - y\| \leq \varepsilon\|x\|$. By Theorem 4.1.5, $y = \mathsf{d}w$ for some $(n + 1)$-chain w on R_k. Regard w as an element of $\mathcal{C}_{n+1}(\mathfrak{A}_k, \mathfrak{A}_k)$: since d_n is open with constant C there exists an $(n + 1)$-chain γ on $\mathfrak{A}_k$ such that $\mathsf{d}\gamma = \mathsf{d}w = y$ and $\|\gamma\| \leq C(1 + \varepsilon)\|y\| \leq C(1 + \varepsilon)^2\|x\|$. This proves our claim and hence concludes the proof of the theorem. □

Lemma 4.1.9. *Let B, C be unital Banach algebras, let M be a left Banach $B\widehat{\otimes}C$-module, let X be a left Banach C-module and let M_C be the left Banach C-module obtained by letting C act via the homomorphism $C \to B\widehat{\otimes}C, c \mapsto 1_B\otimes c$.*

Then for each n,

$$\mathrm{Ext}^n_{B\widehat{\otimes}C}(B\widehat{\otimes}X, M) \cong \mathrm{Ext}^n_C(X, M_C)$$

Proof. Let $0 \leftarrow X \leftarrow P_\bullet$ be the standard "bar resolution" of X by left C-projective modules (see [19, Propn 2.9]). Recall that this complex is split exact in Ban; so by functoriality of $B\widehat{\otimes}\_\_ : \mathsf{Ban} \to {}_B\mathsf{unmod}$, the complex $0 \leftarrow B\widehat{\otimes}X \leftarrow B\widehat{\otimes}P_\bullet$ is an admissible complex of Banach B-modules and module maps. Moreover, since B is B-projective, $B\widehat{\otimes}P_n$ is $B\widehat{\otimes}C$-projective for every n. Thus $B\widehat{\otimes}P_\bullet$ is an admissible $B\widehat{\otimes}C$-projective resolution of $B\widehat{\otimes}X$: since Ext may be calculated using any admissible projective resolution of the first variable,

$$\begin{aligned}\mathrm{Ext}^n_{B\widehat{\otimes}C}(B\widehat{\otimes}X, M) &\cong H^n\left[{}_{B\widehat{\otimes}C}\mathrm{Hom}(B\widehat{\otimes}P_\bullet, M)\right] \\ &= H^n\left[{}_C\mathrm{Hom}(P_\bullet, M)\right] \qquad \cong \mathrm{Ext}^n_C(X, M_C)\end{aligned}$$

as claimed. □

As in the statement of 4.1.3, let us identify $\mathfrak{A}_k$ with the k-fold tensor product $A_1\widehat{\otimes}\ldots\widehat{\otimes}A_k$, where each A_i denotes a copy of the Banach algebra A.

Theorem 4.1.10. *Let M be a unit-linked symmetric $\mathfrak{A}_k$-bimodule. For each $i = 1,\ldots,k$ there is an induced A_i-bimodule structure on M; let us denote the resulting symmetric A_i-bimodule by M_i. Then for $n \geq 1$,*

$$\mathcal{H}ar\mathcal{H}^n(\mathfrak{A}_k, M) \cong \bigoplus_{i=1}^{k} \mathrm{Ext}^{n-1}_{A_i}(\mathcal{H}_1(A_i,A_i), M_i) \cong \bigoplus_{i=1}^{k} \mathcal{H}^n(A_i, M_i)$$

Proof. The second isomorphism follows from 4.1.4, so we need only verify the first one. This is done using Proposition 3.1.1, following a procedure very similar to that in the proof of Theorem 4.1.4.

Consider the Hochschild chain complex $\mathcal{C}_\bullet(\mathfrak{A}_k,\mathfrak{A}_k)$. By Proposition 4.1.8 all the homology has to live in the Lie component of the Hodge decomposition: in particular, the Harrison summand

$$\mathcal{H}ar\mathcal{C}_1(\mathfrak{A}_k,\mathfrak{A}_k) \xleftarrow{\mathsf{d}_1} \mathcal{H}ar\mathcal{C}_2(\mathfrak{A}_k,\mathfrak{A}_k) \xleftarrow{\mathsf{d}_2} \ldots$$

is an exact sequence in Ban. The cokernel of d_1 is $\mathcal{H}ar\mathcal{H}_1(\mathfrak{A}_k,\mathfrak{A}_k)$ and by Proposition 4.1.3 we know this is a Banach space isomorphic to ℓ^1. Hence

$$0 \longleftarrow \mathcal{H}ar\mathcal{H}_1(\mathfrak{A}_k,\mathfrak{A}_k) \xleftarrow{q} \mathcal{H}ar\mathcal{C}_1(\mathfrak{A}_k,\mathfrak{A}_k) \xleftarrow{\mathsf{d}_1} \ldots$$

is an exact sequence in Ban with every term isomorphic to a complemented subspace of ℓ^1: the lifting property of such spaces with respect to surjective linear maps now allows us to inductively construct a *splitting in* Ban for this exact sequence.

Thus the conditions of Proposition 3.1.1 are satisfied, and using that proposition we obtain an isomorphism of seminormed spaces

$$\mathcal{H}ar\mathcal{H}^n(\mathfrak{A}_k, M) \cong \mathrm{Ext}^{n-1}_{\mathfrak{A}_k}(\mathcal{H}_1(\mathfrak{A}_k,\mathfrak{A}_k), M) \ .$$

By Proposition 4.1.3

$$\mathcal{H}_1(\mathfrak{A}_k,\mathfrak{A}_k) \cong \bigoplus_{i=1}^{k} A_1\widehat{\otimes}\ldots\widehat{\otimes}\mathcal{H}_1(A_i,A_i)\widehat{\otimes}\ldots\widehat{\otimes}A_k \quad ;$$

so using Lemma 4.1.9 we deduce that for each i

$$\mathrm{Ext}^{n-1}_{\mathfrak{A}_k}\left[\left(\widehat{\bigotimes}_{j\neq i}A_j\right)\widehat{\otimes}\mathcal{H}_1(A_i,A_i), M\right] \cong \mathrm{Ext}^{n-1}_{A_i}(\mathcal{H}_1(A_i,A_i), M_i) \ .$$

This implies that

$$\mathcal{H}ar\mathcal{H}^n(\mathfrak{A}_k, M) \cong \bigoplus_{i=1}^{k} \mathrm{Ext}_{A_i}^{n-1}(\mathcal{H}_1(A_i, A_i), M_i)$$

and our proof is complete. □

Remark. The proof of Theorem 4.1.10 can be easily modified to yield a parallel result for Harrison *homology* of $\mathfrak{A}_k$, as follows: using the same notation as above, we have

$$\mathcal{H}ar\mathcal{H}_n(\mathfrak{A}_k, M) \cong \bigoplus_{i=1}^{k} \mathrm{Tor}_{n-1}^{A_i}(\mathcal{H}_1(A_i, A_i), M_i) \cong \bigoplus_{i=1}^{k} \mathcal{H}_n(A_i, M_i)$$

for all $n \geq 1$. We omit the details.

4.2 ℓ^1-homology for $\mathbb{Z}_+$-sets

For the rest of this chapter we shall denote the Banach A-module $\mathcal{H}_1(A, A)$ by Ω_A. This is done purely to make the formulas that follow more readable.

Definition 4.2.1. Let Γ be a $\mathbb{Z}_+$-set (i.e. a set equipped with an action of the monoid $\mathbb{Z}_+$). The Banach space $\ell^1(\Gamma)$ then has a natural A-module structure, defined on the standard basis vectors of $\ell^1(\Gamma)$ by

$$z^n \cdot e_x := e_{z^n \cdot x} \qquad (n \in \mathbb{Z}_+; x \in \Gamma)$$

We denote the resulting A-module by $\mathsf{F}(\Gamma)$.

We can regard $\mathsf{F}(\Gamma)$ as a symmetric A-bimodule $\mathsf{F}(\Gamma)_s$, and so obtain a plentiful supply of coefficient modules to which Theorem 4.1.4 applies. In this section and the next we shall develop some techniques that allow us to determine the Hochschild cohomology in several cases.

By Theorem 4.1.4, $\mathcal{H}_n(A, \mathsf{F}(\Gamma)_s) \cong \mathrm{Tor}_{n-1}^{A}(\Omega_A, \mathsf{F}(\Gamma))$ for all $n \geq 1$; therefore one can attempt to calculate Hochschild homology of the *bimodule* $\mathsf{F}(\Gamma)_s$ using the homological properties of $\mathsf{F}(\Gamma)$ as a *1-sided* A-module.

4.2.1 Surgery arguments

Definition 4.2.2 (Notation). We define a pre-order on Γ by

$$x \preceq y \quad \Longleftrightarrow \quad y = z^n \cdot x \text{ for some } n \in \mathbb{Z}_+$$

Given $x \in \Gamma$ we let $\mathsf{prec}(x) := \{w \ : \ w \preceq x\}$ and $\mathsf{succ}(x) := \{y \ : \ y \succeq x\}$.

Definition 4.2.3. Let Γ be a $\mathbb{Z}_+$-set. A loop in Γ is a non-empty $\mathbb{Z}_+$-subset C with the following property: for each $x, y \in C$ we have $x \preceq y \preceq x$.

Definition 4.2.4. Let Γ be a $\mathbb{Z}_+$-set and let $x \in \Gamma$.

- x is a source if $\mathsf{prec}(x) = \{x\}$, i.e. there exists no $w \in \Gamma$ such that $z \cdot w = x$.
- x is a sink if $z \cdot x = x$.
- x is a junction if there exist distinct points $w_1, w_2 \in \Gamma$ such that $z \cdot w_1 = x = z \cdot w_2$.

Definition 4.2.5 (Connected components). Let Γ be a $\mathbb{Z}_+$-set. We define an equivalence relation on Γ by

$$x \sim y \iff \text{there exist } m, n \in \mathbb{Z}_+ \text{ such that } z^m x = z^n y$$

The equivalence classes under this relation are each $\mathbb{Z}_+$-subsets of Γ; we call them the connected components of Γ. A $\mathbb{Z}_+$-set is said to be connected if it has only one connected component.

The following lemma appears to be well-known to specialists: we have been unable to find a suitable reference, and so state and prove the result explicitly for sake of completeness.

Lemma 4.2.6. *Let $\mathbb{Z}_+$ act on $\mathbb{Z}$ in the canonical way. Then $\mathsf{F}(\mathbb{Z})$ is A-flat (on left or right) with constant 1.*

Proof. It suffices to find a contractive $\mathbb{Z}_+$-module map $\Lambda : (A \widehat{\otimes} \mathsf{F}(\mathbb{Z}))' \to \mathsf{F}(\mathbb{Z})'$ which is left inverse to the natural map $\mu' : \mathsf{F}(\mathbb{Z})' \to (A \widehat{\otimes} \mathsf{F}(\mathbb{Z}))'$.

Identify $\mathsf{F}(\mathbb{Z})'$ with $\ell^\infty(\mathbb{Z})$ and $(A \widehat{\otimes} \mathsf{F}(\mathbb{Z}))'$ with $\ell^\infty(\mathbb{Z}_+ \times \mathbb{Z})$. For each $N \in \mathbb{N}$, define a linear contraction $\Lambda_N : \ell^\infty(\mathbb{Z}_+ \times \mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ by

$$(\Lambda_N \psi)_n := \frac{1}{N+1} \sum_{j=0}^{N} \psi_{j,n-j} \qquad (n \in \mathbb{Z}) .$$

The sequence $(\Lambda_N)_{N \geq 1}$ has a weak*-cluster point in the closed unit ball of $\mathcal{L}(\ell^\infty(\mathbb{Z}_+ \times \mathbb{Z}), \ell^\infty(\mathbb{Z}))$ (by compactness of this unit ball in its natural weak*-topology). That is, there exists a linear contraction $\Lambda : \ell^\infty(\mathbb{Z}_+ \times \mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ such that

$$\lim_{N \to \infty} \Lambda_N(\psi)_n = \Lambda(\psi)_n \qquad \text{for all } \psi \in \ell^\infty(\mathbb{Z}_+ \times \mathbb{Z}),\ n \in \mathbb{Z}.$$

Now for each $n \in \mathbb{Z}$

$$\begin{aligned}\Lambda_N(z\cdot\psi)_n &= \frac{1}{N+1}\sum_{j=0}^{N}(z\cdot\psi)_{j,n-j} \\ &= \frac{1}{N+1}\sum_{j=1}^{N}\psi_{j-1,n-j} \qquad = \frac{1}{N+1}\sum_{k=0}^{N-1}\psi_{k,n-1-k}\end{aligned}$$

while

$$[z\cdot(\Lambda_N\psi)]_n = (\Lambda\psi)_{n-1} = \frac{1}{N+1}\sum_{k=0}^{N}\psi_{k,n-1-k} \quad ;$$

hence

$$\begin{aligned}|[\Lambda(z\cdot\psi)]_n - [z\cdot(\Lambda\psi)]_n| &= \left|\lim_{N\to\infty}\left([\Lambda_N(z\cdot\psi)]_n - [z\cdot(\Lambda_N\psi)]_n\right)\right| \\ &\leq \limsup_N \frac{1}{N+1}\|\psi\|_\infty \qquad\qquad = 0\ .\end{aligned}$$

Thus $\Lambda(z\cdot\psi)_n = z\cdot(\Lambda\psi)_n$ for all $n\in\mathbb{Z}$; since ψ was arbitrary this shows that Λ is an A-module map. It remains to observe that for any $\xi\in\ell^\infty(\mathbb{Z})$

$$\Lambda\mu'(\xi)_n = \lim_{N\to\infty}\Lambda_N\mu'(\xi)_n = \lim_{N\to\infty}\xi_n = \xi_n \quad (n\in\mathbb{Z}).$$

Thus $\Lambda_N\mu' = \mathsf{id}$ and our proof is complete. ☐

Corollary 4.2.7. *Let L be any right $\ell^1(\mathbb{Z})$-module whose dual is isomorphic as a Banach space to ℓ^∞. Regard L as an A-module via the inclusion homomorphism $A \hookrightarrow \ell^1(\mathbb{Z})$; then L is A-flat.*

Proof. For this proof we shall denote the Banach algebra $\ell^1(\mathbb{Z})$ by C. Let

$$0\to X\to Y\to Z\to 0$$

be an admissible short exact sequence in ${}_A\mathsf{mod}$: it suffices to show that the sequence

$$0\to L\widehat{\otimes}_A X\to L\widehat{\otimes}_A Y\to L\widehat{\otimes}_A Z\to 0 \tag{4.2}$$

is also short exact.

Since C is A-flat on the right, the sequence

$$0\to C\widehat{\otimes}_A X\to C\widehat{\otimes}_A Y\to C\widehat{\otimes}_A Z\to 0 \tag{4.3}$$

is a short exact sequence of Banach spaces.

Claim: L is *strictly* flat as a right Banach C-module.

Assume for the moment that the claim holds. Then applying $L\hat{\otimes}_C\_\_$ to the short exact sequence (4.3) we deduce that the following sequence is exact:

$$0 \to L\hat{\otimes}_C\left(C\hat{\otimes}_A X\right) \to L\hat{\otimes}_C\left(C\hat{\otimes}_A Y\right) \to L\hat{\otimes}_C\left(C\hat{\otimes}_A Z\right) \to 0 \quad .$$

But this sequence coincides with (4.2), so we conclude that (4.2) is exact as required.

It therefore suffices to prove the claim. We do this in two stages.

(1) Since C is amenable L' is automatically C-injective, i.e. the canonical inclusion $\iota : L' \to \mathfrak{L}(C, L')$ given by $\iota(\psi)(b) = \psi \cdot b$ has a left inverse C-module map.

(2) Since L' is isomorphic as a Banach space to ℓ^∞, $\mathfrak{L}(C, L')$ is complemented as a C-module in $N := \mathfrak{L}(C, \ell^1(\mathbb{B}_L))$, where $\mathbb{B}_L$ denotes the closed unit ball of L.

Hence L' is C-module-complemented in N; but by [31, Propn 3.1] N is strictly C-injective. Therefore L' is also strictly C-injective, which implies by [31, Propn 4.10] that L is strictly C-flat. This proves the claim and so completes the proof of Corollary 4.2.7. □

Thus for such L, $\mathrm{Tor}_n^A(\_\_, L) = 0$ for all $n \geq 1$.

Lemma 4.2.8. *Let M be a left Banach $\ell^1(\mathbb{Z})$-module. Then $\mathrm{Tor}_n^A(\Omega_A, M) = 0$ for all n.*

Proof. Since M is an $\ell^1(\mathbb{Z})$-module, it is A-flat by Corollary 4.2.7. Hence

$$\mathrm{Tor}_n^A(\Omega_A, M) = 0 \qquad \text{for all } n \geq 1$$

and

$$\mathrm{Tor}_0^A(\Omega_A, M) \cong \Omega_A \hat{\otimes}_A M = (\Omega_A \hat{\otimes} M) / N$$

where $N = \overline{\mathrm{lin}}\{(x \cdot a) \otimes m - x \otimes (a \cdot m) \ : \ x \in \Omega_A, a \in A, m \in M\}$.

By Corollary 4.1.2 we may identify Ω_A as a Banach space with $\ell^1(\mathbb{N})$, with the (right) module action given by

$$e_n \cdot z := \frac{n}{n+1} e_{n+1}$$

Fix $n \in \mathbb{N}$ and $m \in M$. Recalling that M is an $\ell^1(\mathbb{Z})$-module: for any $k \in \mathbb{N}$

$$e_n \underset{A}{\otimes} m = (e_n \cdot z^k) \underset{A}{\otimes} z^{-k} m = \frac{n}{n+k} e_{n+k} \underset{A}{\otimes} z^{-k} m$$

so that $\|e_n \underset{A}{\otimes} m\| \leq \frac{n}{n+k} \|e_{n+k}\| \|z^{-k} m\| = \frac{n}{n+k} \|m\| \to 0$ as $k \to \infty$. Hence $e_n \underset{A}{\otimes} m = 0$ for all $n \in \mathbb{N}$ and $m \in M$, and since the e_n span a dense linear subspace of Ω_A we deduce that $\Omega_A \hat{\otimes}_A M = 0$ as required. □

Adding tails to sources

Let $\mathfrak{T}$ denote the quotient module $\mathsf{F}(\mathbb{Z})/\mathsf{F}(\mathbb{Z}_+)$ (which as a Banach space may be identified with $\ell^1(\mathbb{Z} \setminus \mathbb{Z}_+)$. Note that the natural short exact sequence

$$0 \leftarrow \mathfrak{T} \leftarrow \mathsf{F}(\mathbb{Z}) \leftarrow \mathsf{F}(\mathbb{Z}_+) \leftarrow 0 \tag{4.4}$$

splits in Ban (since there is an obvious contractive linear projection from $\mathsf{F}(\mathbb{Z})$ onto the subspace $\mathsf{F}(\mathbb{Z}_+)$).

Lemma 4.2.9. *Let Y be a Banach space: then*

1. $\operatorname{Tor}_0^A(\Omega_A, \mathfrak{T}\widehat{\otimes}Y) = 0$
2. $\operatorname{Tor}_1^A(\Omega_A, \mathfrak{T}\widehat{\otimes}Y) \cong \operatorname{Tor}_0(\Omega_A, \mathsf{F}(\mathbb{Z}_+)\widehat{\otimes}Y) \cong \Omega_A\widehat{\otimes}Y$
3. $\operatorname{Tor}_m^A(\_\_, \mathfrak{T}\widehat{\otimes}Y) = 0$ *for all* $m \geq 2$.

Proof. Tensoring (4.4) with Y gives an admissible short exact sequence in ${}_A\mathsf{mod}$

$$0 \leftarrow \mathfrak{T}\widehat{\otimes}Y \leftarrow \mathsf{F}(\mathbb{Z})\widehat{\otimes}Y) \leftarrow \mathsf{F}(\mathbb{Z}_+)\widehat{\otimes}Y \leftarrow 0 \tag{4.5}$$

where both $\mathsf{F}(\mathbb{Z})\widehat{\otimes}Y$ and $\mathsf{F}(\mathbb{Z}_+)\widehat{\otimes}Y$ are A-flat. For each right Banach A-module M, we may apply the long exact sequence for $\operatorname{Tor}_n^A(M, \_\_)$ to the short exact sequence (4.5) to obtain $\operatorname{Tor}_n^A(M, \mathfrak{T}\widehat{\otimes}Y) = 0$ for $n \geq 2$, and an exact sequence

$$\begin{aligned} 0 \leftarrow \operatorname{Tor}_0^A(M, \mathfrak{T}\widehat{\otimes}Y) \leftarrow \operatorname{Tor}_0^A(M, \mathsf{F}(\mathbb{Z})\widehat{\otimes}Y) \\ \leftarrow \operatorname{Tor}_0^A(M, \mathsf{F}(\mathbb{Z}_+)\widehat{\otimes}Y) \leftarrow \operatorname{Tor}_1^A(M, \mathfrak{T}\widehat{\otimes}Y) \leftarrow 0 \end{aligned}$$

The rest now follows from Lemma 4.2.8. ☐

Let Γ_0 be the set of sources in Γ, and let Γ_T be the set

$$\Gamma \sqcup \coprod_{\alpha \in \Gamma_0} (\mathbb{Z} \setminus \mathbb{Z}_+)$$

equipped with the obvious $\mathbb{Z}_+$-action which extends that on Γ.

We wish to compare $\mathsf{F}(\Gamma_T)$ with $\mathsf{F}(\Gamma)$. The embedding of Γ as a $\mathbb{Z}_+$-subset of Γ_T induces an embedding $\mathbf{J} : \mathsf{F}(\Gamma)$ into $\mathsf{F}(\Gamma_T)$, and clearly there are isomorphisms of A-modules

$$\operatorname{coker}(\mathbf{J}) \cong \bigoplus_{\alpha \in \Gamma_0}^{(1)} \mathsf{F}(\mathbb{Z})/\mathsf{F}(\mathbb{Z}_+) \cong \mathfrak{T}\widehat{\otimes}\ell^1(\Gamma_0)$$

Applying Lemma 4.2.9 gives us the following result.

Proposition 4.2.10 (Adding tails to sources). *If $n \geq 2$ then for each right A-module M, the canonical map $\operatorname{Tor}_n^A(M, \mathsf{F}(\Gamma)) \to \operatorname{Tor}_n^A(M, \mathsf{F}(\Gamma_T))$ is an isomorphism of seminormed spaces.*

Moreover, there is a short exact sequence

$$0 \leftarrow \operatorname{Tor}_0^A(\Omega_A, \mathsf{F}(\Gamma_T)) \leftarrow \operatorname{Tor}_0^A(\Omega_A, \mathsf{F}(\Gamma)) \leftarrow \Omega_A \widehat{\otimes} \ell^1(\Gamma_0) \leftarrow \operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma_T)) \leftarrow \operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma)) \leftarrow 0$$

Loops and sinks

Let Γ be a $\mathbb{Z}_+$-set, and suppose it contains at least one loop. Let $\Lambda = \{C_i\}$ denote the set of all loops in Γ.

Define Γ_* to be the $\mathbb{Z}_+$-set obtained by identifying every C_i to a point $\{\omega_i\}$. The quotient map $\Gamma \to \Gamma_*$ induces a quotient map of A-modules

$$\mathsf{F}(\Gamma) \xrightarrow{q} \mathsf{F}(\Gamma_*)$$

Let $C := \coprod_{i \in \Lambda} C_i \subseteq \Gamma$ and let $\Omega := \coprod_{i \in \Lambda} \{\omega_i\} \subseteq \Gamma_*$. Let ε be the "augmentation map" $\mathsf{F}(C) \xrightarrow{\varepsilon} \mathsf{F}(\Omega)$: then we have a commuting diagram

$$\begin{array}{ccccc}
\ker(q) & \longrightarrow & \mathsf{F}(\Gamma) & \xrightarrow{q} & \mathsf{F}(\Gamma_*) \\
\uparrow \iota & & \uparrow & & \uparrow \\
\ker(\varepsilon) & \longrightarrow & \mathsf{F}(C) & \xrightarrow[\varepsilon]{} & \mathsf{F}(\Omega)
\end{array}$$

where all arrows are A-module maps, both rows are exact and the vertical arrows are inclusion maps. It is easy to check that ι is an isomorphism.

C is a disjoint union of loops, so the A-module action on $\mathsf{F}(C)$ extends to an action of $\ell^1(\mathbb{Z})$; the A-module action on $\mathsf{F}(\Omega)$ is just the trivial action, so ε is then an $\ell^1(\mathbb{Z})$-module map. Therefore $\ker(\varepsilon)$ is also a $\ell^1(\mathbb{Z})$-module. Furthermore, this $\ell^1(\mathbb{Z})$-action extends the original A-action on $\ker(\varepsilon)$, so by Corollary 4.2.7 $\ker(\varepsilon)$ is A-flat.

Therefore $\ker(q)$ is A-flat, and applying the long exact sequence for Tor_*^A we deduce the following:

Proposition 4.2.11.

$$\operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma)) \cong \operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma_*)) \qquad \textit{for all } n \geq 2$$
$$\operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma)) \cong \operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma_*))$$

Sinks and outlets

Recall that if B is a unital algebra and I is any ideal in B with bounded approximate identity, then I is B-flat. (We have not found this mentioned explicitly in [19] but it follows immediately once we recall that B is B-flat and there is a B-module retract of B' onto I'.)

Suppose now that Γ_* is a $\mathbb{Z}_+$-set with a sink; let Ω be the set of all sinks.

Let Γ_{out} be the $\mathbb{Z}_+$-set obtained by gluing an outlet onto each sink in Γ. There is a quotient map of $\mathbb{Z}_+$-sets $\Gamma_{\mathsf{out}} \to \Gamma_*$, and arguing as above we can identify the kernel of the A-module map $q : \mathsf{F}(\Gamma_{\mathsf{out}}) \to \mathsf{F}(\Gamma_*)$ with an ℓ^1-sum

$$\bigoplus\nolimits^{(1)}_{\omega\in\Omega} I_1$$

where I_1 denotes the augmentation ideal in A. I_1 is known to have a bounded approximate identity: take the sequence $(u_n)_{n\geq 1}$, for example, where

$$u_n(z) = 1 - \frac{1}{n}\frac{1-z^n}{1-z} \qquad (z \in \mathbb{T}).$$

Hence I_1 is A-flat, and so $\ker(q)$ is also A-flat. Applying the long exact sequence for Tor^A_* to the short exact sequence

$$0 \to I_1 \to \mathsf{F}(\Gamma_{\mathsf{out}}) \to \mathsf{F}(\Gamma)$$

we arrive at the following.

Proposition 4.2.12. *There are isomorphisms*

$$\mathrm{Tor}^A_n(\_\_, \mathsf{F}(\Gamma_{\mathsf{out}})) \cong \mathrm{Tor}^A_n(\_\_, \mathsf{F}(\Gamma_*)) \qquad \textit{for } n \geq 2$$

$$\mathrm{Tor}^A_1(\Omega_A, \mathsf{F}(\Gamma_{\mathsf{out}})) \cong \mathrm{Tor}^A_1(\Omega_A, \mathsf{F}(\Gamma_*))\ .$$

and a short exact sequence

$$0 \to \Omega_A \mathop{\widehat{\otimes}}_A I_1 \to \mathrm{Tor}^A_0(\Omega_A, \mathsf{F}(\Gamma_{\mathsf{out}})) \to \mathrm{Tor}^A_0(\Omega_A, \mathsf{F}(\Gamma_*)) \to 0$$

4.2.2 Resolutions of some $\mathbb{Z}_+$-sets

To make the proofs of this section a little more readable we shall introduce some *ad hoc* terminology, which should hopefully not cause confusion for the reader.

Definition 4.2.13. Let Γ be a $\mathbb{Z}_+$-set. A cover of Γ is a pair (Ξ, q), where Ξ is a $\mathbb{Z}_+$-set and q is a quotient map of $\mathbb{Z}_+$-sets from Ξ onto Γ.

The kernel of the cover (Ξ, q) is defined to be the kernel of the induced A-module map $\mathsf{F}(\Xi) \to \mathsf{F}(\Gamma)$ (we shall often abuse notation and denote this map also by q).

We say that (Ξ, q) is a free cover if Ξ is a free $\mathbb{Z}_+$-set, i.e. one of the form $\mathbb{Z}_+ \times \mathbb{I}$, and a parallel cover if Ξ of the form $(\mathbb{Z}_+ \times \mathbb{I}) \sqcup (\mathbb{Z} \times \mathbb{J})$, where the action of $\mathbb{Z}_+$ is defined by translation along each copy of $\mathbb{Z}_+$ or $\mathbb{Z}$ (we allow either $\mathbb{I}$ or $\mathbb{J}$ to be empty).

By a Banach resolution of Γ we mean a resolution of the Banach A-module $\mathsf{F}(\Gamma)$ in the category ${}_A\mathsf{mod}$, i.e. a long exact sequence

$$0 \leftarrow \mathsf{F}(\Gamma) \leftarrow C_0 \leftarrow C_1 \leftarrow \dots$$

in ${}_A\mathsf{mod}$. Note that since $\mathsf{F}(\Gamma)$ has ℓ^1 as its underlying Banach space, the lifting property of ℓ^1 with respect to open mappings implies that any Banach resolution of Γ is a split exact complex in Ban, hence is an *admissible* resolution of $\mathsf{F}(\Gamma)$ in ${}_A\mathsf{mod}$.

Lemma 4.2.14. *Let $\mathbb{I}$ be an index set and let $(\Delta_i)_{i\in\mathbb{I}}$ be a family of $\mathbb{Z}_+$-sets. For each i let $q_i : \Xi_i \to \Delta_i$ be a covering map of $\mathbb{Z}_+$sets, with kernel K_i.*

Define Ξ to be the disjoint union $\coprod_{i\in\mathbb{I}} \Xi_i$ and let q be the coproduct of the maps q_i. Then (Ξ, q) is a cover of $\coprod_{i\in\mathbb{I}} \Delta_i$, and its kernel may be identified (up to isometric isomorphism of A-modules) with the ℓ^1-sum $\overset{(1)}{\bigoplus}_{i\in\mathbb{I}} K_i$.

The proof is clear and we omit the details.

The case of finitely many junctions

Definition 4.2.15. We say that a $\mathbb{Z}_+$-set Γ is basic if the following conditions are satisfied:

(a) Γ contains no loops;

(b) Γ contains no infinite tails, i.e. $\mathsf{prec}(x)$ is finite for each x;

(c) Γ contains only finitely many junctions.

Note that each connected component of a basic $\mathbb{Z}_+$-set is itself basic: thus in trying to find Banach resolutions of a basic $\mathbb{Z}_+$-set, it suffices to look for Banach resolutions of each connected component.

Theorem 4.2.16. *Let Γ be a connected, basic $\mathbb{Z}_+$-set with N junctions. Then Γ has a parallel cover whose kernel is projective with constant 2^{4N+1}. Moreover, if Γ has no infinite tails, then we may take our parallel cover to be a free cover.*

It should be possible to prove Theorem 4.2.16 directly (and with better constants) by taking the obvious parallel cover and analysing its kernel more carefully.

However, since the details and notation look messy, we have chosen an inductive argument instead. For the inductive step, it is convenient to isolate the following lemma.

Lemma 4.2.17. *Let B be a Banach algebra and suppose we have Banach B-modules K, J, P such that*

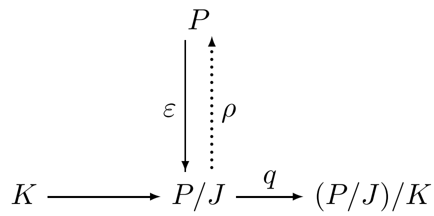
(i) J is a closed submodule of P, and is projective with constant C_J;

(ii) K is a closed, submodule of P/J, and is projective with constant C_K;

(iii) there exists a bounded linear map $\rho : P/J \to P$ which is right inverse to the quotient map.

Let ε denote the quotient map $P \to P/J$ and q the quotient map $P/J \to (P/J)/K$. Then the kernel of $q\varepsilon$ is linearly complemented in P and is projective with constant $(2 + C_K\|\rho\|)^2 \max(C_P, C_K)$.

Remark. It may help to picture the setup of this lemma using the following diagram:



The purely algebraic version of this lemma constructs the pullback C of K and P over P/J, shows that it is isomorphic to the desired kernel, and then observes that $C \cong J \oplus K$ so is certainly projective. (See, for example, [30, Propn 9.3.3].) What follows is merely an explicit hack through the construction to keep track of the norm estimates.

Proof. For notational convenience let $L := P/J$, $M := (P/J)/K$, and $\widetilde{K} := \ker(q\varepsilon)$. Let

$$D := \{(p,k) \in P \overset{\infty}{\oplus} K \ : \ \varepsilon(p) = k\} \ .$$

Clearly D is a closed submodule of $P \overset{\infty}{\oplus} K$; and since ε is surjective

$$\begin{aligned} D &= \{(p,\varepsilon(p)) \in P \overset{\infty}{\oplus} L \ : \ \varepsilon(p) \in K\} \\ &= \{(p,\varepsilon(p)) \in P \overset{\infty}{\oplus} L \ : \ q\varepsilon(p) = 0\} \\ &= \{(p,\varepsilon(p)) \in P \overset{\infty}{\oplus} L \ : \ p \in \widetilde{K}\} \end{aligned}$$

Let $\pi_P : P \overset{\infty}{\oplus} L \to P$ and $\pi_K : P \overset{\infty}{\oplus} L \to K$ be the coordinate projections (which are both contractive, open B-module maps), and let $\widetilde{\pi}_P$, $\widetilde{\pi}_K$ be their restrictions to D.

$$\begin{array}{ccccc} D & \overset{\widetilde{\pi}_P}{\dashrightarrow} & P & \overset{q\varepsilon}{\dashrightarrow} & M \\ {\scriptstyle \widetilde{\pi}_K}\vdots & & \downarrow{\scriptstyle \varepsilon} & & \| \\ K & \longrightarrow & L & \overset{q}{\longrightarrow} & M \end{array}$$

$\widetilde{\pi}_P$ is injective and has range $\widetilde{K}$; moreover it is an isometry (since ε is contractive). Thus D is isometrically isomorphic as a B-module to $\widetilde{K}$, and it suffices to show that D is projective with the appropriate constant.

By assumption there is a bounded linear map $\rho : L \to P$ satisfying $\varepsilon\rho(y) = y$ for all $y \in L$. In particular, $\widetilde{\pi}_K$ is a linearly split surjection of B-modules and so by projectivity of K there exists a B-module map $\sigma : K \to D$ such that $\|\sigma\| \leq C_K\|\rho\|$ and $\widetilde{\pi}_K\sigma = \mathsf{id}_K$.

$$\begin{array}{ccccc} D & \overset{\widetilde{\pi}_P}{\longrightarrow} & P & \overset{q\varepsilon}{\dashrightarrow} & M \\ {\scriptstyle \widetilde{\pi}_K}\downarrow \ \vdots{\scriptstyle \sigma} & & \downarrow{\scriptstyle \varepsilon} & & \| \\ K & \longrightarrow & L & \overset{q}{\dashrightarrow} & M \end{array}$$

Let $\sigma_P = \widetilde{\pi}_P\sigma : K \to P$. We may write σ in the form (σ_P, id), i.e. $\sigma(k) = (\sigma_P(k), k)$. Since σ takes values in K we must have $\varepsilon\sigma_P(k) = k$ for all $k \in K$.

Define $\theta : D \to J \overset{1}{\oplus} K$ and $\phi : J \overset{1}{\oplus} K \to D$ by

$$\theta(p,k) = (p + \sigma_P k, k) \qquad ; \qquad \phi(x,y) = (x - \sigma_P y, y)$$

and note that both θ and ϕ are B-module maps with norm $\leq 2 + \|\sigma\|$. Moreover, it is easily checked that θ and ϕ are mutually inverse maps, so that D is isomorphic as a B-module to $J \overset{1}{\oplus} K$ with distortion constant bounded above by

$$\|\theta\|\|\phi\| \leq (2 + \|\sigma\|)^2 \leq (2 + C_K\|\rho\|)^2 \ .$$

Since $J \overset{1}{\oplus} K$ is projective with constant $\leq \max(C_P, C_K)$ the result follows. $\square$

Proof of Theorem 4.2.16. We use induction on N. The case $N = 0$ is trivial, since in that case Γ is just a copy of $\mathbb{Z}_+$ or $\mathbb{Z}$ and we may choose our parallel cover to be (Γ, id); the kernel of this cover is $\{0\}$, which is trivially projective with constant ≤ 2.

Let $N \geq 1$, and suppose the theorem holds true for all connected basic sets with at most $N-1$ junctions.

Note that since Γ has no loops, the preorder $\preceq$ is actually a partial order, i.e. if $x, y \in \Gamma$ and $x \preceq y \preceq x$ then $x = y$. It follows, since there are finitely many junctions in Γ, that there exists a junction γ_0 such that no other element of $\mathsf{succ}(\gamma_0)$ is a junction.

We claim that $\Gamma = \mathsf{prec}(\gamma_0) \cup \mathsf{succ}(\gamma_0)$. To see this, let $x \in \Gamma$: since Γ is connected there exist $m, n \in \mathbb{Z}_+$ such that $z^m \cdot x = z^n \cdot \gamma_0 = w$, say; now if both m and n are non-zero then w would be a junction in $\mathsf{succ}(\gamma_0) \setminus \{\gamma_0\}$, which contradicts the maximal property of γ_0; hence either $x \preceq \gamma_0$ or $\gamma_0 \preceq x$.

Note also that since Γ has no loops, $\mathsf{succ}(\gamma_0)$ is a $\mathbb{Z}_+$-subset of Γ isomorphic to $\mathbb{Z}_+$.

Let $\Lambda = \{\gamma \in \Gamma \ : \ z \cdot \gamma = \gamma_0\}$: for each $\alpha \in \Lambda$ let S_α be a copy of $\mathbb{N}$ and let $\Gamma_\alpha := \mathsf{prec}(\alpha) \sqcup S_\alpha$. There is a covering map of $\mathbb{Z}_+$-sets $q : \widetilde{\Gamma} \to \Gamma$, obtained by identifying all the copies of S_α with $\mathsf{succ}(\gamma_0)$ (to see that q is surjective, recall that $\Gamma = \mathsf{prec}(\gamma_0) \cup \mathsf{succ}(\gamma_0)$.

Let K be the kernel of the induced map $q : \mathsf{F}(\widetilde{\Gamma}) \to \mathsf{F}(\Gamma)$.

Claim: K is projective with constant 2.

Proof of claim. This is done by constructing an explicit isomorphism from a free A-module onto K. We observe that if $x \in K$ then x is supported on $\coprod_{\alpha \in \Lambda} S_\alpha$. Moreover, for any fixed $\lambda \in \Lambda$,

$$x = \sum_{n \in \mathbb{N}, \alpha \in \Lambda} x_{n,\alpha} z^n \cdot e_\alpha = \sum_{n \in \mathbb{N}, \alpha \in \Lambda \setminus \{\lambda\}} x_{n,\alpha} \left(z^n \cdot (e_\alpha - e_\lambda) \right) \tag{4.6}$$

Fix $\lambda \in \Lambda$; observe that $\Lambda \setminus \{\lambda\}$ is non-empty. Let $\mathcal{F} := \overset{(1)}{\bigoplus}_{\alpha \in \Lambda \setminus \{\lambda\}} \ell^1(\mathbb{N})$ and define $\theta : \mathcal{F} \to \overset{(1)}{\bigoplus}_{\alpha \in \Lambda} \mathsf{F}(\Gamma_\alpha)$ by

$$\theta(e_{n,\alpha}) := z^n \cdot (e_\alpha - e_\lambda)$$

Clearly θ is an A-module map with norm 2; by Equation (4.6), for any $x \in K$ there exists a unique $u \in \mathcal{F}$ such that $\theta(u) = x$, which moreover satisfies $\|u\| = \|x\|$. Hence

K is 2-isomorphic as a Banach A-module to $\mathcal{F}$. Since $\mathcal{F}$ is clearly projective with constant 1, K is projective with constant 2 as claimed. ☐

Now we construct a parallel cover of $\widetilde{\Gamma} = \coprod_\alpha \Gamma_\alpha$. Each Γ_α is a connected, basic $\mathbb{Z}_+$-set with at most $N-1$ junctions; hence by the inductive hypothesis, Γ_α has a parallel cover $(\Xi_\alpha, \varepsilon_\alpha)$ whose kernel K_α is projective with constant $\leq 2^{4N-3}$. Moreover, if Γ has no infinite tails then neither do any of the Γ_α, and so we can assume (by the inductive hypothesis) that each Ξ_α is a free $\mathbb{Z}_+$-set.

Let $\Xi := \coprod_{\alpha\in\Lambda} \Xi_\alpha$, $\widetilde{\Gamma} := \coprod_{\alpha\in\Lambda} \Gamma_\alpha$ and let ε be the coproduct of the maps (ε_α). Clearly (Ξ, ε) is a parallel cover of $\widetilde{\Gamma}$, and so $(\Xi, q\varepsilon)$ is a parallel cover of Γ; if Γ has no infinite tails then Ξ may be chosen to be a free cover.

It remains to show that the kernel of $(\Xi, q\varepsilon)$ is projective with the appropriate constant. Consider the diagram

$$\begin{array}{ccccc} & & \mathsf{F}(\Xi) & & \\ & & \big\downarrow{\scriptstyle \varepsilon} & & \\ K & \longrightarrow & \mathsf{F}(\widetilde{\Gamma}) & \xrightarrow{\ q\ } & \mathsf{F}(\Gamma) \end{array}$$

By Lemma 4.2.14, $\ker(\varepsilon)$ may be identified with the ℓ^1-sum $\overset{(1)}{\bigoplus}_\alpha K_\alpha$, and is therefore projective with constant $\leq 2^{4N-3}$. We saw earlier that K is projective with constant 2, so by Lemma 4.2.17 $\ker q\varepsilon$ is projective with constant

$$(2+2)^2 \max(2^{4N-3}, 2) = 2^4 \cdot 2^{4N-3} = 2^{4N+1}$$

and the inductive step is complete. ☐

Corollary 4.2.18. *If Γ is a basic $\mathbb{Z}_+$-set then $\operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma)) = 0$ for all $n \geq 2$.*

Proof. Write Γ as the disjoint union of its connected components $(\Delta_i)_{i\in\mathbb{I}}$, and let N_i be the number of junctions in Δ_i. Note that if N is the number of junctions in Γ, then $\sup_i N_i \leq \sum_i N_i = N < \infty$.

For each $i \in \mathbb{I}$ Δ_i is a connected basic $\mathbb{Z}_+$-set, and hence by Theorem 4.2.16 has a parallel cover $q_i : \Xi_i \to \Delta_i$ whose kernel K_i is A-projective with constant 2^{4N_i+1}. Therefore by Lemma 4.2.14 we have a short exact sequence of A-modules:

$$0 \to \overset{(1)}{\bigoplus}_{i\in\mathbb{I}} K_i \to \mathsf{F}(\Xi) \xrightarrow{\ q\ } \mathsf{F}(\Gamma) \tag{4.7}$$

where $\Xi := \coprod_{i\in\mathbb{I}} \Xi_i$. The ℓ^1-sum of the K_i is A-projective with constant

$$\sup_i 2^{4N_i+1} = 2 \cdot 16^{\sup_i N_i} \leq 2 \cdot 16^N < \infty$$

and since Ξ is a parallel cover $\mathsf{F}(\Xi)$ is A-flat. Therefore, for any right Banach A-module M we may apply the long exact sequence for $\operatorname{Tor}_*^A(M, \_\_)$ to the short exact sequence (4.7) to get $\operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma)) = 0$ for all $n \geq 2$. □

Combining these results for basic sets with our earlier surgery techniques yields the following:

Theorem 4.2.19. *Let Γ be a $\mathbb{Z}_+$-set with only finitely many junctions; let Γ_* denote the $\mathbb{Z}_+$-set obtained by contracting each loop in Γ to a sink; and let Γ_{out} denote the $\mathbb{Z}_+$-set obtained by adjoining an outlet to each sink in Γ_*. Then*

(i) $\operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma)) \cong \operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma_{\mathsf{out}})) = 0$ *for all* $n \geq 2$*;*

(ii) $\operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma)) \cong \operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma_{\mathsf{out}}))$

(iii) there is a short exact sequence of seminormed spaces

$$0 \to \Omega_A \mathop{\widehat{\otimes}}_A I_1 \to \operatorname{Tor}_0^A(\Omega_A, \mathsf{F}(\Gamma_{\mathsf{out}})) \to \operatorname{Tor}_0^A(\Omega_A, \mathsf{F}(\Gamma)) \to 0$$

Proof. Let $n \geq 2$. By Propositions 4.2.11 and Proposition 4.2.12,

$$\operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma)) \cong \operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma_*)) \cong \operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma_{\mathsf{out}})) .$$

But since Γ_{out} is a basic $\mathbb{Z}_+$-set, Corollary 4.2.18 tells us that $\operatorname{Tor}_n^A(\_\_, \mathsf{F}(\Gamma_{\mathsf{out}})) = 0$, and *(i)* is proved.

Similarly: by Proposition 4.2.11 there are isomorphisms of seminormed spaces

$$\operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma)) \cong \operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma_*))$$
$$\operatorname{Tor}_0^A(\Omega_A, \mathsf{F}(\Gamma)) \cong \operatorname{Tor}_0^A(\Omega_A, \mathsf{F}(\Gamma_*))$$

and by Proposition 4.2.12 there is an isomorphism of seminormed spaces

$$\operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma_{\mathsf{out}})) \cong \operatorname{Tor}_1^A(\Omega_A, \mathsf{F}(\Gamma_*)) .$$

and a short exact sequence

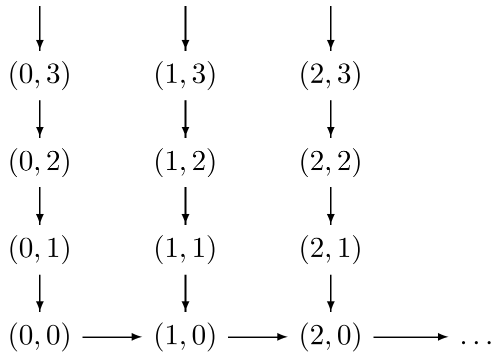
$$0 \to \Omega_A \mathop{\widehat{\otimes}}_A I_1 \to \operatorname{Tor}_0^A(\Omega_A, \mathsf{F}(\Gamma_{\mathsf{out}})) \to \operatorname{Tor}_0^A(\Omega_A, \mathsf{F}(\Gamma_*)) \to 0$$

This yields *(ii)* and *(iii)* as required. □

4.2.3 An example with infinitely many junctions

It is natural to wonder what happens if we consider connected $\mathbb{Z}_+$-sets where there are infinitely many junctions. We have not been able in this thesis to solve the general case, but for particular examples one can do better. We shall consider only one such example to simplify the discussion.

Our example is the $\mathbb{Z}_+$-set T which has underlying set $\mathbb{Z}_+^2$ and action as shown by the following diagram:



Although T has infinitely many junctions, we shall show that it has a parallel cover with projective kernel.

Let $F_0 := \coprod_{i\in\mathbb{Z}_+}\mathbb{Z}$ and define $\varepsilon : F_0 \to T$ by

$$\varepsilon : e_{i,j} \longmapsto \begin{cases} (i,-j) \text{ if } j \leq -1 \\ (i+j,0) \text{ if } j \geq 0 \end{cases}$$

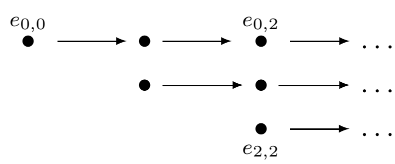
(recall that $z \cdot e_{i,j} = e_{i,j+1}$). It is clear that (F_0, ε) is a parallel cover of T. Consider $K := \ker(\mathsf{F}(F_0) \to \mathsf{F}(T))$.

Proposition 4.2.20. *K is a free A-module: more precisely, if we let F_1 be the free $\mathbb{Z}_+$ set $\coprod_{n\in\mathbb{N}}\mathbb{Z}_+$ then there is an isomorphism of Banach A-modules from $\mathsf{F}(F_1)$ onto K.*

Proof. Let $\iota : \mathsf{F}(F_1) \to \mathsf{F}(F_0)$ be defined by

$$\iota(e_{n,j}) = e_{0,n+j} - e_{n,j} \qquad (j \in \mathbb{Z}_+, n \in \mathbb{N}) \tag{4.8}$$

and observe that ι is a bounded linear A-module map.



Moreover, a quick calculation shows that $\varepsilon_* \iota = 0$, so that $\operatorname{Ran}(\iota) \subseteq K$. To show that ι is a linear isomorphism onto K it therefore suffices to show that $K \subseteq \operatorname{Ran}(\iota)$.

We can describe the elements of K explicitly, as follows: if $\mathbf{a} = (a_{i,j})_{i\in\mathbb{Z}_+ j\in\mathbb{Z}} \in \mathsf{F}(F_0)$ then

$$\mathbf{a} \in K \iff \begin{cases} a_{i,j} = 0 & \forall\, i \in \mathbb{Z}_+ \ \forall\, j \leq -1 \\ \displaystyle\sum_{m=0}^{j} a_{m,j-m} = 0 & \forall\, j \geq 0 \end{cases}$$

This allows us to construct a linear projection $\pi : \mathsf{F}(F_0) \to K$, defined by

$$\pi(e_{i,j}) := \begin{cases} 0 \text{ if } j < 0 \\ e_{i,j} - e_{0,i+j} \text{ if } j \geq 0 \end{cases} \qquad \text{for } i \in \mathbb{Z}_+,\ j \in \mathbb{Z}.$$

(Clearly $\operatorname{Ran}(\pi) \subseteq K$; it is easily checked that π fixes each element of K. Note that π is **not** an A-module map.)

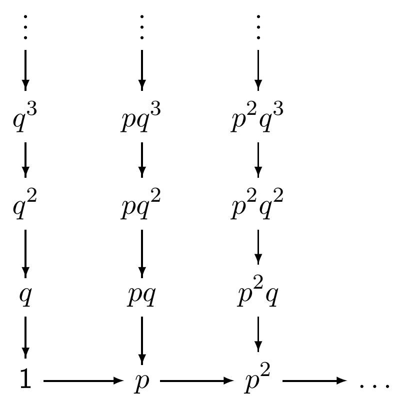
Going back to Equation (4.8), we deduce that

$$\pi(e_{i,j}) = \begin{cases} 0 \text{ if } j < 0 \text{ or } i = 0 \\ \iota(e_{i,j}) \text{ if } j \geq 0 \text{ and } i \geq 1 \end{cases}$$

Hence $\pi(e_{i,j}) \in \operatorname{Ran}(\iota)$ for all i, j; since π surjects onto K, $K \subseteq \operatorname{Ran}(\iota)$ as claimed. □

Remark. The $\mathbb{Z}_+$set T arises from considering the bicyclic semigroup $\mathcal{S}$, which is the monoid defined by the presentation $\mathcal{S} = \langle 1, p, q | qp = 1 \rangle$. More precisely, we regard $\mathfrak{A} = \ell^1(\mathcal{S})$ as a *right A-module* where the generator of $\mathbb{Z}_+$ acts as right multiplication by $p \in \mathfrak{A}$.

A little thought shows that with this A-module structure $\mathfrak{A} \cong \mathsf{F}(T)$; the easiest way to see this is to consider the diagram below.



4.3 Return to Hochschild homology

We close this chapter by briefly recasting the results of the previous sections in terms of Hochschild homology.

Recall that by Theorem 4.1.4,

$$\mathcal{H}_n(A, \mathsf{F}(\Gamma)_s) \cong \operatorname{Tor}^A_{n-1}(\Omega_A, \mathsf{F}(\Gamma)) \qquad \text{for all } n \geq 1$$

Therefore the results of Section 4.2 give us the following:

Proposition 4.3.1. *Let Γ be a basic $\mathbb{Z}_+$-set. Then $\mathcal{H}_n(A, \mathsf{F}(\Gamma)_s) = 0$ for all $n \geq 3$.*

Unfortunately I have not been able to resolve the following natural question:

Question 4.3.2. Is there a $\mathbb{Z}_+$-set Γ such that $\mathcal{H}_3(A, \mathsf{F}(\Gamma)_s) \neq 0$?

As discussed at the start of this chapter we may obtain some partial results for the multi-variable case by applying "soft" methods to our results for the one-variable case. Let X be a set equipped with an action of the monoid $\mathbb{Z}_+^k$, and let $\mathsf{F}(X)$ be the associated $\mathfrak{A}_k$-module with underlying Banach space $\ell^1(X)$.

Proposition 4.3.3. *For each $i \in \{1, \ldots, k\}$, let X_i be the $\mathbb{Z}_+$-set obtained from X by letting z act on X as z_i.*

If each X_i is a basic set, then $\mathcal{H}ar\mathcal{H}_n(\mathfrak{A}_k, \mathsf{F}(X)_s) = 0$ for all $n \geq 3$.

Proof. By the remark after Theorem 4.1.10

$$\mathcal{H}ar\mathcal{H}_n(\mathfrak{A}_k, \mathsf{F}(X)_s) \cong \bigoplus_{i=1}^{k} \operatorname{Tor}^{n-1}_{A_i}(\mathcal{H}_1(A_i, A_i), \mathsf{F}(X_i))$$

□

Chapter 5

Simplicial homology of semilattices of algebras

This chapter concerns a class of Banach algebras that are, loosely, obtained by taking a family of Banach algebras and gluing them together along the points of a semilattice. The prototypical examples are the ℓ^1-convolution algebras of Clifford semigroups.

Most of the chapter will be devoted to stating and proving some general results on the simplicial cohomology groups of such Banach algebras. After doing this we shall present some applications to the ℓ^1-convolution algebras of certain Clifford semigroups.

Remark. It follows as a special case of our results that if L is a semilattice then the simplicial homology groups $\mathcal{H}_*(\ell^1(L), \ell^1(L))$ vanish in degrees 1 and above. This result was originally proved using a simpler version of the main theorem and can be found in the author's article [5]; although that paper is superseded by the results of this chapter, it may be the case that the reader finds the simpler version more transparent.

In what follows, the reader is advised to focus on special cases of the general definitions, to get a better feel for what is going on. Our results are stated in quite a general setting because this seems the best way to isolate the features which make the main proofs work; a side-effect is that the notation is rather off-putting.

5.1 Definitions and preliminaries

Definition 5.1.1. A semilattice is a commutative semigroup each of whose elements is idempotent.

Each semilattice has a canonical partial order on it, defined by

$$e \preceq f \iff ef = e$$

and so may be regarded as a poset in a natural way.

The objects of study in this chapter are "semilattice-shaped diagrams" in the category BAlg_1^+ of unital Banach algebras and contractive unital homomorphisms. An algebraist's definition (following the language of [25], say) might be as follows:

Provisional definition. A strong semilattice of Banach algebras, or a semilattice in BAlg_1^+, is a pair (L, A) where L is a semilattice and A is a functor from the small category $(L, \succeq)$ to the category BAlg_1^+.

(To understand this definition one needs to have seen the interpretation of a poset as a special kind of small category. This can be found in [25, §1.2] for example.)

If the reader is happy with this provisional definition, it should be clear that the "proper" definition to follow is merely an unpacking of the terminology in the "provisional" definition just given.

Definition 5.1.2. A semilattice in BAlg_1^+ consists of: a semilattice L; a family $(A_e)_{e\in L}$ of unital Banach algebras; and *contractive, unital* algebra homomorphisms $\phi_{f,e} : A_e \to A_f$ for each pair $(e, f) \in L \times L$ such that $e \succeq f$, which satisfy the following compatibility conditions:

- $\phi_{e,e}$ is the identity homomorphism on A_e ;
- if $e \succeq f \succeq g$ in L then $\phi_{g,f} \circ \phi_{f,e} = \phi_{g,e}$.

We shall use the following notational shorthand: the expression "let $(L, A) \in \mathbf{SA}$" is henceforth used as an abbreviation for the phrase "let (L, A) be a semilattice in BAlg_1^+", although on occasion we shall revert to the longer phrase in order to state certain results.

Remark. Some remarks:

(a) The reader may wish to think of such a pair (L, A) as a "projective system of Banach algebras" indexed by the partially ordered set $(L, \succeq)$. However, such terminology does not seem apposite, as we shall consider neither the inductive nor projective limit of such a system, and since we later use properties of L that are not shared by arbitrary posets.

Our terminology – indeed, our actual definition – is instead modelled on the standard semigroup-theoretic notion of a "strong semilattice of semigroups", as presented in [20, Ch. IV] for example.

(b) Since A is a functor with respect to the reverse ordering on L, it may be viewed as a functor $(L, \preceq)^{\mathsf{op}} \to \mathsf{BAlg}_1^+$. In more high-powered language this latter formulation exhibits A as a *presheaf in* BAlg_1^+; we shall briefly discuss at the end of the chapter how such a formulation suggests possible generalisations of the results presented here.

(c) The terminology "$(L, A) \in \mathbf{SA}$" strongly suggests that we might view strong semilattices of Banach algebras as objects of a category $\mathbf{SA}$. While this is easy and not at all deep, we feel that such extra abstraction would not make the results of this chapter any easier to follow.

Given $(L, A) \in \mathbf{SA}$, we can endow the ℓ^1-sum $\overset{(1)}{\bigoplus}_{e \in L} A_e$ with a multiplication that turns it into a Banach algebra. Before doing so, we shall introduce some basic notation regarding ℓ^1-sums: the point of our pedantry will be explained in due course.

Definition 5.1.3 (Notation). Let $\mathbb{I}$ be an indexing set and let $(X_i)_{i \in \mathbb{I}}$ be a family of Banach spaces. We write $\mathfrak{X}$ for the ℓ^1-sum $\overset{(1)}{\bigoplus}_{i \in \mathbb{I}} X_i$, and for each $j \in \mathbb{I}$ we let ι_j^X denote the canonical inclusion map $X_j \hookrightarrow \mathfrak{X}$.

If $x \in \coprod_{j \in \mathbb{I}} X_j$ then we shall sometimes write $\operatorname{base}(x)$ for the unique $i \in \mathbb{I}$ such that $x \in X_i$.

Definition 5.1.4. Let $(L, A) \in \mathbf{SA}$. We can form a Banach algebra $\mathfrak{alg}$, whose underlying vector space is the ℓ^1-sum $\overset{(1)}{\bigoplus}_{e \in L} A_e$ and whose multiplication is defined by the following rule:

$$\left(\sum_{e \in L} \iota_e^A a_e\right) \cdot \left(\sum_{f \in L} \iota_f^A b_f\right) := \sum_{g \in L} \left[\sum_{(e,f) \in L \times L \,:\, ef = g} \phi_{g,e}(a_e) \phi_{g,f}(b_f) \right]$$

We say that $\mathfrak{alg}$ is the $\mathsf{convolution\ algebra\ of}$ (L, A). If we need to make the dependence on A and L explicit we write $\mathfrak{alg}_{L,A}$.

Note that although each A_e is unital, $\mathfrak{alg}$ need not be.

Remark. There is canonical, isometric embedding of the vector space $\ell^1(L)$ into $\bigoplus^{(1)}_{e\in L} A_e$, defined by $e \mapsto \iota_e(1_{A_e})$. The way that we have defined multiplication in the convolution algebra $\mathfrak{alg}_{L,A}$ ensures that this isometric embedding is an algebra homomorphism of the convolution algebra $\ell^1(L)$ *into the centre of* $\mathfrak{alg}_{L,A}$. We shall on occasion identify $\ell^1(L)$ with its image under this algebra embedding.

Example 5.1.5 (Key examples). The following special cases of our construction should be kept in mind, and hopefully serve to clarify the general picture.

(a) Fix a unital Banach algebra $\mathfrak{B}$; let $A_e = \mathfrak{B}$ for all $e \in L$; and let each transition map $\phi_{f,e}$ be the identity map on $\mathfrak{B}$. Then there is an isomorphism $\ell^1(L)\hat{\otimes}\mathfrak{B} \to \mathfrak{alg}_{L,A}$, defined by sending $e\hat{\otimes}b$ to $\iota_e b$ for every $e \in L$ and $b \in \mathfrak{B}$.
In particular, if $\mathfrak{B} = \mathbb{C}$ then $\mathfrak{alg} \cong \ell^1(L)$, the usual convolution algebra of the semigroup L.

(b) At the other extreme, suppose that $L = \{1\}$, the trivial semilattice containing only one element. Then clearly $\mathfrak{alg}_{L,A} \cong A_1$.

(c) Let $L = \{1, e\}$ be the two-element semilattice consisting of an identity element 1 and an idempotent e distinct from 1: then $1 \succeq e$. Let $\mathfrak{B}$ be any unital Banach algebra and define the functor $A : L \to \mathsf{BAlg}_1^+$ by taking $A_1 = \mathbb{C}$, $A_e = \mathfrak{B}$ and $\phi_{e,1} : \mathbb{C} \to \mathfrak{B}$ to be the homomorphism $\lambda \mapsto \lambda 1_{\mathfrak{B}}$.

$$\begin{array}{ccc} \mathbb{C} & & \mathfrak{B} \\ 1 & \longrightarrow & e \end{array}$$

Then $\mathfrak{alg}_{L,A}$ is by definition isomorphic as a Banach space to the ℓ^1-sum $\mathbb{C} \oplus \mathfrak{B}$; and the multiplication on $\mathfrak{alg}_{L,A}$ is given by

$$(\lambda, b) \cdot (\mu, c) = (\lambda\mu, \lambda c + \mu b + bc) \qquad (\lambda, \mu \in \mathbb{C}; b, c \in \mathfrak{B}) .$$

Thus in this instance $\mathfrak{alg}_{L,A}$ is nothing but the forced unitisation of $\mathfrak{B}$.

The following example provides the main motivation for our definition of the convolution algebra. (The relevant definition and properties of Clifford semigroupscan be found in most reference texts on semigroups: see [20, §IV.2] for instance.)

Example 5.1.6 (ℓ^1-algebras of Clifford semigroups). Let L be a semilattice and $G_\bullet$ a functor from $(L, \succeq)$ to the category of groups. Composing this with the ℓ^1-group algebra functor gives a functor $\ell^1(G_\bullet)$ from $(L, \succeq)$ to BAlg_1^+. The convolution algebra of $(L, \ell^1(G_\bullet))$ is then just the ℓ^1-semigroup algebra of the Clifford semigroup $\mathcal{G} = \coprod_{e\in L} G_e$.

Later we shall try to determine some of the cohomology groups for convolution algebras of the form $\mathfrak{alg}_{L,A}$. To make things clearer we adopt the following notation.

Definition 5.1.7 (Notation). Let $(L, A) \in \mathbf{SA}$. We shall write

$$\begin{aligned} \mathcal{C}_*[L;A] &\text{ for } \quad \mathcal{C}_*(\mathfrak{alg}_{L,A}, \mathfrak{alg}_{L,A}) \\ \mathcal{Z}_*[L;A] &\text{ for } \quad \mathcal{Z}_*(\mathfrak{alg}_{L,A}, \mathfrak{alg}_{L,A}) \\ \mathcal{B}_*[L;A] &\text{ for } \quad \mathcal{B}_*(\mathfrak{alg}_{L,A}, \mathfrak{alg}_{L,A}) \\ \mathcal{H}_*[L;A] &\text{ for } \quad \mathcal{H}_*(\mathfrak{alg}_{L,A}, \mathfrak{alg}_{L,A}) \end{aligned}$$

and similarly for simplicial cochains, cocycles, coboundaries and cohomology.

(This notation is meant to suggest that the simplicial chain and cochain complexes of $\mathfrak{alg}_{L,A}$ may be defined more directly in terms of the pair (L, A), without introducing the intermediate object $\mathfrak{alg}_{L,A}$.)

When is $\mathfrak{alg}_{L,A}$ unital?

Let $(L, A) \in \mathbf{SA}$. It is clear that if the semilattice L has an identity $\mathbf{1}$, then $\iota_{\mathbf{1}}(1)$ is an identity element for the Banach algebra $\mathfrak{alg}_{L,A}$. Later on in this chapter it will be convenient for technical reasons to know that a stronger result is true.

Lemma 5.1.8. *Let $(L, A) \in \mathbf{SA}$. Suppose the convolution algebra $\ell^1(L)$ has an identity element $u = \sum_{e\in L} \lambda_e e$; then $\sum_{e\in L} \lambda_e \iota_e(e)$ is an identity element for the Banach algebra $\mathfrak{alg}_{L,A}$. In particular, if $\ell^1(L)$ is unital then so is $\mathfrak{alg}_{L,A}$.*

Proof. Let $\widetilde{u} = \sum_{e\in L} \lambda_e \iota_e(e)$. By linearity and continuity, it suffices to show that $\widetilde{u} \cdot \iota_f(a) = \iota_f(a) = \iota_f(a) \cdot \widetilde{u}$ for any $f \in L$ and $a \in A_f$. Direct computation yields

$$\begin{aligned} \widetilde{u} \cdot \iota_f(a) &= \sum_{e\in L} \lambda_e \iota_e(e) \cdot \iota_f(a) \\ &= \sum_{e\in L} \lambda_e \iota_{ef} \phi_{ef,f}(a) \\ &= \sum_{h\in fL} \left(\sum_{e\,:\,ef=h} \lambda_e \right) \iota_h \phi_{h,f}(a) \end{aligned}$$

(manipulations with sums are justified since $\sum_e \lambda_e$ is an absolutely convergent series).

On the other hand, since u is an identity for $\ell^1(F)$ we have

$$f = u \cdot f = \sum_{e \in L} \lambda_e e f = \sum_{h \in fL} \left(\sum_{e\,:\,ef=h} \lambda_e \right) h$$

and by comparing coefficients of h on both sides we deduce that

$$\sum_{e\,:\,ef=h} \lambda_e = \begin{cases} 0 & \text{if } h \neq f \\ 1 & \text{if } h = f \end{cases}$$

Therefore $\widetilde{u} \cdot \iota_f(a) = \iota_f \phi_{f,f}(a) = \iota_f(a)$. The proof that $\iota_f(a) \cdot \widetilde{u} = \iota_f(a)$ is identical save for switching left and right multiplication, and we omit the details. □

Multilinear extensions

Later on we shall need to check whether certain functions defined on $\overbrace{\mathfrak{alg} \times \ldots \times \mathfrak{alg}}^{n}$ are multilinear. In the cases we need it is fairly obvious whether this is the case, but to be precise we include the following lemma. First we set up some terminology:

Definition 5.1.9. Let $\mathbb{I}$ be an indexing set and let $(X_i)_{i \in \mathbb{I}}$ be a family of Banach spaces. Let E be a Banach space: if $\widetilde{T}$ is a function $\left(\coprod_{k \in \mathbb{I}} X_k\right)^n \to E$, an extension of $\widetilde{T}$ to $\mathfrak{X}$ is an n-linear function

$$T : \overbrace{\mathfrak{X} \times \ldots \times \mathfrak{X}}^{n} \to E$$

such that $T \circ (\iota_{k(1)}, \ldots, \iota_{k(n)}) = \widetilde{T}|_{X_{k(1)} \times \ldots \times X_{k(n)}}$ for all $k(1), \ldots, k(n) \in \mathbb{I}$.

Lemma 5.1.10. *Let $n \in \mathbb{N}$ and let $C > 0$; let E be a Banach space and let*

$$\widetilde{T} : \left(\coprod_{i \in \mathbb{I}} X_i \right)^n \to E$$

be an arbitrary function.

Then $\widetilde{T}$ has an extension to $\mathfrak{X}$ of norm $\leq C$, if and only if the following condition is satisfied:

(*) *for every n-tuple $(k(1), \ldots, k(n)) \in \mathbb{I}^n$, the restriction of $\widetilde{T}$ to $X_{k(1)} \times \ldots \times X_{k(n)}$ is a bounded n-linear function with norm $\leq C$.*

Clearly if such an extension exists it will be unique, by linearity and continuity of the extension.

Proof of Lemma 5.1.10. If $\widetilde{T}$ extends to T and $\mathbf{k} = (k(1), \dots, k(n)) \in \mathbb{I}^n$, then for each $j \in \{1, \dots, n\}$ and every $(n-1)$-tuple $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ where $x_r \in X_{k(r)}$ for all r, the function $X_{j(r)} \to E$ given by

$$y \mapsto \widetilde{T}(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n) = T\left(\iota_{k(1)}x_1, \dots, \iota_{k(j)}y, \dots, \iota_{k(n)}x_n\right)$$

is bounded linear with norm $\leq \|T\|$. Hence $(*)$ holds.

Conversely, if $(*)$ holds we define the putative extension T as follows. If $u_1, \dots, u_n \in \mathfrak{X}$ then each u_r has a unique representation as an absolutely convergent sum

$$u_r = \bigoplus_{k \in \mathbb{I}} \iota_k x_k^r \qquad (x_k^r \in X_k \ \forall\, k \in \mathbb{I})$$

and we make the definition

$$T(u_1, \dots, u_n) := \sum_{\mathbf{k} \in \mathbb{I}^n} \widetilde{T}(x_{k(1)}^1, \dots, x_{k(n)}^n)$$

where the sum on the RHS is absolutely convergent and thus well-defined; moreover, we have by condition $(*)$ the following estimate:

$$\sum_{\mathbf{k} \in \mathbb{I}^n} \|\widetilde{T}(x_{k(1)}^1, \dots, x_{k(n)}^n)\| \leq C \sum_{\mathbf{k} \in \mathbb{I}^n} \|x_{k(1)}^1\| \cdots \|x_{k(n)}^n\| = C\|u_1\| \cdots \|u_n\| \,.$$

It remains only to show that T is indeed multilinear. Let $\lambda \in \mathbb{C}$ and $r \in \{1, \dots, n\}$; let $v \in \mathfrak{X}$ have an expansion of the form $v = \sum_{k \in \mathbb{I}} \iota_k y_k$ where $\|v\| = \sum_{k \in \mathbb{I}} \|y_k\| < \infty$; then

$$\begin{aligned}
&T(u_1, \dots, \lambda u_r + v, \dots, u_n) \\
&= \sum_{\mathbf{k} \in \mathbb{I}^n} \widetilde{T}(x_{k(1)}^1, \dots, \lambda x_{k(r)}^r + y_{k(r)}, \dots, x^n k(n)) && \text{(by definition of } T) \\
&= \sum_{\mathbf{e} \in \mathbb{I}^n} \begin{bmatrix} \lambda \widetilde{T}(x_{k(1)}^1, \dots, x_{k(r)}^r, \dots, x_{k(n)}^n) \\ + \widetilde{T}(x_{k(1)}^1, \dots, y_{k(r)}, \dots, x_{k(n)}^n) \end{bmatrix} && \text{(by Condition } (*)) \\
&= \begin{cases} \lambda T(u_1, \dots, u_r, \dots, u_n) \\ + T(u_1, \dots, u_{r-1}, v, \dots, u_n) \end{cases} && \text{(definition of } T)
\end{aligned}$$

and thus T is linear in the rth variable. Since r was arbitrary, T is multilinear as required. □

5.2 Transfer along semilattice homomorphisms

In this section we look at a particular class of homomorphisms between convolution algebras of the form $\mathfrak{alg}_{L,A}$. We shall see later that they allow us to make precise certain "changes of variable" that are required for certain proofs.

Let $(H, B), (L, A) \in \mathbf{SA}$, and suppose that there exists a semigroup homomorphism $\alpha : H \to L$ such that $B = A\alpha$, i.e. such that the diagram

$$\begin{array}{ccc} (H, \succeq) & \xrightarrow{B} & \mathsf{BAlg}_1 \\ {\scriptstyle\alpha}\big\downarrow & \nearrow_{A} & \\ (L, \succeq) & & \end{array}$$

commutes. Then there is a contractive linear map $\tau_\alpha : \mathfrak{alg}_{H,B} \to \mathfrak{alg}_{L,A}$, defined by

$$\tau_\alpha(\iota^B_e b) := \iota^A_{\alpha e} b \qquad (e \in H; b \in B_e). \tag{5.1}$$

(Strictly speaking, τ_α is defined as the unique linear extension of the function $\widetilde{\tau}$: $\coprod_{e \in H} B_e \to \mathfrak{alg}_{L,A}$, where

$$\widetilde{\tau}(b) = \iota^A_{\alpha e}(b) \qquad \text{where } e := \text{base}(b)$$

and the extension is guaranteed by Lemma 5.1.10.)

Remark. Note that the hypothesis that $B = A\alpha$ means not only that $B_e = A_{\alpha e}$ for all $e \in H$, but that whenever $f \preceq e$ in H then we also have equality of transition homomorphisms $\phi^B_{f,e} = \phi^A_{\alpha f, \alpha e}$.

In fact τ_α is an algebra homomorphism.

Proof. Since τ_α is linear and continuous it suffices to show that

$$\tau_\alpha(\iota^B_e b \cdot \iota^B_f c) = \tau_\alpha(\iota^B_e b) \cdot \tau_\alpha(\iota^B_f c) \tag{5.2}$$

for all $b, c \in \coprod_{e \in H} B_e$, where $e := \text{base}(b)$ and $f := \text{base}(c)$. Since

$$\iota^B_e b \cdot \iota^B_f c \equiv \iota^B_{ef} \left[\phi^B_{ef,e}(b)\phi^B_{ef,f}(c)\right]$$

the LHS of (5.2) is, by definition of τ_α,

$$\tau_\alpha\left(\iota^B_{ef}[\phi^B_{ef,e}(b)\phi^B_{ef,f}(c)]\right) = \iota^A_{\alpha(ef)}\left[\phi^A_{\alpha(ef),\alpha e}(b)\phi^A_{\alpha(ef),\alpha f}(c)\right] \quad ,$$

while the RHS of (5.2) is, by the definitions of τ_α and of multiplication in $\mathfrak{alg}_{L,A}$,

$$\iota^A_{\alpha e}(b) \cdot \iota^A_{\alpha f}(c) = \iota^A_{\alpha(e)\alpha(f)}\left[\phi^A_{\alpha(e)\alpha(f),\alpha(e)}(b)\phi^A_{\alpha(e)\alpha(f),\alpha(f)}(c)\right] \quad .$$

(5.2) now follows, since $\alpha(ef) = \alpha(e)\alpha(f)$. ☐

Definition 5.2.1. For $n \geq 0$, we let Tr^α_n denote the contractive linear map $\tau_\alpha{}^{\widehat{\otimes} n+1}$: $\mathcal{C}_n[H, B] \to \mathcal{C}_n[L, A]$. We call Tr^α_n the $\mathsf{transfer\ map\ along}$ α in degree n.

Since τ_α is a homomorphism, it is clear that Tr^α_* is a chain map; this will be crucial in later sections.

Definition 5.2.2. Later on, it will be useful to have the following shorthand: if $a_0, a_1, \dots, a_n$ is an $(n+1)$-tuple of elements of $\coprod_{e\in L} A_e$, where $a_i \in A_{e(i)}$, say, then we write

$$\boldsymbol{\iota}^A_{\mathbf{e}}(\mathbf{a}) := \iota^A_{e(0)}(a_0)\otimes\iota^A_{e(1)}(a_1)\otimes\dots\otimes\iota^A_{e(n)}(a_n)$$

The following lemma is merely a matter of interpreting the new notation correctly.

Lemma 5.2.3.

$$\mathsf{Tr}^\alpha \boldsymbol{\iota}^B_{\mathbf{e}}(\mathbf{a}) = \boldsymbol{\iota}^A_{\alpha\mathbf{e}}(\mathbf{a})$$

The other crucial property of our transfer maps is that they are "functorial" in the following sense.

Lemma 5.2.4 ("Transfer is functorial"). *Let F, H, L be semilattices and let*

$$F \xrightarrow{\beta} H \xrightarrow{\alpha} L$$

be semigroup homomorhpisms. Then whenever $(L, A) \in \mathbf{SA}$,

$$\mathsf{Tr}^\alpha\mathsf{Tr}^\beta \boldsymbol{\iota}^{A\alpha\beta}_\bullet = \mathsf{Tr}^{\alpha\beta}\boldsymbol{\iota}^{A\alpha\beta}_\bullet = \boldsymbol{\iota}^A_{\alpha\beta\bullet} \quad .$$

Remark. There is a more general notion of transfer, where we not only allow ourselves to change the base semilattice but also the algebra structure above.

Specifically: let $(L, A), (H, B) \in \mathbf{SA}$ and let $\alpha : H \to L$ be a homomorphism of semilattices. A transfer morphism $(H, B) \to (L, A)$ is then given by a family $(\mu_h : B_h \to A_{\alpha h})_{h\in H}$ of contractive, unital algebra homomorphisms such that the diagram

$$\begin{array}{ccc} B_h & \xrightarrow{\mu_h} & A_{\alpha h} \\ {\scriptstyle \phi^B_{h,j}}\downarrow & & \downarrow{\scriptstyle \phi^A_{\alpha h,\alpha j}} \\ B_j & \xrightarrow[\mu_j]{} & A_{\alpha j} \end{array}$$

commutes for every $h \le j$ in H.

In more high-powered language, this can be concisely expressed as a natural transformation μ from the functor $B_\bullet$ to the functor $A\alpha_\bullet$.

5.3 Properties of L-normalised chains

We recall that if $(L, A) \in \mathbf{SA}$ then the convolution algebra $\mathfrak{alg}_{L,A}$ contains a closed subalgebra isomorphic to $\ell^1(L)$. Therefore we may consider chains and cochains which are normalised with respect to this subalgebra. For sake of legibility we shall write $\mathcal{C}_n^L$ and $\mathcal{C}_L^n$ instead of $\mathcal{C}_n^{\ell^1(L)}$ and $\mathcal{C}^n_{\ell^1(L)}$, etc, and speak of L-normalised, rather than $\ell^1(L)$-normalised, chains and cochains.

In the following sections we shall obtain a normalisation theorem for simplicial homology of $\mathfrak{alg}_{L,A}$ with respect to the subalgebra $\ell^1(L)$. This is unusal because in general $\ell^1(L)$ is not amenable: indeed, by an old result of Duncan and Namioka, if L is a semilattice and $\ell^1(L)$ is amenable then L is finite (see [8, Thm 10]). In order to achieve our normalisation we will make use of the transfer maps $(\mathsf{Tr}_n^\alpha)_{n\geq 0}$ that were introduced in the previous section.

First let us see why the normalisation theorem is useful. It turns out that from the way $\mathfrak{alg}_{L,A}$ is constructed, L-normalised simplicial chains and cochains have a particularly simple description. In fact we may identify $\mathcal{C}_*^L[L; A]$ with a *subcomplex* (rather than a quotient complex) of $\mathcal{C}_*[L; A]$.

Definition 5.3.1. For each n we let $\mathcal{C}_n^{\text{diag}}[L; A]$ denote the subspace

$$\bigoplus_{e\in L}^{(1)} \mathcal{C}_n(A_e, A_e) \quad \subseteq \quad \mathcal{C}_n[L; A]$$

It is clear that $\mathcal{C}_*^{\text{diag}}[L; A]$ is a subcomplex of $\mathcal{C}_*[L; A]$. We shall sometimes refer to the elements of $\mathcal{C}_*^{\text{diag}}[L; A]$ as L-diagonal chains on $\mathfrak{alg}_{L,A}$.

In fact the subcomplex $\mathcal{C}_*^{\text{diag}}[L; A]$ is a chain summand in $\mathcal{C}_*[L; A]$, and we are able to choose a chain projection onto it with good properties.

Splitting off the diagonal part

Let $(L, A) \in \mathbf{SA}$, and denote the transition homomorphisms by $\phi_{f,e} : A_e \to A_f$ (where $f \preceq e$ in L). For each n, let $\mu_n^{L,A} : \mathcal{C}_n[L; A] \to \mathcal{C}_n^{\text{diag}}[L; A]$ be the linear contraction defined by

$$\boxed{\mu_n^{L,A}(\boldsymbol{\iota}_{\mathbf{e}}\mathbf{a}) = \iota_p\phi_{p,e(0)}a_0\otimes\ldots\otimes\iota_p\phi_{p,e(n)}a_n} \tag{5.3}$$

where $e(0), \ldots, e(n) \in L$, $a_j \in A_{e(j)}$ for all j and $p := e(0)e(1)\cdots e(n)$. We shall occasionally drop the superscript and write μ_n when it is clear which (L, A) we are working with.

Lemma 5.3.2. *For each n, $\mu_n^{L,A}$ is a contractive, linear projection of $\mathcal{C}_n[L;A]$ onto $\mathcal{C}_n^{\mathrm{diag}}[L;A]$.*

Moreover, for any $\mathbf{a} \in \left(\coprod_{e\in L} A_e\right)^{n+1}$, $\boldsymbol{\iota}_{\mathbf{e}}\mathbf{a}$ and $\mu_n \boldsymbol{\iota}_{\mathbf{e}}\mathbf{a}$ have the same image under the canonical quotient map $q : \mathcal{C}_*[L;A] \to \mathcal{C}_*^L[L;A]$.

Proof. The first part is clear by computation on elementary tensors.

The second is a little more fiddly to prove, although the underlying reason is very simple. We shall give a slightly informal proof since the details of a formal induction would obscure the main idea.

If x, y are elementary tensors in $\mathcal{C}_n[L;A]$, let us temporarily say that x and y are *L-equivalent* (denoted $x \sim y$) if $x - y \in \ker(q)$.

Let $\mathbf{a} = (a_0, \dots, a_n)$ be an arbitrary element of $\left(\coprod_{e\in L} A_e\right)^{n+1}$; let $e(j) = \mathrm{base}(a_j)$ and let u_j denote $\iota_{e(j)}(1_{A_{e(j)}})$ for each j. Note that

$$\iota_{e(j)} a_j = u_j \iota_{e(j)}(a_j) = \iota_{e(j)}(a_j) u_j$$

for each j. Let $p = e(0)\dots e(n)$ and let $v = u_0 \dots u_n = \iota_p(1_{A_p})$. By the definition of L-equivalence of tensors we see that

$$\begin{aligned}
\boldsymbol{\iota}_{\mathbf{e}}\mathbf{a} &= \iota_{e(0)} a_0 \otimes \dots \otimes \iota_{e(n-1)} a_{n-1} \otimes \iota_{e(n)} a_n \\
&= u_0 \iota_{e(0)}(a_0) \otimes \dots \otimes u_{n-1} \iota_{e(n-1)} a_{n-1} \otimes u_n \iota_{e(n)}(a_n) \\
&\sim u_0 \iota_{e(0)}(a_0) \otimes \dots \otimes (u_{n-1}u_n) \iota_{e(n-1)} a_{n-1} \otimes u_n \iota_{e(n)}(a_n) \\
&\ \vdots \\
&\sim (u_0 \dots u_n) \iota_{e(0)}(a_0) \otimes (u_1 \dots u_n) \iota_{e(1)} a_1 \otimes \dots \otimes u_n \iota_{e(n)}(a_n) \\
&= v \iota_{e(0)}(a_0) \otimes (u_1 \dots u_n) \iota_{e(1)} a_1 \otimes \dots \otimes u_n \iota_{e(n)}(a_n)
\end{aligned}$$

Repeating this argument to "pass u from left to right", we find that

$$\begin{aligned}
& v \iota_{e(0)}(a_0) \otimes (u_1 \dots u_n) \iota_{e(1)} a_1 \otimes \dots \otimes u_n \iota_{e(n)}(a_n) \\
\sim & v \iota_{e(0)}(a_0) \otimes v \iota_{e(1)} a_1 \otimes \dots \otimes u_n \iota_{e(n)}(a_n) \\
& \vdots \\
\sim & v \iota_{e(0)} a_0 \otimes \dots \otimes v \iota_{e(n)} a_n \\
= & \iota_p(1_{A_p}) \iota_{e(0)} a_0 \otimes \dots \otimes \iota_p(1_{A_p}) \iota_{e(n)} a_n \\
= & \iota_p \phi_{p,e(0)} a_0 \otimes \dots \otimes \iota_p \phi_{p,e(n)} a_n
\end{aligned}$$

and thus $\boldsymbol{\iota}_{\mathbf{e}}\mathbf{a} \sim \mu_n \boldsymbol{\iota}_{\mathbf{e}}\mathbf{a}$, as claimed. ☐

Lemma 5.3.3. $\mu_*^{L,A}$ *is a chain map.*

Proof. It suffices to prove that $\mu_*^{L,A}$ commutes with each face map on the simplicial chain complex $\mathcal{C}_*[L;A]$. Let us do this in full detail for the face map ∂_0.

Let $n \geq 1$; let $e(0), \ldots, e(n) \in L$; and let $a_j \in A_{e(j)}$ for $j = 0, 1, \ldots, n$. We write p for $e(0)e(1)\cdots e(n)$ and f for $e(0)e(1)$; then

$$\begin{aligned}
&\mu_{n-1}^{L,A}\partial_0\left(\iota_{e(0)}a_0 \otimes \ldots \otimes \iota_{e(n)}a_n\right) \\
&= \mu_{n-1}^{L,A}\left((\iota_{e(0)}a_0 \cdot \iota_{e(1)}a_1) \otimes \ldots \otimes \iota_{e(n)}a_n\right) \\
&= \mu_{n-1}^{L,A}\left(\iota_f[\phi_{f,e(0)}(a_0)\phi_{f,e(1)}(a_1)] \otimes \ldots \otimes \iota_{e(n)}a_n\right) && \text{(defn of product in } \mathfrak{alg}_{L,A}) \\
&= \iota_p\left[\phi_{p,e(0)}(a_0)\phi_{p,e(1)}(a_1)\right] \otimes \ldots \otimes \iota_p\phi_{p,e(n)}a_n && \text{(defn of } \mu_{n-1}^{L,A}\text{, since } fe(2)\cdots e(n) = p) \\
&= (\iota_p\phi_{p,e(0)}a_0) \cdot (\iota_p\phi_{p,e(1)}a_1) \otimes \ldots \otimes \iota_p\phi_{p,e(n)}a_n && \text{(since } \iota_p \text{ is a HM)} \\
&= \partial_0\left(\iota_p\phi_{p,e(0)}a_0 \otimes \ldots \otimes \iota_e\phi_{p,e(n)}a_n\right) \\
&= \partial_0\mu_n^{L,A}\left(\iota_{e(0)}a_0 \otimes \ldots \otimes \iota_{e(n)}a_n\right) && \text{(defn of } \mu_n^{L,A})
\end{aligned}$$

By linearity and continuity, we deduce that $\mu_{n-1}^{L,A}\partial_0 = \partial_0\mu_n^{L,A}$ for all $n \geq 1$.

An exactly similar calculation shows that $\mu_{n-1}^{L,A}\partial_i = \partial_i\mu_n^{L,A}$, for each $i = 1, 2, \ldots, n$, and thus $\mu_{n-1}^{L,A}\mathsf{d}_{n-1} = \mathsf{d}_{n-1}\mu_n^{L,A}$ for all $n \geq 1$, as required. ☐

Proposition 5.3.4. *Let* $q : \mathcal{C}_*[L;A] \to \mathcal{C}_*^L[L;A]$ *be the canonical quotient chain map. There exists a chain map* $\widetilde{\mu} : \mathcal{C}_n^L[L;A] \to \mathcal{C}_n[L;A]$*, right inverse to* q*, such that* $\|\widetilde{\mu}_n\| \leq 1$ *and* $\widetilde{\mu}_n q_n = \mu_n$ *for all* n*.*

Proof. We saw earlier (Lemma 5.3.2) that for any $\mathbf{a} \in \left(\coprod_{e\in L} A_e\right)^{n+1}$

$$q_n\mu_n\boldsymbol{\iota}_{\mathbf{e}}\mathbf{a} = q_n\left(\iota_p\phi_{p,e(0)}a_0 \otimes \ldots \otimes \iota_e\phi_{p,e(n)}a_n\right) = q_n\boldsymbol{\iota}_{\mathbf{e}}\mathbf{a}$$

where $a_j \in A_{e(j)}$ for all j and $p := e(0)e(1)\cdots e(n)$. Therefore, by continuity and linearity, $q_n\mu_n = q_n$.

The kernel of q_n is the closed linear span of the set of all tensors that arise by "balancing" an elementary tensor against an element of the form $\iota_f\mathbf{1}_{A_f}$, i.e. all tensors of the form

$$\begin{cases} \iota_{e(0)}a_0 \otimes \ldots \otimes \iota_{e(j)}(a_j)\iota_f(1_{A_f}) \otimes \ldots \otimes \iota_{e(n)}a_n \\ -\iota_{e(0)}a_0 \otimes \ldots \otimes \iota_f(1_{A_f})\iota_{e(j+1)}(a_{j+1}) \otimes \ldots \otimes \iota_{e(n)}a_n \end{cases}$$
$$= \begin{cases} \iota_{e(0)}a_0 \otimes \ldots \otimes \iota_{e(j)f}\phi_{e(j)f,e(j)}a_j \otimes \ldots \otimes \iota_{e(n)}a_n \\ -\iota_{e(0)}a_0 \otimes \ldots \otimes \iota_{fe(j+1)}\phi_{fe(j+1),e(j+1)}a_{j+1} \otimes \ldots \otimes \iota_{e(n)}a_n \end{cases}$$

and a direct computation shows that μ_n vanishes on all such tensors. Hence μ_n factors through q_n and we have a commuting diagram

$$\begin{array}{ccc} \mathcal{C}_n[L;A] & \xrightarrow{\mu_n} & \mathcal{C}_n[L;A] \\ {\scriptstyle q_n}\downarrow & \nearrow {\scriptstyle \widetilde{\mu}_n} & \\ \mathcal{C}_n^L[L;A] & & \end{array}$$

for some bounded linear map $\widetilde{\mu_n}$. Since $\|\mu_n\| \leq 1$ and q_n is a quotient map of Banach spaces, $\|\widetilde{\mu}_n\| \leq 1$. We observed earlier that $q_n\mu_n = q_n$: hence $q_n\widetilde{\mu}_n q_n = q_n$, and since q_n is surjective we deduce that $q_n\widetilde{\mu}_n = \mathsf{id}$.

That $\widetilde{\mu}$ is a chain map follows from the facts that μ is a chain map and q is a *surjective* chain map, just by the following diagram-chasing argument: in *any category*, if we have a diagram of the form

$$\begin{array}{ccc} C_{n-1} & \longleftarrow & C_n \\ \downarrow & & \downarrow {\scriptstyle q_n} \\ D_{n-1} & \longleftarrow & D_n \\ \downarrow & & \downarrow \\ C_{n-1} & \longleftarrow & C_n \end{array}$$

where q_n is an epimorphism and the outer square and the top square both commute, then the bottom square commutes. □

Note that since q is surjective and $\mu = \widetilde{\mu}q$ takes values in $\mathcal{C}_*^{\text{diag}}[L;A]$, the chain map $\widetilde{\mu}$ also takes values in $\mathcal{C}_*^{\text{diag}}[L;A]$.

Proposition 5.3.5 (Disintegration of normalised simplicial homology). *Let* $(L,A) \in \mathbf{SA}$. *There is a chain isomorphism*

$$\mathcal{C}_*^{\text{diag}}[L;A] \underset{\widetilde{\mu}}{\overset{q\iota}{\rightleftarrows}} \mathcal{C}_*^L[L;A]$$

Proof. It suffices to prove that for all n, $\widetilde{\mu}_n$ and $q_n\iota_n$ are mutually inverse maps.

We know already that each μ_n is a projection of $\mathcal{C}_n[L;A]$ onto the summand $\mathcal{C}_n^{\text{diag}}[L;A]$, i.e. $\mu_n\iota = \mathsf{id}$. Hence $\widetilde{\mu}_n q_n\iota = \mathsf{id}$. By the previous proposition $q_n\widetilde{\mu}_n = \mathsf{id}$. □

Recall that the L-normalised simplicial cochain complex $\mathcal{C}^*_L[L;A]$ is the dual of the L-normalised simplicial chain complex $\mathcal{C}^L_*[L;A]$. Therefore, if we let

$$\mathcal{C}^*_{\mathrm{diag}}[L;A] := \mathcal{C}^{\mathrm{diag}}_*[L;A]' = \left(\bigoplus_{\ell^\infty}\right)_{e\in L} \mathcal{C}^*(A_e, A'_e)$$

then dualising the previous proposition yields the following:

Proposition 5.3.6 (Disintegration of normalised simplicial cohomology). *Let $(L,A)\in\mathbf{SA}$. There is a chain isomorphism*

$$\mathcal{C}^*_L[L;A] \underset{\widetilde{\mu}'}{\overset{\text{restr.}}{\rightleftarrows}} \mathcal{C}^*_{\mathrm{diag}}[L;A]$$

where restr. *denotes the restriction of an L-normalised simplicial cochain on* $\mathsf{alg}[L;A]$ *to the space of L-diagonal chains.*

We have seen that for given $(L,A)\in\mathbf{SA}$ the maps $(\mu_n^{L,A})_{n\geq 0}$ have good properties. It will be vital for the technical arguments used later that these maps also depend on (L,A) in a well-behaved way.

Proposition 5.3.7 (μ is natural). *$\mu_n^\bullet$ commutes with transfer in degree n. That is: given $(H,B)\in\mathbf{SA}$ and $(L,A)\in\mathbf{SA}$, and a semigroup homomorphism $\alpha: H\to L$ such that $B = A\alpha$, we have a commuting diagram*

$$\begin{array}{ccc} \mathcal{C}_n[H;B] & \xrightarrow{\mu_n^{H,B}} & \mathcal{C}_n[H;B] \\ \Big\downarrow{\scriptstyle \mathsf{Tr}_n^\alpha} & & \Big\downarrow{\scriptstyle \mathsf{Tr}_n^\alpha} \\ \mathcal{C}_n[L;A] & \xrightarrow[\mu_n^{L,A}]{} & \mathcal{C}_n[L;A] \end{array}$$

Proof. This is just book-keeping. Recall that Tr_n^α is just alternative notation for $\tau_\alpha^{\widehat{\otimes} n+1}$. Recall also that since $B = A\alpha$,

$$\text{whenever } y \preceq x \text{ in } H, \qquad \phi^B_{y,x} = \phi^A_{\alpha y, \alpha x}\,.$$

Let $n\geq 0$; let $e(0),\dots,e(n)\in L$; and let $a_j\in A_{e(j)}$ for $j=0,\dots,n$. We write p for $e(0)e(1)\cdots e(n)$ and observe that, since $\alpha(p) = \alpha(e(0))\cdots\alpha(e(n))$,

$$\begin{aligned}
\mathsf{Tr}^{\alpha}_{n}\mu^{H,B}_{n}\big(\iota^{B}_{e(0)}a_0\otimes\ldots\otimes\iota^{B}_{e(n)}a_n\big) &= \mathsf{Tr}^{\alpha}_{n}\big(\iota^{B}_{p}\phi^{B}_{p,e(0)}a_0\otimes\ldots\otimes\iota^{B}_{p}\phi^{B}_{p,e(n)}a_n\big)\\
&= \tau_\alpha\iota^{B}_{p}\phi^{B}_{p,e(0)}a_0\otimes\ldots\otimes\tau_\alpha\iota^{B}_{p}\phi^{B}_{p,e(n)}a_n\\
&= \iota^{A}_{\alpha p}\phi^{B}_{p,e(0)}a_0\otimes\ldots\otimes\iota^{A}_{\alpha p}\phi^{B}_{p,e(n)}a_n\\
&= \iota^{A}_{\alpha p}\phi^{A}_{\alpha p,\alpha e(0)}a_0\otimes\ldots\otimes\iota^{A}_{\alpha p}\phi^{A}_{\alpha p,\alpha e(n)}a_n\\
&= \mu^{L,A}_{n}\big(\iota^{A}_{\alpha e(0)}a_0\otimes\ldots\otimes\iota^{A}_{\alpha e(n)}a_n\big)\\
&= \mu^{L,A}_{n}\big(\tau_\alpha\iota^{A}_{e(0)}a_0\otimes\ldots\otimes\tau_\alpha\iota^{A}_{e(n)}a_n\big)\\
&= \mu^{L,A}_{n}\mathsf{Tr}^{\alpha}_{n}\big(\iota^{B}_{e(0)}a_0\otimes\ldots\otimes\iota^{B}_{e(n)}a_n\big)\quad.
\end{aligned}$$

By linearity and continuity we deduce that $\mathsf{Tr}^{\alpha}_{n}\mu^{H,B}_{n}=\mu^{L,A}_{n}\mathsf{Tr}^{\alpha}_{n}$ as required. □

5.4 Statement of the main normalisation theorem

It would be of limited interest or use if we could not relate the L-normalised cohomology groups $\mathcal{H}^n_L$ to the "genuine" cohomology groups $\mathcal{H}^n$. In the remaining sections of this chapter we shall establish such a relation for the simplicial cohomology groups. The methods used are inspired by calculations in the paper [4] and the preprint [14] for ℓ^1-algebras of Clifford semigroups; they constitute a more complicated vesion of the techniques used in the author's paper [5].

To be precise, we shall prove the following:

Theorem 5.4.1. *Let* $(L,A)\in\mathbf{SA}$*. Then the quotient map from* $\mathcal{C}_*[L;A]$ *onto* $\mathcal{C}^L_*[L;A]$ *induces an isomorphism of homology groups.*

More precisely, there exists a chain map $\nu:\mathcal{C}^L_*[L;A]\to\mathcal{C}_*[L;A]$ *and bounded linear maps* $\sigma_n:\mathcal{C}_n[L;A]\to\mathcal{C}_{n+1}[L;A]$ *such that*

- $q_n\nu_n=\mathsf{id}$*;*
- $\mathsf{id}-\nu_nq_n=\mathsf{d}_n\sigma_n+\sigma_{n-1}\mathsf{d}_{n-1}$

for all $n\geq 0$*.*

Remark. The proofs of [4, Thm 2.1] and [14, Thm 4.6] may be easily extended to recover the cases $n=1$ and $n=2$ of our Theorem 5.4.1. The novelty of our result lies not so much in the generalisation from Clifford semigoup algebras to semilattices of algebras, as in systematically solving the higher-degree cohomology problems.

In fact, to call our result a normalisation result is a little misleading, because the proof goes via the identification of L-normalised chains with L-diagonal ones:

Theorem 5.4.2. *Let* $\pi_n := \mathsf{id}_n - \mu_n : \mathcal{C}_n[L;A] \to \mathcal{C}_n[L;A]$. *Then the chain projection* $\pi_* : \mathcal{C}_*[L;A] \to \mathcal{C}_*[L;A]$ *is null-homotopic: that is, there exist bounded linear maps* $\sigma_n : \mathcal{C}_n[L;A] \to \mathcal{C}_{n+1}[L;A]$ *for each* $n \geq 0$ *such that*

$$\mathsf{d}_n \sigma_n + \sigma_{n-1}\mathsf{d}_{n-1} = \pi_n \qquad (n \geq 1)$$

and $\mathsf{d}_0\sigma_0 = \pi_0 = 0$.

The strategy for proving Theorem 5.4.2 is to combine two lines of attack: the naturality of π_* with respect to transfer; and the fact that we can prove the theorem in the special case where L is a *finite, free* semilattice. We then set up an inductive argument to construct a splitting homotopy for π_*, by inductively "transferring" known splitting formulas from the finite free case over to "natural" splitting formulas for the general case.

5.5 π is null-homotopic for finitely generated F

In this section we give a leisurely account of how one can construct a splitting homotopy for $\pi^{F,-}$ when F is a finitely generated semilattice. Our starting point is to observe that for such F the subalgebra $\ell^1(F)$ is homologically very well-behaved.

Lemma 5.5.1. *Let* F *be a finitely generated semilattice and let* $\mathbb{I}$ *be a finite set of generators for* F. *Then the Banach algebra* $\ell^1(F)$ *is finite-dimensional and contractible, and it has a diagonal* Δ_F *whose norm is bounded by a constant depending only on* $\mathbb{I}$.

Proof. For this proof we use π to denote the product map $\ell^1(F)\widehat{\otimes}\ell^1(F) \to \ell^1(F)$; this should not cause any confusion with the chain map $\pi \equiv \pi^{L,A}$ which has already been introduced.

We first observe that $\ell^1(F)$ is a quotient of the finite-dimensional algebra $\ell^1(\widetilde{F})$, where $\widetilde{F}$ is the free semilattice on the generating set $\mathbb{I}$, and is therefore itself finite-dimensional.

To construct Δ it is convenient to adjoin an identity element $\mathsf{1}$ to F and consider the algebra $\ell^1(F \sqcup \{\mathsf{1}\}) \underset{1}{\cong} \ell^1(F)^{\#}$. For every $J \subseteq \mathbb{I}$ we let

$$u_J := \prod_{i\in J} e_i \prod_{k\notin J} (e_1 - e_k) \in \ell^1(F \sqcup \{\mathsf{1}\})$$

and note that if J is non-empty then $u_J \in \ell^1(F)$.

Since F is commutative, u_J is a product of commuting idempotents and is therefore itself an idempotent; moreover, for any $s \in F$ a direct computation yields

$$e_s \cdot u_J = \left\{ \begin{matrix} u_J & \text{if } s \in J \\ 0 & \text{if } s \notin J \end{matrix} \right\} = u_J \cdot e_s \quad .$$

It follows that $e_s \cdot (u_J \otimes u_J) = (u_J \otimes u_J) \cdot e_s$ for all $s \in \mathbb{I}$, and since every $x \in F$ can be written as a product of finitely many elements of $\mathbb{I}$, we deduce that

$$e_x \cdot (u_J \otimes u_J) = (u_J \otimes u_J) \cdot e_x \qquad \text{for all } x \in F \text{ and } J \subseteq \mathbb{I} \, .$$

We now set $\Delta_F = \sum_{\emptyset \neq J \subseteq \mathbb{I}} u_J \otimes u_J \in \ell^1(F) \widehat{\otimes} \ell^1(F)$: then

$$e_x \cdot \Delta_F = \Delta_F \cdot e_x \qquad \text{for all } x \in F \quad .$$

Since each u_J is an idempotent, $\pi(\Delta_F) = \sum_{\emptyset \neq J \subseteq \mathbb{I}} u_J$, and so

$$u_\emptyset + \pi(\Delta_F) = \sum_{J \subseteq \mathbb{I}} u_J = \sum_{J \subseteq \mathbb{I}} \prod_{i \in J} e_i \prod_{k \notin J} (e_1 - e_k) = [e_i + (e_1 - e_i)]^n = e_1 \, .$$

Let $s \in \mathbb{I}$: then $e_s \cdot e_1 = e_s = e_1 \cdot e_s$, and $e_s \cdot u_\emptyset = 0 = u_\emptyset \cdot e_s$, so that $e_s \cdot \pi(\Delta_F) = e_s = \pi(\Delta_F) \cdot e_s$; hence

$$e_x \cdot \pi(\Delta_F) = e_x = \pi(\Delta_F) \cdot e_x \qquad (x \in F) \, .$$

This completes the proof that Δ_F is a diagonal for $\ell^1(F)$. It remains only to bound the norm of Δ_F in terms of $|F|$, and for this the following crude calculation suffices:

$$\begin{aligned} \|\Delta_F\| &\leq \sum_{\emptyset \neq J \subseteq \mathbb{I}} \|u_J\|^2 \\ &\leq \sum_{\emptyset \neq J \subseteq \mathbb{I}} \left\| \prod_{k \notin J} (e_1 - e_k) \right\|^2 \\ &\leq \sum_{\emptyset \neq J \subseteq \mathbb{I}} \left(2^{|\mathbb{I}| - |J|} \right)^2 \\ &\leq \sum_{j=1}^{|\mathbb{I}|} \binom{|\mathbb{I}|}{j} 4^{|\mathbb{I}| - |J|} \qquad = 5^{|\mathbb{I}|} - 1 \end{aligned}$$

so that $\|\Delta_F\| < 5^{|\mathbb{I}|}$. □

Remark. In effect, this proof works by constructing a diagonal $\widetilde{\Delta}$ for the algebra $\ell^1(\widetilde{F})$; Δ is then obtained as the image of Δ under the quotient homomorphism $\ell^1(\widetilde{F}) \widehat{\otimes} \ell^1(\widetilde{F}) \to \ell^1(F) \widehat{\otimes} \ell^1(F)$.

We also remark that although the idempotents u_J may seem a case of *deus ex machina* they arise very naturally by the following argument: the forced unitisation $\widetilde{F} \sqcup \{1\}$ is isomorphic as a monoid to the direct product $S^{\mathbb{I}}$ where S is the unital semilattice generated by a single idempotent e; hence

$$\ell^1(\widetilde{F})^{\#} \underset{1}{\cong} \ell^1(\widetilde{F} \sqcup \{1\}) \underset{1}{\cong} \ell^1(S)^{\otimes |\mathbb{I}|}$$

is the tensor power of a contractible algebra, and hence is itself contractible. If Δ_S is a diagonal for the two-dimensional algebra $\ell^1(S)$ then the tensor power $(\Delta_S)^{\otimes|\mathbb{I}|}$ is a diagonal for $\ell^1(\widetilde{F})^{\#}$, and this can be cut down to a diagonal for $\ell^1(\widetilde{F})$ itself; on taking

$$\Delta_S = e \otimes e + (1-e) \otimes (1-e)$$

we recover the formula for Δ (and thence for the idempotents u_J) which was obtained in the proof of Lemma 5.5.1.

Remark. It is known that if F is a finitely generated semilattice then the semigroup algebra $\mathbb{C}F$ is finite-dimensional and semisimple: the Gelfand transform then yields an algebra isomorphism of $\mathbb{C}F$ onto the algebra $\mathbb{C}^{|F|}$ equipped with pointwise multiplication. The latter algebra has an obvious diagonal, and so we obtain an alternative proof that $\ell^1(F) = \mathbb{C}F$ has a diagonal.

Proposition 5.5.2. *Let* $(F, B) \in \mathbf{SA}$ *where* F *is a finitely generated semilattice, let* $\mathcal{B}$ *denote the convolution algebra* $\mathfrak{alg}_{F,B}$*, and let* X *be a Banach* $\mathcal{B}$*-bimodule. Then there exists a chain map* $\alpha : \mathcal{C}_*(\mathcal{B}, X) \to \mathcal{C}_*(\mathcal{B}, X)$ *with the following properties:*

(a) *each* α_n *factors through the quotient map* $q_n : \mathcal{C}_*[F; B] \to \mathcal{C}^F_*[F; B]$*;*

(b) *there exists a chain homotopy from* id *to* α*, given by bounded linear maps* $t_n : \mathcal{C}_n(\mathcal{B}, X) \to \mathcal{C}_{n+1}(\mathcal{B}, X)$ *satisfying* $\mathsf{d}_n t_n + t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \alpha_n$ *for all* n*;*

(c) *the norm of each* t_n *is bounded above by some constant depending only on* n *and the number of generators for* $|F|$*.*

Proof. By Lemma 5.5.1 the algebra $\ell^1(F)$ is contractible, so in particular has an identity element. By Lemma 5.1.8, the canonical inclusion of $\ell^1(F)$ into $\mathfrak{alg}_{F,B}$ sends the identity element of $\ell^1(F)$ to an identity element for $\mathfrak{alg}_{F,B}$, and the results *(a)*–*(c)* now follow as a special case of Theorem 1.4.6. ☐

In particular, on taking $X = \mathcal{B}$ we see that every simplicial cycle on $\mathfrak{alg}_{F,B}$ is homologous to an F-diagonal one. However, this is not quite enough: we wish to show that every n-cycle $x \in \mathcal{Z}_n[F;B]$ is homologous to $\mu_n^{F,B}x$, so an extra step is needed.

This extra step can be done at a slightly more general level, and is given by the following trivial lemma. (It is stated for Hochschild homology but there is clearly a dual version for cohomology.)

Lemma 5.5.3 (Combining normalising projections). *Let A be a Banach algebra, X a Banach A-bimodule. Suppose we have two chain maps $\alpha, \mu : \mathcal{C}_*(A,X) \to \mathcal{C}_*(A,X)$ such that*

- *α is chain-homotopic to the identity: i.e. there exist bounded linear maps $t_n : \mathcal{C}_n(A,X) \to \mathcal{C}_{n+1}(A,X)$ such that $\mathsf{d}_n t_n + t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \alpha_n$ for all n;*
- *$\mu_n \alpha_n = \alpha_n$.*

Then μ_n is chain-homotopic to the identity: more precisely, we have

$$\mathsf{d}_n s_n + s_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \mu_n$$

where $\|s_n\| \leq (1 + \|\mu_{n+1}\|)\|t_n\|$.

Proof. Let $n \geq 0$. We know that $\mathsf{d}_n t_n + t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \alpha_n$; composing on the left with μ_n on both sides, and recalling that μ is a chain map, we have

$$\mathsf{d}_n \mu_{n+1} t_n + \mu_n t_{n-1}\mathsf{d}_{n-1} = \mu_n - \mu_n\alpha_n = \mu_n - \alpha_n$$

Subtracting this equation from the previous one yields

$$\mathsf{d}_n(\mathsf{id}_{n+1} - \mu_{n+1})t_n + (\mathsf{id}_n - \mu_n)t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \mu_n\ ,$$

and on taking $s_n := (\mathsf{id}_{n+1} - \mu_{n+1})t_n$ we have proved the lemma. □

Proposition 5.5.4. *Let $(F,B) \in$ **SA** where F is a finitely generated semilattice. Let $\mu_*^{F,B} : \mathcal{C}_*[F;B] \to \mathcal{C}_*[F;B]$ be the chain map defined earlier (5.3). Then there exists a chain homotopy from $\mu^{F,B}$ to id: more precisely, there exist bounded linear maps $s_n^F : \mathcal{C}_n[F;B] \to \mathcal{C}_{n+1}[F;B]$ such that*

$$\mathsf{d}_n^{F,B} s_n^F + s_{n-1}^F \mathsf{d}_{n-1}^{F,B} = \mathsf{id}_n - \mu_n^{F,B}$$

and such that $\|s_n^F\|$ is bounded above by a constant depending only on $|F|$ and n.

Proof. By Proposition 5.5.2 there exists a chain map $\alpha : \mathcal{C}_*[F;B] \to \mathcal{C}_*[F,B]$ which factors through the quotient map $q : \mathcal{C}_*[F;B] \to \mathcal{C}^F_*[F;B]$ and which is homotopic to the identity. Since α factors through q_n, $\mu_n^{F,B}\alpha_n = \alpha_n$ for all n. Hence Lemma 5.5.3 applies. ☐

Remark. Note that for fixed F the maps $s_n^{F,\text{—}}$ are natural in the second variable, in some sense ... but to make this precise, we need a more general notion of transfer.

We shall use maps of the form $s^{F,\text{—}}$ to inductively construct splitting maps for general semilattices of algebras, using the transfer maps. It is worth pointing out that when we come to apply Proposition 5.5.4 it will only be in the particular case where the semilattice F is *finite free*, and therefore we could have done without the estimate on $\|\Delta_F\|$ (since all we in fact need is norm estimates on $s^{F,\text{—}}$ that depend only on F).

The fact that Proposition 5.5.4 holds for all finitely generated semilattices, and not just the finite free ones, is in some sense incidental to our main argument. We have chosen to prove the more general statement since the proofs are no harder; but arguably this obscures the main idea of the proof that we are building up to. (In contrast, the presentation in the paper [5] only uses the finite free case.)

5.6 Proving the main splitting theorem

5.6.1 Formulating the inductive step

The following schematic is meant to give some intuitive idea of the strategy behind the rather messy calculations in the section.

Idea/motivation:

NATURAL PROOF that $\mathcal{H}^{n-1}_{\text{offdiag}}[L;\_\_] = 0$ for all L

↓

NATURAL PROOF that $\mathcal{H}^{n}_{\text{offdiag}}[L;\_\_]$ is Banach for all L

↓

proof that $\mathcal{H}^{n}_{\text{offdiag}}[F;\_\_] = 0$ for FINITE FREE F ⟶ NATURAL PROOF that $\mathcal{H}^{n}_{\text{offdiag}}[L;\_\_] = 0$ for all L

Definition 5.6.1. Let j be a non-negative integer and let

$$\left(\sigma_j^{L,A} : \mathcal{C}_j[L;A] \longrightarrow \mathcal{C}_{j+1}[L;A]\right)_{(L,A)\in\mathbf{SA}}$$

be a family of bounded linear maps. We consider the following four conditions that $\sigma^\bullet$ may or may not satisfy:

(R) for each $(L,A) \in \mathbf{SA}$,

$$\pi_{j+1}^{L,A}\sigma_j^{L,A} = \sigma_j^{L,A} \ .$$

(S) for each $(L,A) \in \mathbf{SA}$

$$\mathsf{d}_j^{L,A}\sigma_j^{L,A}\mathsf{d}_j^{L,A} = \mathsf{d}_j^{L,A}\pi_{j+1}^{L,A} \ .$$

(T) whenever $(L,A) \in \mathbf{SA}$, $(H,B) \in \mathbf{SA}$, and $\alpha : H \to L$ is a semilattice homomorphism such that $B = A\alpha$, then the diagram

$$\begin{array}{ccc}
\mathcal{C}_j[H;B] & \xrightarrow{\ \sigma_j^{H,B}\ } & \mathcal{C}_{j+1}[H;B] \\
\Big\downarrow{\scriptstyle \mathsf{Tr}_n^\alpha} & & \Big\downarrow{\scriptstyle \mathsf{Tr}_{n+1}^\alpha} \\
\mathcal{C}_j[L;A] & \xrightarrow[\ \sigma_j^{L,A}\ ]{} & \mathcal{C}_{j+1}[L;A]
\end{array}$$

commutes.

(U) there is a constant $K_j > 0$ such that $\|\sigma_j^{L,A}\| \leq K_j$ for all $(L,A) \in \mathbf{SA}$.

Here (R) stands for *range* fixed by π; (S) for *semilattice-normalised splitting*; (T) for *transferable* map; and (U) for *uniformly bounded*.

Remark. Note that condition (R) tells us that our splitting map σ_j should take "off-diagonal" cycles to "off-diagonal" cycles.

Lemma 5.6.2 (Base for induction). *For each* $(L,A) \in \mathbf{SA}$ *let* $\sigma_0^{L,A} : \mathcal{C}_0[L;A] \to \mathcal{C}_1[L;A]$ *be the zero map. Then* $\sigma_0^\bullet$ *satisfies conditions* (R)–(U).

Proof. It is clear that taking $\sigma_0^{L,A} = 0$ for all $(L,A) \in \mathbf{SA}$ will satisfy conditions (R), (T) and (U). To show that condition (S) is satisfied we need only show that

$$0 = d_0^{L,A}\pi_1^{L,A} \qquad \text{for all } (L,A) \in \mathbf{SA}.$$

By linearity and continuity it suffices to verify this identity on block elements. So, let $e, f \in L$ and let $a \in A_e$, $b \in B_f$. Omitting superscripts for sake of clarity, we see that

$$\mu_1(\iota_e a \otimes \iota_f b) = \iota_{ef}\phi_{ef,e}(a) \otimes \iota_{ef}\phi_{ef,f}(b) \equiv (f \cdot \iota_e a) \otimes (e \cdot \iota_f b)$$

and so

$$\begin{aligned} \mathsf{d}_0\mu_1(\iota_e a \otimes \iota_f b) &= \mathsf{d}_0\left((f \cdot \iota_e a) \otimes (e \cdot \iota_f b)\right) \\ &= (\iota_f b \cdot \iota_e a) - (\iota_e a \cdot \iota_f b) \quad = \mathsf{d}_0(\iota_e a \otimes \iota_f b) \end{aligned}$$

which gives us $\mathsf{d}_0\pi_1(\iota_e a \otimes \iota_f b) = \mathsf{d}_0(\iota_e a \otimes \iota_f b) - \mathsf{d}_0\mu_1(\iota_e a \otimes \iota_f b) = 0$ as required. □

Remark. It may help the reader to think of the above proof as the "predual version" of the following statement: *every derivation $\mathfrak{alg}_{L,A} \to (\mathfrak{alg}_{L,A})'$ is automatically L-normalised.*

The following proposition is our inductive step, and is the heart of the main normalisation theorem.

Proposition 5.6.3. *Let $n \geq 1$, and suppose that there exists a family of linear maps $\sigma_{n-1}^{L,A} : \mathcal{C}_{n-1}[L;A] \to \mathcal{C}_n[L;A]$ which satisfies conditions* (R)*,* (S)*,* (T) *and* (U)*.*

Then there exists a family of linear maps $\sigma_n^{L,A} : \mathcal{C}_n[L;A] \to \mathcal{C}_{n+1}[L;A]$ which satisfies conditions (R)*,* (T) *and* (U)*, and also satisfies*

$$d_n^{L,A}\sigma_n^{L,A} + \sigma_{n-1}^{L,A}d_{n-1}^{L,A} = \pi_n^{L,A} \qquad \textit{for all } (L,A) \in \mathbf{SA}\ . \tag{5.4}$$

Let us first see how this proposition gives Theorem 5.4.2.

Proof of Theorem 5.4.2, using Proposition 5.6.3. We shall prove a stronger statement, namely that there exists a family of bounded linear maps $\sigma_n^{L,A} : \mathcal{C}_n[L;A] \to \mathcal{C}_{n+1}[L;A]$ for each $n \geq 0$ and $(L,A) \in \mathbf{SA}$, such that

$$\mathsf{d}_n^{L,A}\sigma_n^{L,A} + \sigma_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A} = \pi_n^{L,A} \qquad (n \geq 1) \qquad \text{and} \qquad \mathsf{d}_0^{L,A}\sigma_0^{L,A} = 0$$

for all $(L,A) \in \mathbf{SA}$, *and* such that for each $n \in \mathbb{Z}_+$ the family $(\sigma_n^{L,A})_{(L,A)\in\mathbf{SA}}$ satisfies conditions (R), (T) and (U).

The proof is by induction on n.

We have seen (Lemma 5.6.2) that taking $\sigma_0^\bullet = 0$ gives a family $(\sigma_0^{L,A})_{(L,A)\in\mathbf{SA}}$ which satisfies conditions (R), (S), (T) and (U), and which satisfies

$$\mathsf{d}_0^{L,A}\sigma_0^{L,A} = 0\ .$$

Therefore by Proposition 5.6.3 there exists a family $(\sigma_1^{L,A})_{(L,A)\in\mathbf{SA}}$ which satisfies conditions (R), (T) and (U), and such that

$$\mathsf{d}_1^{L,A}\sigma_1^{L,A} + \sigma_0^{L,A}\mathsf{d}_0^{L,A} = \pi_1^{L,A}$$

for all $(L,A) \in \mathbf{SA}$. Hence

$$\mathsf{d}_1^{L,A}\sigma_1^{L,A}\mathsf{d}_1^{L,A} = \left(\mathsf{d}_1^{L,A}\sigma_1^{L,A} + \sigma_0^{L,A}\mathsf{d}_0^{L,A}\right)\mathsf{d}_1^{L,A} = \pi_1^{L,A}\mathsf{d}_1^{L,A} = \mathsf{d}_1^{L,A}\pi_2^{L,A}$$

so that $\sigma_1^\bullet$ also satisfies condition (S).

Now suppose that there exists $n \geq 2$ and families $\sigma_{n-2}^\bullet$, $\sigma_{n-1}^\bullet$ which both satisfy conditions (R), (T) and (U), and which also satisfy

$$\mathsf{d}_{n-1}^{L,A}\sigma_{n-1}^{L,A} + \sigma_{n-2}^{L,A}\mathsf{d}_{n-2}^{L,A} = \pi_{n-1}^{L,A}$$

for all $(L,A) \in \mathbf{SA}$. Then

$$\mathsf{d}_{n-1}^{L,A}\sigma_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A} = \left(\mathsf{d}_{n-1}^{L,A}\sigma_{n-1}^{L,A} + \sigma_{n-2}^{L,A}\mathsf{d}_{n-2}^{L,A}\right)\mathsf{d}_{n-1}^{L,A} = \pi_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A} = \mathsf{d}_{n-1}^{L,A}\pi_n^{L,A}$$

and so $\sigma_{n-1}^\bullet$ satisfies condition (S). Hence by Proposition 5.6.3 there exists a family $(\sigma_n^{L,A})_{(L,A)\in\mathbf{SA}}$ which satisfies conditions (R),(T) and (U) and which also satisfies

$$\mathsf{d}_n^{L,A}\sigma_n^{L,A} + \sigma_{n-1}^{L,A}\mathsf{d}_{n-1}^{L,A} = \pi_n^{L,A} \quad .$$

This completes the induction, and hence Theorem 5.4.2 is proved *assuming that Proposition 5.6.3 is valid.* ☐

The proof of Proposition 5.6.3 occupies the next section.

5.6.2 Proof of the inductive step (Proposition 5.6.3)

Throughout this section F denotes the finite semilattice freely generated by $n+1$ idempotents $f(0), \dots, f(n)$. Initially we will work on $\mathcal{C}_*[F,T]$ where T is an arbitrary functor $F \to \mathsf{BAlg}_1$, and later specialise to particular kinds of T.

Recall that the finite free case works in all degrees, for any T. More precisely, by Proposition 5.5.4 there exist bounded linear maps s_n^T and s_{n-1}^T such that:

$$\mathsf{d}_n^{F,T}s_n^T = \pi_n^{F,T} - s_{n-1}^T\mathsf{d}_{n-1}^{F,T} \ . \tag{5.5}$$

Recall also that $\|s_n^T\| \leq C_n$ for some constant C_n that depends only on n.

The formula (5.5) says that every n-chain z in the image of $\pi_n^{F,T}$ can be written as the sum of an n-boundary and a chain which is a linear image of $\mathsf{d}_{n-1}^{F,T}z$. The next step is aimed, roughly speaking, at improving this decomposition so that z is the sum of an n-boundary and a chain which is the linear image of $\mathsf{d}_{n-1}^{F,T}z$ *under a natural map* (Equation (5.7) below).

Since $\pi^{F,T}$ is a chain projection, (5.5) implies that

$$\mathsf{d}_n^{F,T}\pi_{n+1}^{F,T}s_n^T = \pi_n^{F,T}\mathsf{d}_n^{F,T}s_n^T = \pi_n^{F,T} - \pi_n^{F,T}s_{n-1}^T\mathsf{d}_{n-1}^{F,T}\ ; \tag{5.6}$$

but, since condition (S) holds for $\sigma_{n-1}^\bullet$, we also know that

$$\mathsf{d}_{n-1}^{F,T}\left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right) = 0\ .$$

Hence post-multiplying on both sides of (5.6) by $\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}$ gives us

$$\begin{aligned}
&\mathsf{d}_n^{F,T}\pi_{n+1}^{F,T}s_n^T\left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right)\\
&= \pi_n^{F,T}\left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right)\\
&= \pi_n^{F,T} - \pi_n^{F,T}\sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T} && \text{(since } \pi_n \text{ is a projection)}\\
&= \pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T} && \text{(since (R) holds for } \sigma_{n-1}^\bullet\text{)}
\end{aligned}$$

We have thus arrived at the following, crucial identity, which should be compared with Equation (5.5):

$$\mathsf{d}_n^{F,T}\pi_{n+1}^{F,T}s_n^T\left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right) = \pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\quad . \tag{5.7}$$

We shall apply the identity (5.7) to particular elements of $\mathfrak{alg}_{F,T}$. Firstly, let us introduce some auxiliary notation to ease the congestion of indices: let $\psi_T : \mathcal{C}_n[F;T] \to \mathcal{C}_n[F;T]$ denote the *bounded linear* map

$$\psi_T := \pi_{n+1}^{F,T}s_n^T\left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right)$$

With this notation, the previous "crucial identity" of linear maps now implies the following identity in $\mathfrak{alg}_{F,T}$:

$$\boxed{\mathsf{d}_n^{F,T}\psi_T(\boldsymbol{\iota}_{\mathbf{f}}^T\mathbf{b}) = \left(\pi_n^{F,T} - \sigma_{n-1}^{F,T}\mathsf{d}_{n-1}^{F,T}\right)(\boldsymbol{\iota}_{\mathbf{f}}^T\mathbf{b})} \tag{5.8}$$

for any $(n+1)$-tuple $\mathbf{b} = (b_0, \ldots, b_n)$ such that $\operatorname{base}(b_j) = f(j)$ for $j = 0, 1, \ldots, n$.

We observe for later reference that

$$\pi_{n+1}^{F,T}\psi_T = \psi_T\ . \tag{5.9}$$

Also, since condition (U) is assumed to hold, there is a constant K_{n-1} such that $\|\sigma_{n-1}^{F,T}\| \leq K_{n-1}$. Therefore

$$\|\psi_T\| \leq \|s_n^T\|(1 + K_{n-1}\|\mathsf{d}_{n-1}^{F,T}\|) \leq C_n(1 + (n+1)K_{n-1}$$

where the RHS depends only on n.

Now let $(L, A) \in \mathbf{SA}$. Let $x(0), \ldots, x(n+1) \in L$, and observe that there is a unique, well-defined homomorphism of semilattices $F \to L$ which sends $f(j) \mapsto x(j)$ for $j = 0, \ldots, n+1$. We denote this homomorphism by $\widehat{\mathbf{x}} : F \to L$ – the notation is meant to suggest that we think of $\widehat{\mathbf{x}}$ as "evaluation" of the "free variables" $f(j)$ at particular values $x(j)$ – and write $A\widehat{\mathbf{x}}$ for $A_\bullet \circ \widehat{\mathbf{x}} : F \to \mathsf{BAlg}_1$.

Since we have a homomorphism $\widehat{\mathbf{x}} : F \to L$, we are in a position to bring in the transfer maps from Section 5.2. Consider the semilattice in BAlg_1 given by $(F, A\widehat{\mathbf{x}})$, and the associated transfer chain map

$$\mathcal{C}_*[F; A\widehat{\mathbf{x}}] \xrightarrow{\mathsf{Tr}_*^{\widehat{\mathbf{x}}}} \mathcal{C}_*[L; A]$$

Now suppose that for $j = 0, 1 \ldots, n$ we have $a_j \in (A\widehat{\mathbf{x}})_{f(j)} = A_{x(j)}$, so that $\boldsymbol{\iota}_{\mathbf{x}}^A$ is an element of $\mathcal{C}_n[L; A]$. We shall define $\sigma_n^{L,A}$ on elements of this form and extend using Lemma 5.1.10.

The key observation is this: intuitively, Equation (5.8) is a kind of formal identity in the unknowns $f(0), \ldots, f(n)$ and $a(0), \ldots, a(n)$, subject to the $f(j)$ being commuting idempotents; therefore equality must be preserved when we "evaluate" each $f(j)$ at $x(j)$. With this observation in mind, we define a function $\widetilde{\sigma}_n^{L,A} : \left(\coprod_{x \in L} A_x\right)^{n+1} \to \mathcal{C}_{n+1}[L; A]$ as follows: given $a_0, \ldots, a_n \in \coprod_{x \in L}$, let $e(j) = \text{base}(a_j)$ for $j = 0, \ldots, n$ and set

$$\boxed{\widetilde{\sigma}_n^{L,A}(a_0, \ldots, a_n) := \mathsf{Tr}_{n+1}^{\widehat{\mathbf{e}}}\psi_{A\widehat{\mathbf{e}}}\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{e}}}(a_0, a_1, \ldots, a_n)}$$

We claim that $\widetilde{\sigma}_n^{L,A}$ has a bounded linear extension to $\mathfrak{alg}_{L,A}$. To see this, recall from Lemma 5.1.10 that $\widetilde{\sigma}_n^{L,A}$ has a linear extension of norm $\leq K$ if and only if, for every $(x(0), \ldots, x(n)) \in L^{n+1}$, the restriction of $\widetilde{\sigma}_n^{L,A}$ to $A_{x(0)} \times \ldots \times A_{x(n)}$ is multilinear with norm $\leq K$. But by the way we have defined $\widetilde{\sigma}_n^{L,A}$, for fixed $(x(0), \ldots, x(n)) \in L^{n+1}$ we have

$$\widetilde{\sigma}_n^{L,A}\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}} = \mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}}\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}$$

and the RHS is clearly multilinear, since $\boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}$ is multilinear and $\mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}}\psi_{A\widehat{\mathbf{x}}}$ is linear.

Moreover, for any $(a_0, \dots, a_n)$ we have

$$\left\| \widetilde{\sigma}_n^{L,A} \boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}(a_0, \dots, a_n) \right\| \leq \|\psi_{A\widehat{\mathbf{x}}}\| \, \|a_0\| \dots \|a_n\|$$

and we saw earlier that $\|\psi_{A\widehat{\mathbf{x}}}\| \leq K_n$ for some constant K_n that depends only on n. We may now invoke Lemma 5.1.10 to deduce that $\widetilde{\sigma}_n^{L,A}$ has a bounded $(n+1)$-linear extension to $\mathfrak{alg}_{L,A}$, as claimed. This extension may be canonically identified with a bounded *linear* map $\mathcal{C}_n[L;A] \to \mathcal{C}_{n+1}[L;A]$, which we denote by $\sigma_n^{L,A}$.

At this point, we have constructed for each $(L, A) \in \mathbf{SA}$ a bounded linear map $\sigma_n^{L,A} : \mathcal{C}_n[L;A] \to \mathcal{C}_{n+1}[L;A]$, which satisfies $\|\sigma_n^{L,A}\| \leq K_n$ for some constant K_n independent of L and A. So in particular our family $\sigma_n^{\bullet}$ satisfies condition (U).

Next, we shall prove that the family $\sigma_n^{\bullet}$ satisfies condition (T). To do this we must show that the diagram

$$\begin{array}{ccc} \mathcal{C}_n[H;B] & \xrightarrow{\quad \sigma_n^{H,B} \quad} & \mathcal{C}_{n+1}[H;B] \\ {\scriptstyle \mathsf{Tr}_n^{\alpha}} \downarrow & & \downarrow {\scriptstyle \mathsf{Tr}_{n+1}^{\alpha}} \\ \mathcal{C}_n[L;A] & \xrightarrow[\quad \sigma_n^{L,A} \quad]{} & \mathcal{C}_{n+1}[L;A] \end{array}$$

commutes whenever we have a semilattice homomorphism $H \xrightarrow{\alpha} L$ such that $B = A\alpha$.

By linearity and continuity it suffices to check this on block elements. Let $y(0),\dots,$ $y(n+1) \in H$: for each j, let $x(j)$ denote $\alpha y(j) \in L$, and let $a_j \in B_{y(j)} = A_{x(j)}$. Recall that by Lemma 5.2.4

$$\mathsf{Tr}_{n+1}^{\alpha} \mathsf{Tr}_{n+1}^{\widehat{\mathbf{y}}} = \mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}}$$

and

$$\mathsf{Tr}_n^{\alpha} \boldsymbol{\iota}_{\mathbf{y}}^{B}(\mathbf{a}) = \boldsymbol{\iota}_{\mathbf{x}}^{A}(\mathbf{a}) \ .$$

Therefore,

$$\mathsf{Tr}_{n+1}^{\alpha} \sigma_n^{H,B} \boldsymbol{\iota}_{\mathbf{y}}^{B}(\mathbf{a}) = \mathsf{Tr}_{n+1}^{\alpha} \mathsf{Tr}_{n+1}^{\widehat{\mathbf{y}}} \psi_{B\mathbf{y}} \boldsymbol{\iota}_{\mathbf{f}}^{B\widehat{\mathbf{y}}}(\mathbf{a}) = \mathsf{Tr}_{n+1}^{\alpha\widehat{\mathbf{y}}} \psi_{A\alpha\widehat{\mathbf{y}}} \boldsymbol{\iota}_{\mathbf{f}}^{A\alpha\widehat{\mathbf{y}}}(\mathbf{a})$$

and

$$\sigma_n^{L,A} \mathsf{Tr}_n^{\alpha} \boldsymbol{\iota}_{\mathbf{y}}^{B}(\mathbf{a}) = \sigma_n^{L,A} \boldsymbol{\iota}_{\mathbf{x}}^{A}(\mathbf{a}) = \mathsf{Tr}_{n+1}^{\widehat{\mathbf{x}}} \psi_{A\widehat{\mathbf{x}}} \boldsymbol{\iota}_{\mathbf{f}}^{A\widehat{\mathbf{x}}}(\mathbf{a})$$

Since $\alpha\widehat{\mathbf{y}} = \widehat{\mathbf{x}}$, the previous two equations combine to give

$$\mathsf{Tr}_{n+1}^{\alpha} \sigma_n^{H,B} \boldsymbol{\iota}_{\mathbf{y}}^{B}(\mathbf{a}) = \sigma_n^{L,A} \mathsf{Tr}_n^{\alpha} \boldsymbol{\iota}_{\mathbf{y}}^{B}(\mathbf{a})$$

as required.

It remains only to verify conditions (R) and Equation (5.4). Let $(L, A) \in \mathbf{SA}$: we must show that $\pi^{L,A}_{n+1}\sigma_n = \sigma^{L,A}_n$ and $\mathsf{d}^{L,A}_n\sigma^{L,A}_n = \pi^{L,A}_n - \sigma^{L,A}_{n-1}\mathsf{d}^{L,A}$.

Since we now know that $\sigma^{L,A}_n$ is a bounded multilinear map $\mathcal{C}_n[L; A] \to \mathcal{C}_{n+1}[L; A]$, it suffices by linearity and continuity to check both putative identities on elementary tensors of the form $\boldsymbol{\iota}^A_{\mathbf{x}}\mathbf{a}$.

This is now mere diagram-chasing, given the machinery set up earlier. Keep in mind the following diagram:

$$\begin{array}{ccccc}
\mathcal{C}_{n-1}[F; A\widehat{\mathbf{x}}] & \underset{\sigma^{F,A\widehat{\mathbf{x}}}_{n-1}}{\overset{\mathsf{d}^{F,A\widehat{\mathbf{x}}}_{n-1}}{\rightleftarrows}} & \mathcal{C}_n[F; A\widehat{\mathbf{x}}] & \xleftarrow{\mathsf{d}^{F,A\widehat{\mathbf{x}}}_{n}} & \mathcal{C}_{n+1}[F; A\widehat{\mathbf{x}}] \\
\downarrow{\scriptstyle \mathsf{Tr}^{\widehat{\mathbf{x}}}_{n-1}} & & \downarrow{\scriptstyle \mathsf{Tr}^{\widehat{\mathbf{x}}}_{n}} & & \\
\mathcal{C}_{n-1}[L; A] & \underset{\sigma^{L,A}_{n-1}}{\overset{\mathsf{d}^{L,A}_{n-1}}{\rightleftarrows}} & \mathcal{C}_n[L; A] & \underset{- - -}{\overset{\mathsf{d}^{L,A}_{n}}{\rightleftarrows}} & \mathcal{C}_{n+1}[L; A]
\end{array}$$

So:

$$\begin{aligned}
&\pi^{L,A}_{n+1}\sigma^{L,A}_n(\boldsymbol{\iota}^A_{\mathbf{x}}\mathbf{a}) \\
&= \pi^{L,A}_{n+1}\mathsf{Tr}^{\widehat{\mathbf{x}}}_{n+1}\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}^{A\widehat{\mathbf{x}}}_{\mathbf{f}}(\mathbf{a}) && \text{(by definition)} \\
&= \mathsf{Tr}^{\widehat{\mathbf{x}}}_{n+1}\pi^{F,A\widehat{\mathbf{x}}}\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}^{A\widehat{\mathbf{x}}}_{\mathbf{f}}(\mathbf{a}) && (\pi \text{ commutes with transfer}) \\
&= \mathsf{Tr}^{\widehat{\mathbf{x}}}_{n+1}\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}^{A\widehat{\mathbf{x}}}_{\mathbf{f}}(\mathbf{a}) && \text{(Equation (5.9))} \\
&= \sigma^{L,A}_n(\boldsymbol{\iota}^A_{\mathbf{x}}\mathbf{a})
\end{aligned}$$

Secondly (and *this* is where we finally make use of condition (T) in our induction):

$$\begin{aligned}
&\mathsf{d}^{L,A}_n\sigma^{L,A}_n(\boldsymbol{\iota}^A_{\mathbf{x}}\mathbf{a}) \\
&= \mathsf{d}^{L,A}_n\mathsf{Tr}^{\widehat{\mathbf{x}}}_{n+1}\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}^{A\widehat{\mathbf{x}}}_{\mathbf{f}}(\mathbf{a}) && \text{(by definition)} \\
&= \mathsf{Tr}^{\widehat{\mathbf{x}}}_{n}\mathsf{d}^{F,A\widehat{\mathbf{x}}}_n\psi_{A\widehat{\mathbf{x}}}\boldsymbol{\iota}^{A\widehat{\mathbf{x}}}_{\mathbf{f}}(\mathbf{a}) && \text{(transfer is a chain map)} \\
&= \mathsf{Tr}^{\widehat{\mathbf{x}}}_{n}\left(\pi^{F,A\widehat{\mathbf{x}}}_n - \sigma^{F,A\widehat{\mathbf{x}}}_{n-1}\mathsf{d}^{F,A\widehat{\mathbf{x}}}_{n-1}\right)(\boldsymbol{\iota}^B_{\mathbf{f}}\mathbf{a}) && \text{(by the “formal identity” (5.8))} \\
&= \left(\mathsf{Tr}^{\widehat{\mathbf{x}}}_{n}\pi^{F,A\widehat{\mathbf{x}}}_n - \mathsf{Tr}^{\widehat{\mathbf{x}}}_{n}\sigma^{F,A\widehat{\mathbf{x}}}_{n-1}\mathsf{d}^{F,A\widehat{\mathbf{x}}}_{n-1}\right)(\boldsymbol{\iota}^B_{\mathbf{f}}\mathbf{a}) \\
&= \left(\mathsf{Tr}^{\widehat{\mathbf{x}}}_{n}\pi^{F,A\widehat{\mathbf{x}}}_n - \sigma^{L,A}_{n-1}\mathsf{Tr}^{\widehat{\mathbf{x}}}_{n-1}\mathsf{d}^{F,A\widehat{\mathbf{x}}}_{n-1}\right)(\boldsymbol{\iota}^B_{\mathbf{f}}\mathbf{a}) && \text{(condition (T) for } \sigma^{\bullet}_{n-1}) \\
&= \left(\pi^{L,A}_n\mathsf{Tr}^{\widehat{\mathbf{x}}}_{n} - \sigma^{L,A}_{n-1}\mathsf{d}^{L,A}_{n-1}\mathsf{Tr}^{\widehat{\mathbf{x}}}_{n}\right)(\boldsymbol{\iota}^B_{\mathbf{f}}\mathbf{a}) && (\pi \text{ and } \mathsf{d} \text{ commute with transfer}) \\
&= \left(\pi^{L,A}_n - \sigma^{L,A}_{n-1}\mathsf{d}^{L,A}_{n-1}\right)\mathsf{Tr}^{\widehat{\mathbf{x}}}_{n}(\boldsymbol{\iota}^{A\widehat{\mathbf{x}}}_{\mathbf{f}}\mathbf{a}) \\
&= \left(\pi^{L,A}_n - \sigma^{L,A}_{n-1}\mathsf{d}^{L,A}_{n-1}\right)(\boldsymbol{\iota}^{A}_{\mathbf{x}}\mathbf{a}) && \text{(defining property of } \widehat{\mathbf{x}})
\end{aligned}$$

Thus the family $(\sigma^{L,A}_n)_{(L,A)\in\mathbf{SA}}$ satisfies Equation (5.4), and this concludes the proof of Proposition 5.6.3. In view of the previous section, this completes the proof of Theorem 5.4.2.

5.7 ℓ^1-algebras of Clifford semigroups

In this section we apply the preceding work on $\mathfrak{alg}_{L,A}$ to obtain generalisations of some results in Bowling and Duncan's paper [4].

By [4, Thm 2.1] the ℓ^1-algebra of *any* Clifford semigroup is weakly amenable. Using the results of the previous section, we can extend some of the results of Bowling and Duncan to higher cohomology groups, if we make further assumptions on the Clifford semigroup S.

Corollary 5.7.1 (Simplicial triviality for Clifford semigroups of amenable groups). *Let $S = \coprod_{e\in L} G_e$ be a Clifford semigroup over the semilattice L. Suppose that each G_e is amenable. Then $\mathcal{H}^n(\ell^1(S), \ell^1(S)') = 0$.*

Proof. Recall that if G_e is amenable then $\ell^1(G_e)$ is amenable with constant 1, so that the simplicial chain complex $\mathcal{C}_*(\ell^1(G_e, \ell^1(G_e))$ is weakly split in degrees 1 and above with constants independent of e. Since

$$\mathcal{C}_n^{\rm diag}(\ell^1(S), \ell^1(S)) = \bigoplus^{(1)}_{e\in L} \mathcal{C}_n(\ell^1(G_e), \ell^1(G_e))$$

it follows that $\mathcal{H}_n^{\rm diag}(\ell^1(S), \ell^1(S)) = 0$ for all $n \geq 1$. Hence by Theorem 5.4.1,

$$\mathcal{H}_n(\ell^1(S), \ell^1(S)) = \mathcal{H}_n^{\rm diag}(\ell^1(S), \ell^1(S)) = 0 \qquad \text{for all } n \geq 1$$

and on dualising we obtain the desired result for simplicial cohomology. □

Back in Section 2.3.3, we observed (Corollary 2.3.5) that simplicial triviality of a unital, commutative Banach algebra was almost enough to imply the vanishing of cohomology with arbitrary symmetric coefficients. In the current setting of ℓ^1-convolution algebras we can now eliminate the word "almost" and obtain the following result, which appears to be new.

Theorem 5.7.2. *Let S be a commutative Clifford semigroup and let X be any symmetric Banach $\ell^1(S)$-bimodule. Then $\mathcal{H}^n(\ell^1(S), X) = 0$ for all $n \geq 1$.*

Proof. We first note that $\mathcal{H}^n(\ell^1(S), X) \cong \mathcal{H}^n(\ell^1(S)^\#, X_1)$ where X_1 has underlying Banach space X and is the natural *unital* bimodule induced from X.

Now $\ell^1(S)^\# = \ell^1(S^\#)$ where $S^\#$ is itself a (unital) commutative Clifford semigroup. By Corollary 5.7.1, $\ell^1(S^\#)$ is simplicially trivial, i.e. the complex

$$\mathcal{C}_0(\ell^1(S^\#), \ell^1(S^\#)) \xleftarrow{\mathsf{d}_0} \mathcal{C}_1(S^\#), \ell^1(S^\#)) \xleftarrow{\mathsf{d}_1} \mathcal{C}_2(S^\#), \ell^1(S^\#)) \xleftarrow{\mathsf{d}_2} \dots$$

is an exact sequence in Ban.

Since $\ell^1(S^\#)$ is commutative $\mathsf{d}_0 = 0$, so d_1 surjects onto $\mathcal{C}_1(\ell^1(S^\#), \ell^1(S^\#))$; then, since each $\mathcal{C}_n(\ell^1(S^\#), \ell^1(S^\#))$ is an ℓ^1-space, we may inductively construct a bounded linear splitting of the complex $(*)$. Therefore Corollary 2.3.5 applies and we deduce that $\mathcal{H}^n(\ell^1(S)^\#, X_1) = 0$ for all $n \geq 1$, as required. ☐

Remark. Note that the case $n = 1$ follows from [4, Thm 2.1] and the well-known fact that weak amenability for commutative Banach algebras forces all bounded derivations with symmetric coefficients to vanish.

What can be said for more general coefficients? Here matters are more delicate, and one difficulty seems to be the lack of a chain projection onto the subcomplex of normalised cochains (recall that in the case of *simplicial* cochains we could find such a projection which had additional good properties).

We note that in [4, Example 3.3] the authors give a telling construction of a Clifford semigroup S with the following properties:

1. each constituent group of S is amenable (in fact, can be taken to be the symmetric group on three objects);
2. there exists a non-inner bounded derivation $\ell^1(S) \to \ell^1(S)$.

Thus in Theorem 5.7.2 the condition of commutativity is essential. On the other hand, if we impose restrictions on the semilattice of idempotents in S then one can say more. We close this section with some further results from Bowling and Duncan's paper.

Theorem 5.7.3 (after [4, Thm 3.1]). *Let $S = \bigcup_{e\in L} G_e$ be a Clifford semigroup with identity 1_S, and suppose that $eG_1 = G_e$ (or, equivalently, that each transition homomorphism $G_e \to G_f$ is surjective). Let X be an $\ell^1(S)$-bimodule on which $\ell^1(L)$ acts centrally, and suppose that $\mathcal{H}^1(\ell^1(G_1), X) = 0$. Then $\mathcal{H}^1(\ell^1(S), X) = 0$.*

Proof. The special case where $X = \ell^1(S)$ is [4, Thm 3.1]; on examining the proof it is clear that we can relax the hypotheses on the coefficient module X to the ones stated above. ☐

Theorem 5.7.4 ([4, Thm 3.2]). *Let S be a Clifford semigroup whose underlying semilattice is finite. Then $\mathcal{H}^1(\ell^1(S), \ell^1(S)) = 0$.*

5.8 Future generalisations?

Early on in this chapter we noted that a strong semilattice $(L, \succeq) \xrightarrow{A} \mathsf{BAlg}_1^+$ could be viewed as a presheaf of Banach algebras on the partially ordered set $(L, \preceq)$. In a series of papers in the 1980s, Gerstenhaber and Schack showed that to any presheaf of algebras on a poset, one can associate a single algebra which encodes some of the properties of the individual algebras and of the underlying poset. (See for example the paper [11].)

In this context, our "normalisation result" can be interpreted *very loosely* as follows: our underlying poset $(L, \preceq)$ is homologically trivial, and so in the cohomology of our algebra $\mathfrak{alg}_{L,A}$ the contribution from $\ell^1(L)$ collapses to zero; thus all the cohomology for $\mathfrak{alg}_{L,A}$ must come from homological obstructions for each constituent algebra A_e.

Of course the previous paragraph is fallacious as mathematical reasoning, but it suggests that we could attempt to extend the techniques of this chapter to attempt a Banach-algebraic analogue of the results in [11]. We leave this as future work to be pursued.

Chapter 6

Homological smoothness for commutative Banach algebras

6.1 Motivation

In commutative algebra there is a notion of "smoothness" for k-algebras, which admits many equivalent characterisations (see the appendix by M. Ronco in [23] for a comprehensive account). Two of these characterisations suggest they might lend themselves to Banach-algebraic analogues:

Definition 6.1.1. Let k be a field and R a commutative k-algebra. We say that R is smooth over k if it satisfies either of the following equivalent conditions:

(a) given any commutative k-algebra C with a square-zero ideal J, and any k-algebra homomorphism $f : \mathsf{R} \to \mathsf{C}/\mathsf{J}$, there exists a lift of f to a k-algebra homomorphism $g : \mathsf{R} \to \mathsf{C}$;

(b) for every symmetric R-bimodule M, the second symmetric Hochschild cohomology group of R with coefficients in M is zero.

In this chapter we shall investigate the natural Banach-algebraic analogues of conditions (a) and (b), eventually proving that they are equivalent. We shall see that perhaps these conditions are too stringent; this will be discussed in due course.

Throughout this chapter we shall abbreviate the phrase "commutative Banach algebra" to CBA. I would like to acknowledge helpful comments from Prof. Niels Grønbæk and Dr. Zinaida Lykova which led to clarifications of some of the following material, especially the discussion of free products at the end of Section 6.2. Of course any remaining defects in the terminology to follow are solely my responsibility.

6.2 A definition in terms of lifting problems

In this section we introduce an analogous definition of smoothness for CBAs, in terms of certain lifting problems, and show how this definition is equivalent to a vanishing condition on 2nd symmetric cohomology.

We first adopt some *ad hoc* terminology to avoid unnecessary repetition.

Definition 6.2.1 (Abelian lifting problems). Let A be a CBA: then an abelian lifting problem for A is a triple $(I, R; f)$ satisfying the following conditions:

- R is a CBA
- I is a closed ideal in R
- f is a continuous algebra homomorphism from A to R/I.

We shall abbreviate the phrase "abelian lifting problem" to ALP.

Given an ALP $(I, R; f)$ for A, we say it is:

- square-zero if the ideal I is square-zero, i.e. if $x^2 = 0$ for all $x \in I$;
- linear if there exists a bounded linear map $h : A \to R$ such that $qh = \mathsf{id}$, where $q : R \to R/I$ is the quotient homomorphism;
- solvable if there exists a continuous algebra homomorphism $g : A \to R$ such that $qg = f$.

Remark (Examples of linear lifting problems). Let $(I, R; f)$ be an ALP for the CBA A. The lifting problem will be linear if I is complemented in R; it will also be linear if A is isomorphic as a Banach space to ℓ^1.

Definition 6.2.2 (Smooth CBAs). Let A be a CBA. We say A is smooth if each *linear, square-zero* ALP for A is solvable.

A natural question is under what conditions such a lift will be unique. The following result gives a sufficient condition, which is surely known in the purely algebraic case but for which I cannot find a reference.

Lemma 6.2.3. *Let A, R be a CBA, and let I be a closed square-zero ideal in R. Let θ, ϕ be continuous homomorphisms from A to R such that $\theta - \phi$ takes values in I. Then $\theta - \phi$ is a bounded derivation from A to I_ϕ, where I_ϕ denotes the symmetric A-bimodule with underlying set I and action given by*

$$a \cdot x = x \cdot a = \phi(a)x \qquad (a \in A, x \in I) \ .$$

In particular, if A is weakly amenable then a linear, square-zero lifting problem $(I, R; f)$ admits at most one solution.

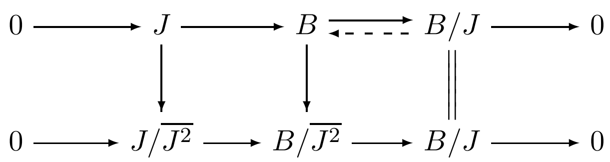
Proof. This is an easy calculation: if $a, b \in A$ then

$$\begin{aligned} &\phi(a)(\theta-\phi)(b) - (\theta-\phi)(ab) + (\theta-\phi)(a)\phi(b) \\ &= \phi(a)\theta(b) - \theta(a)\theta(b) + \theta(a)\phi(b) - \phi(a)\phi(b) \\ &= -(\theta-\phi)(a)\cdot(\theta-\phi)(b) \\ &= 0 \end{aligned}$$

(where the last step follows because $\theta - \phi$ takes values in a square-zero ideal). Thus $\theta - \phi$ is indeed a derivation, and the final remark of the lemma follows merely by unpacking the definitions involved. □

Remark. One idea behind Definition 6.2.2 is that one can iterate the lifting process to handle nilpotent ALPs. To be more precise, let J be a closed, linearly complemented ideal in a CBA B, and let $f : A \to B/J$ be a continuous homomorphism where A is smooth.

A simple and standard diagram chase shows that the following diagram has exact rows and commutes:



The bottom row is thus a singular split extension: since A is smooth, we can therefore lift f to a continuous homomorphism $f_1 : A \to B/\overline{J^2}$.

Iterating this procedure one gets successive lifts of f to continuous homomorphisms $f_n : A \to B/\overline{J^{2^n}}$. If J is imoreover a nilpotent ideal, then for some n $f_n : A \to B$ is a lift of f.

(All this is standard knowledge in the purely algebraic case.)

It is not hard to find finite-dimensional commutative algebras that are not smooth in the "purely algebraic" sense, and which are therefore not smooth as CBAs. Here is one example.

Example 6.2.4 (A finite-dimensional CBA that is not smooth). Let A be the algebra

$$\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \;:\; a, b \in \mathbb{C} \right\}$$

and let t denote the element $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

Let J be the ideal spanned by t^2, i.e. the set of all $x \in A$ which are zero in all entries except possibly the top-right. An easy computation shows that $\{x \in A : x^2 = 0\} = J$: in particular, J is a square-zero ideal in A.

We claim that A/J is not smooth. To do this, consider the linear, square-zero ALP $(J, A; \mathsf{id})$ for A/J. Writing q for the quotient homomorphism $A \xrightarrow{q} A/J$, we observe that $q(t)$ is a non-zero square-zero element in A/J. If our ALP was solvable there would exist a square-zero element $y \in A$ such that $q(y) = q(t)$; but since q annihilates the space of all square-zero elements this would imply $0 = q(t)$, a contradiction.

One can easily generalise this construction to show that for any $n \geq 2$, the algebra generated by a single nilpotent of order n fails to be smooth. We omit the details.

Smoothness and unitisations

We do not assume in Definitions 6.2.1 and 6.2.2 that our CBAs are unital.

The following simple calculation shows that if our lifting problem is unital and has a solution, then the solution is automatically unital.

Lemma 6.2.5. *Let B, C be unital CBAs; let I be a square-zero ideal in C and let $f : B \to C/I$ be a continuous unital homomorphism. Suppose $g : B \to C$ is a continuous homomorphism which lifts f; then g is automatically unital.*

Proof. Let $u = g(1_B)$; then $q(u) = 1_{(C/I)} = q(1_C)$. Hence $u - 1_C \in I$, and since $I^2 = 0$ this implies that

$$0 = (u - 1_C)^2 = u^2 - 2u + 1_C$$

Since u is an idempotent we conclude that $1_C - u = 0$, and therefore g is unital as claimed. ∎

Proposition 6.2.6. *If A is a smooth CBA, so is its forced unitisation $A^{\#}$.*

We shall see later that one can deduce this from a cohomological characterisation of smoothness. However, it seems worth providing a self-contained proof which uses only the original definition in terms of lifting problems.

Proof. Suppose then that A is a smooth CBA, and let $(I, R; f)$ be a ALP for $A^{\#}$. We know there exists a bounded linear map $h : A \to R$ such that $qh = f$.

Let $\iota : A \to A^{\#}$ be the inclusion homomorphism; then clearly $(I, R; f\iota)$ is a square-zero ALP for A, and since $qh\iota = f\iota$ this ALP is linear. By smoothness of A we deduce that there exists a continuous homomorphism $g : A \to R$ such that $qg = f\iota$.

Let $\mathsf{1}$ denotes the adjoined unit of $A^{\#}$, and observe that $f(\mathsf{1})$ is an idempotent element in R/I. Suppose we can find an *idempotent* $e \in R$ such that $q(e) = f(\mathsf{1})$: then if we define $\widetilde{g} : A^{\#} \to R$ by

$$\widetilde{g}(a, \lambda\mathsf{1}) := g(a)e + \lambda e$$

an easy calculation shows that $\widetilde{g}$ is a continuous homomorphism and that it lifts f.

Therefore it remains only to construct such an idempotent e. This could be done by appeal to general results on lifting idempotents modulo the radical (an old idea going as far back as Jacobson's notion of *SBI rings*) but we give a direct and rather *ad hoc* construction as follows. (The calculations are slightly obscure because we do not assume R has an identity.)

Let y be any pre-image of $f(\mathsf{1})$ in R. *If* there exists e in R with $e^2 = e$ and $q(e) = \mathsf{1} = q(y)$ then $e - y \in I$; in particular,

$$0 = (e - y)^2 = e^2 - 2ey + y^2 = e - 2ey + y^2$$

so that

$$e = 2ey - y^2 \tag{6.1}$$

Multiplying through by e gives $e = 2ey - ey^2$; subtracting this from (6.1) gives

$$y^2 = ey^2 \; . \tag{6.2}$$

Multiplying (6.1) through by y, and simplifying using (6.2), we get

$$ey = (2e - y)y^2 = 2ey^2 - y^3 = 2y^2 - y^3 \; ;$$

substituting back into (6.1) finally yields

$$e = 2(2y^2 - y^3) - y^2 = 3y^2 - 2y^3 \ .$$

Thus if an e with the required properties exists, it is uniquely determined by y. This suggests that as an *Ansatz* we *define* e to be $3y^2 - 2y^3$. Clearly

$$q(3y^2 - 2y^3) = 3f(1)^2 - 2f(1)^2 = 3f(1) - 2f(1) = f(1)$$

so we need only check that $3y^2 - 2y^3$ is an idempotent. To do this, observe that since $q(y^2 - y) = 0$, $y^2 - y$ lies in I and hence squares to zero, i.e.

$$0 = (y^2 - y)^2 = y^4 - 2y^3 + y^2 \ .$$

Hence

$$(3y - 2y^2)^2 = 9y^2 - 12y^3 + 4y^4 = 5y^2 - 4y^3 \ ,$$

which gives

$$\begin{aligned}(3y^2 - 2y^3)^2 = y^2(3y - 2y^2)^2 &= y^2(5y^2 - 4y^3) \\ &= y(5y^3 - 4(2y^3 - y^2)) \\ &= y(4y^2 - 3y^3) \\ &= 4y^3 - 3(2y^3 - y^2) \qquad = 3y^2 - 2y^3\end{aligned}$$

as required. □

Proposition 6.2.7. *If $A^\#$ is smooth, then so is A.*

Proof. Let $(I, R; f)$ be a linear, square-zero ALP for A. The homomorphism $f : A \to R/I$ has a unique extension to a unital homomorphism $f^\# : A^\# \to (R/I)^\# \cong R^\#/I$. Since $A^\#$ is smooth there exists a homomorphism $\widetilde{g} : A^\# \to R^\#/I$ which lifts $f^\#$.

Let $\widetilde{g}(a) = (g(a), \psi(a))$ where $g : A \to R$ and $\psi : A \to \mathbb{C}$ are bounded linear. Since $q\widetilde{g}(a) = f(a) \in R/I$, it follows that $\psi(a) = 0$. Hence the restriction of $\widetilde{g}$ to A yields a well-defined homomorphism $g : A \to R/I$ which lifts f. Thus the original ALP $(R, I; f)$ is solvable, and therefore A is smooth. □

Smoothness of tensor products

Proposition 6.2.8. *If A, B are smooth unital CBAs, then so is $A \widehat{\otimes} B$.*

Proof. Let $(I, R; f)$ be a linear square-zero ALP for $A\widehat{\otimes}B$: then f is a continuous homomorphism from $A\widehat{\otimes}B$ to R/I and by assumption there exists a bounded linear map $h : A\widehat{\otimes}B \to R$ such that $qh = f$ (q denoting the quotient homomorphism from R to R/I).

Define $f_A : A \to R/I$ and $h_A : A \to R$ by

$$f_A(a) := f(a\otimes 1_B) \quad \text{and} \quad h_A(a) := h(a\otimes 1_B) \qquad (a \in A)\,;$$

then f_A is a continuous homomorphism, h_A is a continuous linear map and $qh_A = f_A$. Thus $(I, R; f_A)$ is a linear-square-zero ALP for A and so by smoothness of A we can lift f_A to a continuous homomorphism $g_A : A \to R$.

By symmetry, if we define $f_B(b) = f(1_A\widehat{\otimes}b)$ then we can lift f_B to a continuous homomorphism $g_B : B \to R$. We then define $g : A\widehat{\otimes}B \to R$ by

$$g(a\otimes b) := g_A(a)g_B(b) \qquad (a \in A, b \in B)\,;$$

a quick check shows that g is a continuous homomorphism, and

$$\begin{aligned} qg(a\otimes b) = q\big[g_A(a)g_B(b)\big] &= qg_A(a)\, qg_B(b) \\ &= f_A(a)f_B(b) \\ &= f(a\otimes 1_B)f(1_A\otimes b) = f(a\otimes b) \qquad (a \in A, b \in B) \end{aligned}$$

so that g is a lift of f. Thus our original ALP was solvable and we conclude that $A\widehat{\otimes}B$ is smooth, as required. □

Remark. The core of this proof is the observation that $A\widehat{\otimes}B$ is the coproduct of the unital CBAs A and B. One might hope, then, to obtain an analogous result for "noncommutatively smooth Banach algebras" where the tensor product is replaced by some kind of "free product" of Banach algebras.

We shall return to this problem once we have established the connections between lifting problems and cohomology (Corollary 6.2.10). In particular it will be shown that the naive attempt to generalise Proposition 6.2.8 to free products does not work.

Smoothness via special lifting problems

As is often the case with properties defined in terms of a class of "diagram completion problems", it suffices to consider a certain subclass. The next lemma is again one for which I can find no explicit reference, but which is essentially a trivial modification of a familiar argument (see e.g. [30, Propn 9.3.3]).

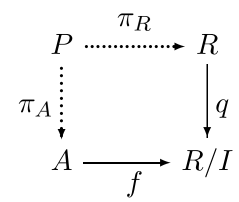
Lemma 6.2.9. *Let A be a CBA. The following are equivalent:*

(1) A is smooth;

(2) every abelian, singular-split extension of A splits as a semidirect product.

Proof. Clearly (1) implies (2), so it suffices to prove the converse implication. Suppose then that Condition (2) holds, and let $(I, R; f)$ be a linear lifting problem for A.

We construct a singular split extension of A as follows. Consider the *pullback*



where $P := \{(a, r) \in A \oplus R \ : \ f(a) = q(r)\}$ and π_A, π_R are the coordinate projections of P to A, R respectively. Clearly P is a closed subspace of $A \oplus R$; and since f and q are homomorphisms, P is a subalgebra of the direct-sum algebra $A \oplus R$. It is immediate that $\ker(\pi_A) \subseteq \{0\} \oplus R$ is a square-zero subalgebra of $A \oplus R$, and hence is a square-zero ideal in P.

By asumption there exists a bounded linear lift of f, which we denote $h : A \to R$. Since h lifts f, $qh(a) = f(a)$ for all $a \in A$; hence we may define a bounded linear map $\rho : A \to P$ by

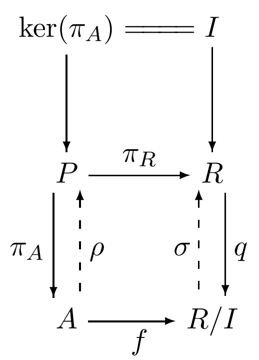
$$\rho(a) := (a, h(a)) \ ,$$

and by construction $\pi_A \rho = \mathsf{id}$.

Therefore $(\ker \pi_A, P; \mathsf{id})$ is a linear, square-zero ALP for A. A quick check shows that

$$\ker(\pi_A) = \{(0, r) \in A \oplus R \ : \ q(r) = 0\} = \{0\} \oplus I$$

giving us a diagram



where all squares commute. By Condition (2) the left-hand extension splits, i.e. there exists a continuous homomorphism $\theta : A \to P$ such that $\pi_A\theta = \mathsf{id}$. Then $\pi_R\theta : A \to R$ is a lift of f, and thus the ALP $(I, R; f)$ is solvable. □

In view of the well-known correspondence between abelian, singular extensions of an algebra and symmetric 2-cocycles on it (see e.g. [19, §I.1.2], [30, Thm 9.3.1]), we have the following.

Corollary 6.2.10. *Let A be a CBA. The following are equivalent:*

(1) A is smooth;

(2) $\mathcal{H}ar\mathcal{H}^2(A, M) = 0$ for every symmetric Banach A-bimodule M.

We shall make this result quantitative in Section 6.4.

Remark. Since $\mathcal{H}ar\mathcal{H}^2(A^\#, M) \cong \mathcal{H}ar\mathcal{H}^2(A, M)$, Corollary 6.2.10 gives an alternative proof that the unitisation of a smooth CBA is itself smooth (Proposition 6.2.6 above).

Remark. Let us note that by combining Corollary 6.2.10 with our earlier result on smoothness of $A\widehat{\otimes}B$ (Proposition 6.2.8), we obtain the following: if A_1, A_2 are unital CBAs such that $\mathcal{H}ar\mathcal{H}^2(A_i, \_\_) = 0$ $(i = 1, 2)$, then $\mathcal{H}ar\mathcal{H}^2(A_1\widehat{\otimes}A_2, \_\_) = 0$. While this could probably be proved more directly, being able to switch between the cohomological and the ALP viewpoints makes the result far more transparent.

Lifting problems in the noncommutative setting?

Although we have restricted attention in this chapter to commutative Banach algebras, it should be clear that Definitions 6.2.1 and 6.2.2 admit generalisations to the noncommutative setting, as follows. Let us say that a Banach algebra $\mathfrak{A}$ is *NC-smooth* if for every square-zero ideal $\mathfrak{I}$ in a Banach algebra $\mathfrak{R}$, and every continuous homomorphism $f : \mathfrak{A} \to \mathfrak{R}/\mathfrak{I}$ which admits a bounded linear lift $h : \mathfrak{A} \to \mathfrak{R}$, there exists a continuous homomorphism $g : \mathfrak{A} \to \mathfrak{R}$ which lifts f. (Without the continuity restrictions, this corresponds to the notion of a *quasi-free algebra*: see for instance [30, 9.3.2].

It can then be shown that there is an analogue of Corollary 6.2.10 for NC-smooth algebras, as follows:

a Banach algebra $\mathfrak{A}$ is NC-smooth if and only if $\mathcal{H}^2(\mathfrak{A}, M) = 0$ for every Banach $\mathfrak{A}$-bimodule M.

We omit the proof since it is almost identical to that of Corollary 6.2.10: see also [30, Propn 9.3.3].

Using this cohomological characterisation, we now see that trying to rerun Proposition 6.2.8 with free products rather than tensor products is doomed. Consider the finite-dimensional, unital commutative algebras

$$\mathfrak{A} = \ell^1(\mathbb{Z}/2\mathbb{Z}) \quad , \quad \mathfrak{B} = \ell^1(\mathbb{Z}/3\mathbb{Z})$$

and note that both $\mathfrak{A}$ and $\mathfrak{B}$ are NC-smooth (this can be seen directly from the definition). Now under any reasonable notion of "free product of Banach algebras" we would expect the free product of $\mathfrak{A}$ and $\mathfrak{B}$ to be the group algebra $\mathfrak{C} = \ell^1((\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}))$. However, $\mathfrak{C}$ *is not smooth*: for by [15, Remark 4.8]

$$\mathcal{H}^2(\mathfrak{C}, \mathbb{C}) \cong \mathcal{H}^2(\ell^1(F_2), \mathbb{C})$$

and as we have observed before in the thesis, $\mathcal{H}^2(\ell^1(F_2), \mathbb{C}) \neq 0$ ([21, Propn 2.8]).

Remark. The underlying reason for this failure to generalise Proposition 6.2.8 is that $A\widehat{\otimes}B$ is the coproduct in the category of unital CBAs and bounded, unital homomorphisms, and not merely in the subcategory of unital CBAs and *contractive* unital homomorphisms. It is the former category in which our notions of smoothness and NC-smoothness are formulated.

In contrast, one can define binary coproducts in the category of unital Banach algebras and contractive unital homomorphisms, but *not always* in the larger category BAlg^+ of unital Banach algebras and bounded unital homomorphisms.

This last point seems to be glossed over in parts of the literature, so we shall briefly give some details. If BAlg^+ had all binary coproducts, then given a unital Banach algebra $\mathfrak{A}$ and *any* doubly power-bounded invertible elements $a, b \in \mathfrak{A}$, there would exist a bounded algebra homomorphism $\ell^1(\mathbb{Z} * \mathbb{Z}) \to \mathfrak{A}$ which sent the standard generators of $\mathbb{Z} * \mathbb{Z}$ to a and b respectively; in particular ab would be doubly power-bounded. As the following example shows, this is usually not the case, and consequently one cannot always form binary coproducts in BAlg^+.

Example 6.2.11 (Doubly power-bounded matrices whose product is not

doubly power-bounded). Let

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad , \quad S = TRT^{-1} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

Clearly $R^2 = I$ and $S^2 = TR^2T^{-1} = I$; in particular R and S are doubly power-bounded. However,

$$RS = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

is *not* power-bounded.

6.3 Examples of smooth and nonsmooth CBAs

Corollary 6.2.10 gives another way for checking that certain CBAs are smooth. In particular, it allows us to exploit a classic result of B.E. Johnson to show that *all* amenable CBAs are smooth.

Corollary 6.3.1 ([21, Propn 8.2]). *If A is an amenable CBA then $\mathcal{H}^2(A, M) = 0$ for all symmetric coefficient modules M. In particular, A is smooth.*

Note that Johnson's result does not follow automatically from the definition of amenability, since we are not restricting ourselves to dual modules.

Remark. In particular $C_0(X)$ is smooth for every locally compact Hausdorff X (this special case is due to Kamowitz). Rephrasing this, every linear, square-zero ALP for $C_0(X)$ is solvable. In this context it seems worth noting that by a result of Albrecht and Ermert, every square-zero ALP for $C_0(X)$ is in fact linear. Combining these results we see that *every nilpotent ALP for $C_0(X)$ is solvable* – which seems a striking reformulation of existing results.

Non-amenable smooth CBAs are harder to find, but we have already seen as a special case of Theorem 5.7.2 that the convolution algebra of any commutative Clifford semigroup $\mathcal{G}$ satisfies $\mathcal{H}^2(\ell^1(\mathcal{G}), M) = 0$. Hence:

Corollary 6.3.2. *Let $\mathcal{G}$ be a commutative Clifford semigroup. Then the convolution algebra $\ell^1(\mathcal{G})$ is smooth.*

We note that the special case of Corollary 6.3.2 where $\mathcal{G}$ is a *semilattice* was already known: see [6, Thm 2.5] and the remark that follows it. Moreover, although

Corollary 6.3.2 appears to be unpublished at the time of writing, it can be read out of the calculations in the preprint [14] for second simplicial cohomology groups of *arbitrary* Clifford semigroups.

Corollary 6.2.10 can also be used to show that certain CBAs are not smooth, by exploiting known calculations of various second symmetric cohomology groups. The following examples are taken from [2]:

Proposition 6.3.3. *The following CBAs are not smooth:*

(i) the disc algebra $A(\mathbb{D})$, and $H^\infty(\mathbb{D})$;

(ii) $\ell^1(\mathbb{Z}_+,\omega)$ whenever ω is an increasing weight sequence;

(iii) $\ell^1(\mathbb{Z}_+,\omega)$ whenever ω is a radical weight sequence.

(iv) $\ell^1((0,1))$ or $\ell^1((0,1)\cap\mathbb{Q})$ with (truncated) convolution multiplication.

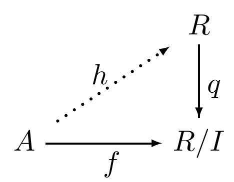
Proof. Let A denote one of the examples in (i) or (ii): then by [2, Thm 3.22] $\mathcal{H}ar\mathcal{H}^2(A,A)\neq 0$. (In fact, the proof in [2] constructs an explicit symmetric 2-cocycle which does not cobound.)

Example (iii) is given by [2, Thm 5.11], which shows that if ω is a radical weight then $\mathcal{H}ar\mathcal{H}^2(\ell^1(\mathbb{Z}_+,\omega),\mathbb{C}_0)\neq 0$, where $\mathbb{C}_0$ denotes the point module corresponding to the character $f\mapsto f(0)$. Similarly, if A= $\ell^1((0,1))$ or $\ell^1((0,1)\cap\mathbb{Q})$, Theorem 5.13 of [2] shows that $\mathcal{H}ar\mathcal{H}^2(A,\mathbb{C}_{\text{ann}})\neq 0$ where $\mathbb{C}_{\text{ann}}$ denotes the 1-dimensional annihilator bimodule, and this gives us the examples in (iv). ☐

Remark. To put the examples just given in some context: in the purely algebraic setting, polynomial rings $\mathbb{C}[z_1,\dots,z_k]$ are the prototypical examples of smooth algebras. Thus our version for Banach algebras is a much more restrictive notion.

6.4 Smoothness and $\mathcal{H}ar\mathcal{H}^2$ (quantitative aspects)

Let



be a linear, square-zero ALP for A. Then I can be made into a symmetric A-bimodule I_h, where the action is defined by

$$a \cdot x = x \cdot a := h(a)x \qquad (a \in A, x \in I) .$$

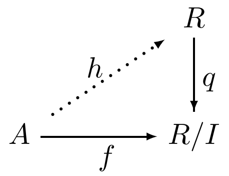
(This is a well-defined action, since for any $a, b \in A$ $h(a)h(b) - h(ab) \in \ker(q) = I$, so that $h(a)h(b)x - h(ab)x \in I^2 = \{0\}$ for any $x \in I$.)

Define $\psi : A \times A \to I_h$ by $\psi(a,b) := h(a)h(b) - h(ab)$. Then as in the purely algebraic case, it follows immediately that ψ satisfies the 2-cocycle identity, and clearly ψ is symmetric. Also immediate from the definitions is the following.

Lemma 6.4.1. *Let $\phi \in \mathcal{L}(A, I)$. Then $\psi = \delta\phi$ if and only if $h+\phi$ is a homomorphism.*

We now give a quantitative formulation of smoothness.

Definition 6.4.2. Let A be a CBA and let $K > 0$. We say that A is universally K-smooth if for every square-zero, linear ALP



where q is a quotient homomorphism, there exists a homomorphism $g : A \to R$ which lifts f and satisfies the estimate $\|g - h\| \leq K\|f\| \max(\|h\|, \|h\|^2)$.

For reference in what follows, we record the following well-known lemma.

Lemma 6.4.3. *Let A be a Banach algebra. Let M be a Banach space, equipped with a left action of A which satisfies*

$$\|a \cdot m\| \leq K\|a\|\|m\| \qquad (a \in A, m \in M)$$

for some constant K independent of a and m.

Define

$$|||m||| := \max\left(\|m\|, \sup_{\|a\| \leq 1} \|am\|\right)$$

Then $|||\_\_|||$ is a norm on M, equivalent to the original one via the inequalities

$$\|m\| \leq |||m||| \leq K\|m\|$$

and $(M, |||\_\_|||)$ is a contractive, left Banach A-module.

Proposition 6.4.4. *Let A be a CBA. Suppose that $\mathcal{H}^2(A, M) = 0$ for every symmetric, contractive Banach A-bimodule M. Then there exists a constant $K_{\rm cob}$, independent of M, such that every symmetric 2-cocycle $\psi : A \times A \to M$ can be cobounded by some cochain ϕ satisfying $\|\phi\| \leq K_{\rm cob}\|\psi\|$.*

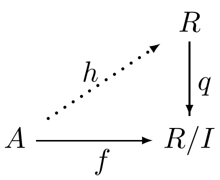
Moreover, A is smooth with constant $2K_{\rm cob}$.

Proof. For the first part, suppose no such constant $K_{\rm cob}$ exists. Then there exists a sequence (X_n, ψ_n), where each X_n is a symmetric, contractive A-bimodule and $\psi_n \in \mathcal{H}ar\mathcal{Z}^2(A, X_n)$, with the following properties:

- $\|\psi_n\| = 1$ for all n;
- if $\psi_n = \delta\varphi_n$ then $\|\varphi_n\| \geq n$.

Take X to be the direct product $\bigoplus^{(\infty)}_{n\in\mathbb{N}} X_n$ (this is where we need the hypothesis that A acts contractively on each X_n) and let $\psi : A \to X$ be the direct product of all the ψ_n. Then $\psi \in \mathcal{H}ar\mathcal{Z}^2(A, X)$, so by the hypothesis on A there exists $\phi \in \mathcal{L}(A, X)$ such that $\delta\phi = \psi$. But if we let $\phi = (\phi_1, \phi_2, \ldots)$, then $\delta\phi_n = \psi_n$ and $n \leq \|\phi_n\| \leq \|\phi\|$ for all n, which gives us a contradiction. Hence there exists some constant $K_{\rm cob}$ with the required properties.

For the second part: let



be a linear, square-zero ALP for A, and just as in the discussion before Lemma 6.4.1 equip I with the structure of a symmetric A-bimodule I_h. Since $I^2 = 0$, for all $a \in A$ and $v, w \in W$ we have $a \cdot w = h(a)w = (h(a) + v)w$. Hence

$$\begin{aligned}
\|a \cdot w\| &= \inf_{v\in V} \|(h(a) + v)w\| \\
&\leq \inf_{\|v\|\leq 1} \|h(a) + v\|\|w\| \\
&= \|qh(a)\|\|w\| \\
&= \|f(a)\|\|w\| \qquad\qquad \leq \|f\|\|a\|\|w\|
\end{aligned}$$

We can renorm I_h to get a contractive A-bimodule $\widetilde{I_h} = (I_h, \|\_\_\|_f)$, where the new norm $\|\_\_\|_f$ is defined by

$$\|w\|_f := \sup_{\|a\|\leq 1} \max\left(\|w\|, \sup_{\|a\|\leq 1} \|a \cdot w\|\right) \quad ;$$

note that by the preceding calculation, $\|w\| \leq \|w\|_f \leq \|f\|\|w\|$ for all $w \in I_h$.

Let $\psi(a,b) := h(a)h(b) - h(ab)$. By the discussion before Lemma 6.4.1 $\psi \in \mathcal{H}ar\mathcal{Z}^2(A, \widetilde{I_h})$; by our hypothesis on A, $\psi - \delta\phi$ for some $\phi \in \mathcal{L}(A, \widetilde{I_h})$ satisfying $\|\phi\| \leq K_{\rm cob}\|\psi\|$. Note that

$$\|\psi(a,b)\|_f \leq \|f\|\|\psi(a,b)\| \leq \|f\| \left(\|h\|^2 + \|h\|\right) \|a\|\|b\|$$

so that $\|\psi\| \leq 2\|f\| \max(\|h\|, \|h\|^2)$.

Let $\sigma(a) := h(a) - \phi(a)$. By Lemma 6.4.1 σ is a homomorphism, and

$$\|\sigma - h\| = \|\phi\| \leq 2K_{\rm cob}\|f\| \max(\|h\|, \|h\|^2)$$

as required. □

Proposition 6.4.5. *Let A be a universally $K_{\rm sm}$-smooth CBA, and let X be a symmetric, contractive A-bimodule. Then every symmetric 2-cocycle with values in X can be cobounded with constant $4K_{\rm sm}$.*

Proof. Let $\psi \in \mathcal{H}ar\mathcal{Z}^2(A, X)$: we aim to produce $\phi \in \mathcal{L}(A, X)$ such that $\delta\phi = \psi$ and $\|\phi\| \leq 4K_{\rm sm}\|\psi\|$. This is trivial if $\psi = 0$ so we may WLOG assume that $\|\psi\| \neq 0$; by considering the graph of the function

$$[1, \infty) \to [0, \infty) \quad , \quad r \mapsto r^2 - r$$

there exists $t > 1$ such that $\|\psi\| = t^2 - t > 0$.

Let C_t be the Banach algebra whose underlying Banach space is the direct sum $A \oplus X$ equipped with the norm

$$\|(a,x)\|_t := t\|a\| + \|x\| \qquad (a \in A, x \in X)$$

and whose multiplication is defined by

$$(a_1, x_1) \cdot (a_2, x_2) := (a_1a_2, a_1x_2 + x_1a_2 + \psi(a_1, a_2))$$

To show this is a well-defined Banach algebra we must check associativity of the multiplication and submultiplicativity of the norm $\|\_\_\|_t$. Associativity follows by a direct use of the 2-cocycle identity (just as in the algebraic case). The submultiplicative property for $\|\_\_\|_t$ is proved as follows: since $t + \|\psi\| = t^2$,

$$\begin{aligned}\|(a_1,x_1)\cdot(a_2,x_2)\|_t &\leq t\|a_1a_2\| + \|a_1x_2\| + \|x_1a_2\| + \|\psi(a_1,a_2)\| \\ &\leq (t+\|\psi\|)\,\|a_1\|\|a_2\| + \|a_1\|\|x_2\| + \|x_1\|\|a_2\| \\ &= t^2\|a_1\|\|a_2\| + \|a_1\|\|x_2\| + \|x_1\|\|a_2\| \\ &\leq (t\|a_1\| + \|x_1\|)(t\|a_2\| + \|x_2\|) \\ &= \|(a_1,x_1)\|_t\|(a_2,x_2)\|_t\end{aligned}$$

for any $a_1, a_2 \in A$ and $x_1, x_2 \in X$, as required.

Thus C_t is indeed a Banach algebra as claimed: it is a *commutative* Banach algebra since X is a symmetric bimodule and ψ a symmetric 2-cocycle. Note for future reference that for any $a \in A$, $x \in X$ we have

$$(a,0)\cdot(0,x) = ax = xa = (0,x)\cdot(0,a) \quad .$$

Let $\iota : X \to C_t$ be the isometric inclusion $x \mapsto (0,x)$; then $\iota(X)$ is a square-zero ideal in C_t. Let A_t denote the Banach algebra which has the same underlying algebra as A but is equipped with the norm

$$\|a\|_t := t\|a\| \ .$$

The projection $q : C_t \to A_t$ given by $(a,x) \mapsto a$ is a quotient homomorphism, with kernel $\iota(X)$. Moreover, if we let $\mathbf{j}$ denote the "identity homomorphism" from A to A_t, then there is a bounded linear map $\rho : A \to C_t$ such that $q\rho = \mathbf{j}$, given by $\rho : a \mapsto (a,0)$.

Therefore, since A is smooth with constant $K_{\rm sm}$, there exists a homomorphism $\sigma : A \to C_t$ such that $q\sigma(a) = a$ for all $a \in A$ and

$$\|\sigma - \rho\| \leq K_{\rm sm}\|\mathbf{j}\| \max(\|\rho\|^2, \|\rho\|) = K_{\rm sm} t \max(t^2, t) = K_{\rm sm} t^3$$

Since $q\rho - q\sigma = 0$, $\rho - \sigma$ takes values in $\ker(q) = \iota(X)$. Therefore there is a bounded linear map $\phi : A \to X$ such that

$$\sigma(a) = \rho(a) - \iota\phi(a) = (a, -\phi(a)) \qquad \text{for all } a \in A.$$

Since σ is a homomorphism, for every $a_1, a_2 \in A$ we have

$$\begin{aligned}0 &= \sigma(a_1)\sigma(a_2) - \sigma(a_1a_2) \\ &= (a_1, -\phi(a_1))\cdot(a_2,-\phi(a_2)) - (a_1a_2, \phi(a_1a_2)) \\ &= \big(a_1a_2, \psi(a_1,a_2) - a_1\phi(a_2) - \phi(a_1)a_2\big) - \big(a_1a_2, \phi(a_1a_2)\big) \\ &= \big(0, \psi(a_1,a_2) - \delta\phi(a_1,a_2)\big) \ ;\end{aligned}$$

hence $\psi = \delta\phi$, where

$$\|\phi\| = \|\iota\phi\| \leq K_{\rm sm} t^3 \ .$$

Recall that $\|\psi\| = t^2 - t > 0$, and that by rescaling ψ we were free to fix t in advance. In particular we could have chosen t so as to minimise the function

$$F(t) = \frac{t^3}{t^2 - t} = \frac{t^2}{t-1} = t + 1 + (t-1)^{-1} \quad ;$$

the minimum turns out to occur at $t = 2$, with $F(2) = 4$. Therefore, running our argument with $t = 2$ we have proved that for every $\psi \in \mathcal{H}ar\mathcal{Z}^2(A, X)$ with $\|\psi\| = 2$ there exists ϕ with $\delta\phi = \psi$ and $\|\phi\| \leq 8K_{\rm sm}$, and the result follows. □

Corollary 6.4.6 (Smooth implies universally smooth). *Let A be a smooth CBA. Then there exists a constant $K > 0$ such that A is K-universally smooth.*

Proof. Let M be a symmetric, contractive A-bimodule; then the proof of Proposition 6.4.5 shows that $\mathcal{H}ar\mathcal{H}^2(A, M) = 0$. By Proposition 6.4.4 there exists a constant $K_{\rm cob}$ such that A is $2K_{\rm cob}$-universally smooth. □

Remark. It should be possible to prove this corollary directly from the definition of "K-universal smoothness", but the obvious attempts seem to get rather messy – the problem is that one has to do various renormings of algebras and homomorphisms.

6.5 Smoothness and $\mathcal{H}_1$

In abstract algebra, smooth algebras are of interest because their Kähler modules are forced to have good properties. We now attempt to mimic such results in the Banach setting, with the Kähler module replaced with the A-module $\mathcal{H}_1(A, A)$.

The first simplicial homology group of an arbitrary CBA need not be Banach, and so the theory of Banach modules might not be applicable. Let us hasten to point out that for smooth CBAs this particular problem never arises:

Lemma 6.5.1. *Let A be a unital, smooth CBA and suppose that $\mathcal{H}ar\mathcal{H}^2(A, A') = 0$. Then $\mathcal{H}_1(A, A)$ is a Banach A-module.*

Proof. Since A is smooth, the second symmetric cohomology vanishes. In particular $\mathcal{H}ar\mathcal{H}^2(A, A') = 0$, which means that

$$\mathcal{C}^1(A, A') \xrightarrow{\delta^1} \mathcal{H}ar\mathcal{C}^2(A, A') \longrightarrow \mathcal{H}ar\mathcal{C}^3(A, A')$$

is exact in the middle. In particular δ^1 has closed range. Now the sequence above is just the dual of the sequence

$$\mathcal{C}_1(A,A) \xleftarrow{\mathsf{d}_1} \mathcal{H}ar\mathcal{C}_2(A,A) \longleftarrow \mathcal{H}ar\mathcal{C}_3(A,A)$$

and so since $(\mathsf{d}_1)' = \delta_1$ has closed range, so does d_1. □

Next we show that smoothness has strong implications not just for the topology on $\mathcal{H}_1(A,A)$ but for the A-module structure.

Proposition 6.5.2. *Suppose A is a smooth, unital CBA. Then*

(i) $\mathcal{H}^2(A,A') = 0$;

(ii) $\mathcal{H}_1(A,A)$ *is Banach, and is a left-projective Banach A-module.*

The proof in [30, Propn 9.3.14] of the algebraic version seems rather indirect; rather than follow its lead, we use Corollary 6.2.10 to get a slicker argument. (The algebraic counterpart of the proof to follow is surely known, but I could not find it in the literature.)

Proof. Part (i) is immediate from our earlier cohomological characterisation of smoothness. Let $0 \to L \to M \xrightarrow{q} N \to 0$ be a linearly split short exact sequence of Banach A-modules. The long exact sequence for *Harrison* cohomology (Lemma 1.6.1) gives us

$$0 \to \mathcal{H}^1(A,L) \to \mathcal{H}^1(A,M) \xrightarrow{q_*} \mathcal{H}^1(A,N) \to \mathcal{H}ar\mathcal{H}^2(A,L) \to \dots$$

and so since A is smooth, we deduce from Corollary 6.2.10 that q_* is surjective. But since ${}_A\mathrm{Hom}(\mathcal{H}_1(A,A), \_\_) \cong \mathsf{Der}(A, \_\_) \cong \mathcal{H}^1(A, \_\_)$, the natural map

$$\widetilde{q_*} : {}_A\mathrm{Hom}(\mathcal{H}_1(A,A), M) \to {}_A\mathrm{Hom}(\mathcal{H}_1(A,A), N)$$

is surjective. This is precisely what it means for $\mathcal{H}_1(A,A)$ to be A-projective. □

At this point it is convenient to introduce the following notation.

Definition 6.5.3 (Notation). If A is a unital CBA we write Ω_A for the Hausdorffification of $\mathcal{H}_1(A,A)$, regarded as a Banach A-module. The map $D_A : A \to \Omega_A$, which is defined by sending a to the equivalence class of $a \otimes 1 - 1 \otimes a$, is called the canonical derivation.

The notation $\Omega_\bullet$ is meant to be suggestive of the Kähler module of a commutative ring. In fact, using Proposition 3.2.1 and Corollary 3.2.5 one can easily check that Ω_A as we have defined it is isomorphic to Runde's "Banach module of differentials" in [28]. We have chosen a different definition from [28] because our interest is in $\mathcal{H}_1(A, A)$, and our definition is meant to reflect this. However, the discussion in [28] gives us the following (which could also be proved directly from our definition of Ω_A):

Lemma 6.5.4 (Universal property of D_A). *Let A be a unital CBA, M a symmetric A-bimodule, and let $\psi : A \to M$ be a bounded derivation. If we write ${}_LM$ for the left A-module obtained by restricting the action on M to one side, then there exists a unique A-module map $g : \Omega_A \to {}_LM$ such that $\psi = g \circ D_A$.*

Intuitively, requiring Ω_A to be 1-sided A-projective is very restrictive (and requiring also that $\Omega_A = \mathcal{H}_1(A, A)$ is even more restrictive). The following calculation makes this more precise, and suggests that our definition of smoothness may be too restrictive for Banach algebras.

Proposition 6.5.5. *Let A be a unital CBA, R its Jacobson radical, and suppose that A/R has the approximation property. If Ω_A is A-projective, then the canonical derivation $D : A \to \Omega_A$ takes values in $\overline{R \cdot \Omega_A}$.*

Corollary 6.5.6. *If A is a smooth, unital, semisimple CBA with the approximation property, then $\mathcal{H}_1(A, A) = \Omega_A = 0$. In particular, A is weakly amenable.*

Proof of Corollary. Since $\mathrm{Rad}(A) = 0$, we deduce from Proposition 6.5.5 that the canonical derivation $D_A : A \to \Omega_A$ is identically zero. By the universal property of D_A (Lemma 6.5.4), this forces $\Omega_A = 0$. □

This gives another tool to show that many familiar CBAs cannot be smooth. For example, the algebra $C^n([0, 1]^m)$ of n-times continuously differentiable functions on the m-cube is semisimple (as are all Banach algebras of functions); it has the approximation property (take a smooth partition of unity and use *uniform* continuity of nth derivatives of elements in the algebra); and it has non-zero, bounded point derivations. Therefore, by Corollary 6.5.6 $C^n([0, 1]^m)$ cannot be smooth.

(Note however that there are many weakly amenable CBAs with the approximation property whose second (symmetric) cohomology is non-zero for certain coefficients, and which therefore cannot be smooth.)

We prove Proposition 6.5.5 via a little lemma that explains where the AP-condition comes in.

Lemma 6.5.7. *Let X, Y be Banach spaces, at least one of which has the approximation property, and suppose that $v \in X \widehat{\otimes} Y$ is such that*

$$(\mathsf{id}_X \widehat{\otimes} \psi) v = 0 \qquad \text{for all } \psi \in Y' \ .$$

Then $v = 0$.

Proof. The hypothesis on v is equivalent to it lying in the kernel of the natural map $X \widehat{\otimes} Y \to X \check{\otimes} Y$; but this map is injective whenever either X or Y has the approximation property ([7, Thm 5.6]; [29, Propn 4.6]). ☐

We also need the following fact about Banach spaces:

Lemma 6.5.8. *Let E be a closed subspace of a Banach space F, and let $\iota : E \to F$ and $q : F \to F/E$ denote the inclusion and quotient maps respectively. Then for any Banach space G, $\iota(E) \otimes G$ is a dense subspace of* $\ker \left(F \widehat{\otimes} G \xrightarrow{q \widehat{\otimes} \mathsf{id}_G} (F/E) \widehat{\otimes} G \right)$.

This result – perhaps part of the folklore of Banach-space theory – may be deduced as a special case of [17, Lemma 3.1]. (I would like to thank Dr. Matthew Daws for helpful conversations on this result, and Prof. Niels Grønbæk for pointing out the reference [17].)

Proof of Proposition 6.5.5. We write $q : A \to A/R$ for the canonical quotient homomorphism. Let $\mu : A \widehat{\otimes} \Omega_A \to \Omega_A$ be the canonical A-module map, where $A \widehat{\otimes} \Omega_A$ is given the structure of a free A-module. Since Ω_A is A-projective, there is an A-module map $\sigma : \Omega_A \to A \widehat{\otimes} \Omega_A$ such that $\mu\sigma = \mathsf{id}$. We can regard σ as an A-bimodule map between *symmetric* A-bimodules.

Let $D_A : A \to \Omega_A$ be the canonical derivation, and consider the derivation $\sigma D_A : A \to A \widehat{\otimes} \Omega_A$. If $\psi \in \Omega_A'$ then $(\mathsf{id} \widehat{\otimes} \psi)\sigma D_A$ is a bounded derivation from A to itself: by the Singer-Wermer theorem for bounded derivations, the range of this derivation is contained in R. Hence

$$q(\mathsf{id} \widehat{\otimes} \psi)\sigma D_A = 0 \qquad \text{for all } \psi \in \Omega_A' \ ; .$$

and since the diagram

$$\begin{array}{ccc} A\widehat{\otimes}\Omega_A & \xrightarrow{\mathsf{id}\widehat{\otimes}\psi} & A \\ \downarrow{\scriptstyle q\widehat{\otimes}\mathsf{id}} & & \downarrow{\scriptstyle q} \\ (A/R)\widehat{\otimes}\Omega_A & \xrightarrow[\mathsf{id}\widehat{\otimes}\psi]{} & A/R \end{array}$$

commutes, it follows that $(\mathsf{id}\widehat{\otimes}\psi)(q\widehat{\otimes}\mathsf{id})\sigma D_A = 0$ for all $\psi \in \Omega'_A$.

By Lemma 6.5.7 we deduce that $(q\widehat{\otimes}\mathsf{id})\sigma D_A = 0$, so that σD_A takes values in $K := \ker\left(A\widehat{\otimes}\Omega_A \xrightarrow{q\widehat{\otimes}\mathsf{id}} (A/R)\widehat{\otimes}\Omega_A\right)$. But then by Lemma 6.5.8 $R\otimes\Omega_A$ is dense in K, so in particuar $\overline{\mu(K)} = \overline{R\cdot\Omega_A}$. We conclude that

$$\mathrm{im}(D_A) = \mathrm{im}(\mu\sigma D_A) \subseteq \mu\left(\mathrm{im}\,\sigma D_A\right) \subseteq \mu(K) \subseteq \overline{R\cdot\Omega_A}$$

as required. □

These results show that semisimple smooth CBAs are quite rare. It is natural to consider what can be said for radical CBAs. We have already seen that finite-dimensional nilpotent CBAs (Example 6.2.4), radical completions of $\mathbb{C}[X]$ (Proposition 6.3.3(iii)), and the convolution algebras $\ell^1((0,1))$ and $\ell^1((0,1)\cap\mathbb{Q})$ (Proposition 6.3.3(iv)) all fail to be smooth.

The obvious examples not covered by the preceding results are the Volterra algebra $\mathcal{V} = L_1([0,1])$ and its discrete version $\ell^1([0,1])$. It has been shown [2, Thm 5.14] that $\mathcal{H}^2(\ell^1([0,1]), M) = 0$ for every *finite-dimensional* bimodule M, but it seems to be unknown what can be said for general symmetric coefficients.

It is claimed in [2, Thm 3.11(vi)] that there exists a Banach $\mathcal{V}$-bimodule X such that $\mathcal{H}^2(A, X) \neq 0$, but the reference given there is to a preprint of Selivanov which I have been unable to track down. In any case the question of whether one can find such an X which is a *symmetric* bimodule appears to still be open.

Question 6.5.9. Is the Volterra algebra smooth?

As mentioned before, the Kähler module is insufficient on its own to characterise smoothness. However, we have some partial converses to Proposition 6.5.2.

Recall that when A is a CBA (as is assumed throughout this chapter) then for each n the Hochschild boundary map $\mathsf{d}_n : \mathcal{C}_{n+1}(A, A) \to \mathcal{C}_n(A, A)$ is a left A-module map.

Lemma 6.5.10. *Suppose that there exist A-module maps*

$$\mathcal{H}ar\mathcal{C}_1(A,A) \xrightarrow{\sigma_1} \mathcal{H}ar\mathcal{C}_2(A,A) \xrightarrow{\sigma_2} \mathcal{H}ar\mathcal{C}_3(A,A)$$

such that $\sigma_1\mathsf{d}_1 + \mathsf{d}_2\sigma_2 = \mathsf{id}$. *Then A is smooth.*

Proof. We use the cohomological characterisation of smoothness (Corollary 6.2.10). Thus, let M be a symmetric A-bimodule: we need to prove that $\mathcal{H}ar\mathcal{H}^2(A,M) = 0$.

This essentially follows from observing that the chain complexes $\mathcal{H}ar\mathcal{C}^*(A,M)$ and ${}_A\mathrm{Hom}\,(\mathcal{H}ar\mathcal{C}_*(A,A),M)$ are isomorphic. We shall do the calculation explicitly in order to make things transparent. Let $\psi \in \mathcal{H}ar\mathcal{Z}^2(A,M)$ and define $\widetilde{\psi} : \mathcal{H}ar\mathcal{C}_2(A,A) \to M$ by

$$\widetilde{\psi}(a_0 \otimes a_1 \otimes a_2) = a_0 \cdot \psi(a_1,a_2) \qquad (a_0,a_1,a_2 \in A)$$

so that $\widetilde{\psi}$ is a (bounded, linear) A-module map. The cocycle condition, together with the fact that M is a symmetric bimodule, yields

$$\begin{aligned} 0 = \delta\psi(b_1,b_2,b_3) &= b_1\psi(b_2,b_3) - \psi(b_1b_2,b_3) + \psi(b_1,b_2b_3) - b_3\psi(b_1,b_2) \\ &= \widetilde{\psi}(b_1\otimes b_2\otimes b_3 - 1_A\otimes b_1b_2\otimes b_3 + 1_A\otimes b_1\otimes b_2b_3 + b_3\otimes b_1\otimes b_2) \\ &= \widetilde{\psi}\mathsf{d}_2(1_A\otimes b_1\otimes b_2\otimes b_3) \end{aligned}$$

for all $b_1,b_2,b_3 \in A$. Hence by linearity and continuity, $\widetilde{\psi}\mathsf{d}_2 = 0$, whence

$$\begin{aligned} \psi(a_1,a_2) &= \widetilde{\psi}(1_A\otimes a_1\otimes a_2) && \text{(by definition)} \\ &= \widetilde{\psi}(1_A\otimes a_1\otimes a_2) - \widetilde{\psi}\mathsf{d}_2\sigma_2(1_A\otimes a_1\otimes a_2) \\ &= \widetilde{\psi}\sigma_1\mathsf{d}_1(1_A\otimes a_1\otimes a_2) && \text{(initial assumption of lemma)} \\ &= \widetilde{\psi}\sigma_1(a_1\otimes a_2 - 1_A\otimes a_1a_2 + a_2\otimes a_1) \end{aligned}$$

for all $a_1,a_2 \in A$. We therefore define $\varphi \in \mathcal{C}^1(A,M)$ by

$$\varphi(x) := \widetilde{\psi}\sigma_1(1_A\otimes x) \qquad (x \in X)$$

and observe that

$$\begin{aligned} \delta\varphi(a_1,a_2) &= a_1\cdot\varphi(a_2) - \varphi(a_1a_2) + a_2\cdot\varphi(a_1) \\ &\qquad \text{(since } M \text{ is symmetric)} \\ &= a_1\cdot\widetilde{\psi}\sigma_1(1_A\otimes a_2) - \widetilde{\psi}\sigma_1(1_A\otimes a_1a_2) + a_2\cdot\widetilde{\psi}\sigma_1(1_A\otimes a_1) \\ &= \widetilde{\psi}\sigma_1(a_1\otimes a_2) - \widetilde{\psi}\sigma_1(1_A\otimes a_1a_2) + \widetilde{\psi}\sigma_1(a_2\otimes a_1) \\ &\qquad \text{(since } \widetilde{\psi} \text{ and } \sigma_1 \text{ are } A\text{-module maps)} \\ &= \psi(a_1,a_2) \\ &\qquad \text{(by the calculation above)} \end{aligned}$$

Hence ψ is a coboundary; since ψ was arbitrary in $\mathcal{H}ar\mathcal{Z}^2(A, M)$, we deduce that $\mathcal{H}ar\mathcal{H}^2(A, M) = 0$ as required. □

Of course it may not be easy to find such maps σ_1, σ_2. The following result builds on Lemma 6.5.10.

Proposition 6.5.11. *Let A be a unital CBA satisfying the following conditions:*

(1) A and Ω_A are both isomorphic as Banach spaces to ℓ^1-spaces;

(2) Ω_A is A-projective;

(3) $\mathcal{H}ar\mathcal{H}^2(A, A') = 0$ and $\mathcal{H}ar\mathcal{H}^3(A, A')$ is Banach.

Then A is smooth.

Remark. Note that if B is a unital, smooth CBA then (by Proposition 6.5.2) Ω_B is B-projective and $\mathcal{H}ar\mathcal{H}^2(B, B') = 0$.

Proof. By assumption (3) $\mathcal{H}ar\mathcal{H}^2(A, A') = 0$: hence by Lemma 6.5.1, $\mathcal{H}_1(A, A)$ is Banach. Writing q for the canonical quotient map $\mathcal{H}ar\mathcal{C}_1(A, A) \to \mathcal{H}_1(A, A)$, we have a complex in ${}_A\mathsf{mod}$

$$0 \leftarrow \mathcal{H}_1(A, A) \xleftarrow{q} \mathcal{H}ar\mathcal{C}_1(A, A) \xleftarrow{\mathsf{d}_1} \mathcal{H}ar\mathcal{C}_2(A, A) \xleftarrow{\mathsf{d}_2} \mathcal{H}ar\mathcal{C}_3(A, A) \tag{6.3}$$

which is exact as a complex of vector spaces, by assumption (3).

Claim. There exist bounded linear maps

$$\mathcal{H}_1(A, A) \xrightarrow{r} \mathcal{H}ar\mathcal{C}_1(A, A) \xrightarrow{s_1} \mathcal{H}ar\mathcal{C}_2(A, A) \xrightarrow{s_2} \mathcal{H}ar\mathcal{C}_3(A, A)$$

such that $qr = \mathsf{id}$, $rq + \mathsf{d}_1 s_1 = \mathsf{id}$ and $s_1\mathsf{d}_1 + \mathsf{d}_2 s_2 = \mathsf{id}$.

Proof of claim. Since $\mathcal{H}_1(A, A)$ is Banach it coincides with Ω_A and is in particular isomorphic as a Banach space to $\ell^1(S)$ for some set S (by assumption (1)). The lifting property of ℓ^1 with respect to the surjection $\mathcal{H}ar\mathcal{C}_1(A, A) \xrightarrow{q} \mathcal{H}_1(A, A)$ allows us to find a bounded linear map r such that $qr = \mathsf{id}$. Then $\mathsf{id} - rq$ is a bounded linear projection from $\mathcal{H}ar\mathcal{C}_1(A, A)$ onto the closed subspace $\ker(q) = \mathcal{H}ar\mathcal{B}_1(A, A)$.

By assumption (1) A is isomorphic as a Banach space to $\ell^1(T)$ for some set T: hence $\mathcal{H}ar\mathcal{C}_1(A, A)$ is isomorphic as a Banach space to $\ell^1(T \times T)$, and so satisfies the lifting property with respect to the surjection $\mathcal{H}ar\mathcal{C}_2(A, A) \xrightarrow{\mathsf{d}_1} \mathcal{H}ar\mathcal{B}_1(A, A) \cong$

$\mathcal{H}ar\mathcal{Z}_1(A,A)$. Therefore we can lift the map $\mathsf{id} - rq$ to a bounded linear map s_1 : $\mathcal{H}ar\mathcal{C}_1(A,A) \to \mathcal{H}ar\mathcal{C}_2(A,A)$; by construction $\mathsf{d}_1 s_1 = \mathsf{id} - rq$.

Finally: the asumption (3) implies that $\mathcal{H}ar\mathcal{H}_2(A,A) = 0$, and so d_2 surjects from $\mathcal{H}ar\mathcal{C}_3(A,A)$ onto $\mathcal{H}ar\mathcal{Z}_2(A,A)$. Moreover, $\mathcal{H}ar\mathcal{C}_2(A,A)$ is the space of symmetric 2-chains on A with coefficients in A, and is therefore isomorphic as a Banach space to $\ell^1(T \times R)$ where R is the quotient of $T \times T$ by the relation $(x,y) \sim (y,x)$. Hence the same lifting argument may be applied as before to show that the projection $\mathsf{id} - s_1\mathsf{d}_1 : \mathcal{H}ar\mathcal{C}_3(A,A) \to \mathcal{H}ar\mathcal{Z}_2(A,A)$ can be lifted to a bounded linear map $s_2 : \mathcal{H}ar\mathcal{C}_2(A,A) \to \mathcal{H}ar\mathcal{C}_3(A,A)$, and the claim is proved. ☐

Having now established that the complex (6.3) splits in Ban, we observe that $\mathcal{H}ar\mathcal{H}_1(A,A) \cong \Omega_A$ is A-projective (by assumption (1)) and that $\mathcal{H}ar\mathcal{C}_1(A,A)$, $\mathcal{H}ar\mathcal{C}_2(A,A)$ are also A-projective (both are module summands of projective A-modules, since A is *unital*). It follows that (6.3) splits in ${}_A\mathsf{mod}$: the argument is almost identical to that used to obtain our splitting in Ban, and we shall omit the full details. In any case, to say that (6.3) splits in ${}_A\mathsf{mod}$ is to say that there exist A-module maps

$$\mathcal{H}_1(A,A) \xrightarrow{\rho} \mathcal{H}ar\mathcal{C}_1(A,A) \xrightarrow{\sigma_1} \mathcal{H}ar\mathcal{C}_2(A,A) \xrightarrow{\sigma_2} \mathcal{H}ar\mathcal{C}_3(A,A)$$

such that $q\rho = \mathsf{id}$, $\rho q + \mathsf{d}_1\sigma_1 = \mathsf{id}$ and $\sigma_1\mathsf{d}_1 + \mathsf{d}_2\sigma_2 = \mathsf{id}$. ☐

Remark. The proof of Proposition 6.5.11 could be made a little shorter if we appealed systematically to the notions of a *projective Banach space* – that is, one which has the lifting property with respect to all quotient maps of Banach spaces – and a *strictly projective* A-module. Since introducing such concepts properly would require extra definitions and lemmas and not yield much extra in return, we have chosen a more hands-on argument here.

6.6 Remarks and questions

The notion of smoothness, as defined above, remains mysterious; without further work on examples it is not clear whether the definition we have chosen is fruitful. One possible generalisation is to require merely that $\mathcal{H}^2(A,M) = 0$ whenever M is a symmetric, *dual* Banach A-bimodule, but I have been unable to determine if this leads to a genuinely larger class of CBAs.

We shall list three obvious general questions which have not been resolved by the work of this chapter:

Question 6.6.1. What are the hereditary properties (if any) of smoothness? (Even in the purely algebraic case, quotients of smooth algebras are in general not smooth.)

Question 6.6.2. Are there any smooth CBAs which are not weakly amenable? If so, can we drop the AP hypothesis in Corollary 6.5.6?

(Our use of the approximation property is in cutting down from a derivation with values in an A-module of the form $A\widehat{\otimes}E$ to a derivation with values in A, so that we can appeal to the Singer-Wermer theorem. It may be possible to generalise the proofs of the Singer-Wermer theorem to deal with coefficients of the form $A\widehat{\otimes}E$, but we have not managed to achieve this.)

Question 6.6.3. Let A be a smooth uniform algebra with the approximation property. Is A isomorphic to $C(X)$ (where X is the maximal ideal space of A)? By Corollary 6.5.6 this question would be answered positively if one could prove the infamous conjecture that no proper uniform algebra can be weakly amenable.

Note also that the proof of Sheinberg's theorem says, in effect, that if A is a uniform algebra and $\mathcal{H}^2(A, \mathcal{L}(K, L)) = 0$ for all Hilbert modules K, L, then $A = C(X)$: this should be compared with the cohomological characterisation of smoothness by the vanishing of all second *symmetric* cohomology groups.

Chapter 7

The third cohomology of some Beurling algebras

In this short chapter we return to the Hodge decomposition and show how it may be used to prove a non-vanishing result for certain simplicial cohomology groups. More precisely, the main result (Theorem 7.1.2) states that for every $\alpha > 0$,

$$\mathcal{H}^{2,1}\left(\ell^1(\mathbb{Z},(1+|\_\_|)^\alpha),\ell^1(\mathbb{Z},(1+|\_\_|)^\alpha)'\right) \neq 0 \ .$$

Here $\mathcal{H}^{2,1}$ is the middle summand in the Hodge decomposition of $\mathcal{H}^3$.

In particular, the third simplicial cohomology of $\ell^1(\mathbb{Z},(1+|\_\_|)^\alpha)$ is non-zero; this result appears to be new, although not particularly surprising.

7.1 Statement of main result

We start in the general setting of Beurling algebras on $\mathbb{Z}^k$ for $k \geq 1$. The cohomology of such a Beurling algebra $\ell^1(\mathbb{Z}^k,\omega)$ is not very well understood : however, we do have the following result as observed in [22].

Proposition 7.1.1. *$\mathcal{H}^{n,0}(\ell^1(\mathbb{Z}^k,\omega),\ell^1(\mathbb{Z}^k,\omega)') = 0$ for all $n > k$.*

Essentially this is because the algebra is topologically generated by k elements, and so can contain at most k "independent" derivations into a given module; therefore by Proposition 1.6.5 the only alternating n-derivation on the algebra is zero. (For a fuller account, see the remarks in [22] that precede Theorem 4.2.)

In particular

$$\mathcal{H}^{3,0}(\ell^1(\mathbb{Z},\omega),\ell^1(\mathbb{Z},\omega)') = 0$$

and it is natural to enquire for what weights (if any) the other summands of $\mathcal{H}^3$ vanish. More precisely, we have the following problem:

Problem. Find conditions on the weight ω that are necessary and sufficient for

(i) $\mathcal{H}^{1,2}(\ell^1(\mathbb{Z},\omega),\ell^1(\mathbb{Z},\omega)') = 0$;

(ii) $\mathcal{H}^{2,1}(\ell^1(\mathbb{Z},\omega),\ell^1(\mathbb{Z},\omega)') = 0$.

Part (i) of this problem seems quite hard at present; part (ii) seems more tractable, and can be answered completely for the special case where $\omega(n) = (1+|n|)^\alpha$, $\alpha > 0$. Given $\alpha \geq 0$, let A_α denote the Beurling algebra on $\mathbb{Z}$ with weight function $\omega(n) = (1+|n|)^\alpha$.

Theorem 7.1.2. *For any $\alpha > 0$, $\mathcal{H}^{2,1}(A_\alpha, A_\alpha{}') \neq 0$.*

Theorem 7.1.2 appears to be new. We shall deduce it from a more precise statement to be given in the next section.

7.2 Constructing cocycles: preliminaries

Before stating the result, we introduce some notation. Let A be the complex group ring $\mathbb{C}\mathbb{Z}$, which can be regarded as the dense subalgebra of $\ell^1(\mathbb{Z},\omega)$ consisting of all $a = \sum_{n\in\mathbb{Z}} a_n e_n$ such that $a_n = 0$ for all but finitely many n. We shall write $\mathcal{C}^*_{\mathsf{alg}}(\mathsf{A})$ for the purely algebraic Hochschild cochain complex of A with coefficients in its algebraic dual. Note that the Hodge decomposition carries over to this complex.

Definition 7.2.1 (Notation). If ω is a weight function on $\mathbb{Z}$ and $n \in \mathbb{N}$, we define a "total weight function" Ω by

$$\Omega(x_1,\ldots,x_n) := \omega(x_1)\ldots\omega(x_n)\omega\left(-\sum_{i=1}^n x_i\right)$$

(This is just a notational convenience to make the formulas below more legible.)

Definition 7.2.2 (Notation). Given $\mathbf{a},\mathbf{b} \in \mathbb{R}^2$ the wedge product $\mathbf{a}\wedge\mathbf{b}$ is

$$\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1b_2 - a_2b_1 \in \mathbb{R}$$

It is convenient to first make some observations about *algebraic cocycles* on A.

Lemma 7.2.3. $\mathcal{Z}^{2,0}_{\mathsf{alg}}(\mathsf{A}) = 0$.

Proof. Let $\psi \in \mathcal{Z}^{2,0}_{\mathsf{alg}}$. By Proposition 1.6.5, ψ must be a biderivation – i.e. a derivation in the first and second variables – and must also satisfy $\psi(z^j, z^k)(z^l) = -\psi(z^k, z^j)(z^l)$ for all $j, k, l \in \mathbb{Z}$. But since the biderivation property implies that

$$\psi(z^j, z^k)(z^l) = jk\psi(z,z)(z^{j+k+l-2}) \qquad \text{for all } j,k,l \in \mathbb{Z},$$

the anti-symmetry of ψ forces $\psi(z,z)(\bullet) = 0$, whence $\psi = 0$ as required. $\square$

The next proposition gives a general method for attempting to construct non-trivial elements of $\mathcal{H}^{2,1}(\ell^1(\mathbb{Z},\omega), \ell^1(\mathbb{Z},\omega)')$.

Proposition 7.2.4. *Suppose there exists a function $g : \mathbb{Z} \to \mathbb{R}^2$ such that*

$$\sup_{x,y\in\mathbb{Z}} \frac{|g(x) \wedge g(y)|}{\Omega(x,y)} = +\infty \tag{7.1a}$$

$$\sup_{a,b,x\in\mathbb{Z}} \frac{|\big(g(a) - g(a+b) + g(b)\big) \wedge g(x)|}{\Omega(a,b,x)} < \infty \tag{7.1b}$$

Define $G \in \mathcal{C}^2_{\mathsf{alg}}(\mathsf{A})$ by

$$G(x_1,x_2)(x_0) = \begin{cases} g(x_1) \wedge g(x_2) & \textit{if } x_1 + x_2 + x_0 = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Then δG extends to a continuous simplicial 3-cocycle on $\ell^1(\mathbb{Z},\omega)$. If we denote this cocycle by F, then $e_3(2)^*F = F$ and $F \notin \mathcal{B}^3(\ell^1(\mathbb{Z},\omega), \ell^1(\mathbb{Z},\omega)')$.

Recall that for each i the projections $e_n(i)^*$, $n = 1, 2, \ldots$, assemble to form a chain map $e_\bullet(i)^* : \mathcal{C}^\bullet(\ell^1(\mathbb{Z},\omega), \ell^1(\mathbb{Z},\omega)')$ (see Section 1.5 for the relevant details). This will be important in the proof that follows.

Proof of Proposition 7.2.4. Clearly $\delta G(x_1,x_2,x_3)(y) = 0$ unless $x_1 + x_2 + x_3 + y = 0$, and

$$\begin{aligned}\delta G(x_1,x_2,x_3)(-x_1-x_2-x_3) &= \left\{ \begin{array}{l} g(x_2)\wedge g(x_3) - g(x_1+x_2)\wedge g(x_3) \\ +g(x_1)\wedge g(x_2+x_3) - g(x_1)\wedge g(x_2) \end{array} \right. \\ &= \left\{ \begin{array}{l} \big[g(x_2) - g(x_1+x_2) + g(x_1)\big] \wedge g(x_3) \\ -g(x_1) \wedge \big[g(x_2) - g(x_2+x_3) + g(x_3)\big] \; . \end{array} \right.\end{aligned}$$

Therefore condition (7.1b) ensures the existence of some $C > 0$ such that

$$|\delta G(x_1,x_2,x_3)(y)| \le C\Omega(x_1,x_2,x_3) \quad \text{ for all } x_1,x_2,x_3,y \in \mathbb{Z} \; .$$

The 3-cochain δG thus extends to a bounded simplicial 3-cochain on $\ell^1(\mathbb{Z},\omega)$ (with norm $\leq C$). Denote this element of $\mathcal{C}^3(\ell^1(\mathbb{Z},\omega),\ell^1(\mathbb{Z},\omega)')$ by F. Then the restriction of δF to A is just $\delta^2 G = 0$, and since A is a dense subset of $\ell^1(\mathbb{Z},\omega)$ this forces $\delta F = 0$, i.e. F is a cocycle.

The way we have constructed G ensures that it is an alternating 2-cochain, i.e. $e_2(2)^*G = G$. Therefore, *since* $e_\bullet(2)^*$ *is a cochain map*,

$$e_3(2)^*\delta G = \delta e_2(2)^* G = \delta G\ .$$

As F is a bounded cochain it follows by continuity that $e_3(2)^*F = F$.

For the final statement, suppose that $F = \delta S$ for some continuous 2-cochain $S \in \mathcal{C}^2(\ell^1(\mathbb{Z},\omega),\ell^1(\mathbb{Z},\omega)')$; we obtain a contradiction as follows. Let $T = e_2(2)^*S$: then

$$\begin{aligned}\delta G = \delta e_2(2)^* G &= e_3(2)^*\delta G \\ &= e_3(2)^*\delta S \quad = \delta e_2(2)^* S = \delta T\ ;\end{aligned}$$

hence $G - T \in \mathcal{Z}^{2,0}_{\mathsf{alg}}(\mathsf{A})$. But $\mathcal{Z}^{2,0}_{\mathsf{alg}}(\mathsf{A}) = 0$ by Lemma 7.2.3, and so $T|_{\mathsf{A}} = G$. Thus

$$|g(x_1)\wedge g(x_2)| = |G(x_1,x_2)(-x_1-x_2)| = |T(x_1,x_2)(-x_1-x_2)| \leq \|T\|\Omega(x_1,x_2)$$

for all $x_1,x_2\in\mathbb{Z}$, which contradicts condition (7.1a) of the proposition. This contradiction shows that no such S exists, and therefore F is not a simplicial coboundary. ☐

We have thus reduced the problem of finding non-trivial simplicial 3-cocycles on $\ell^1(\mathbb{Z},\omega)$ to the task of cooking up an appropriate $\mathbb{R}^2$-valued function. The next lemma breaks up this task into simpler pieces.

Lemma 7.2.5. *Suppose that we can find functions* $\eta_1,\eta_2:\mathbb{Z}\to\mathbb{R}$ *and* $\lambda_1,\lambda_2\in[0,1]$ *such that the following conditions hold:*

$$|\eta_1(a)-\eta_1(a+b)+\eta_1(b)| \lesssim \frac{\omega(a)\omega(b)}{\omega(a+b)^{\lambda_2}}\ , \tag{7.2a}$$

$$|\eta_2(a)-\eta_2(a+b)+\eta_2(b)| \lesssim \frac{\omega(a)\omega(b)}{\omega(a+b)^{\lambda_1}}\ ; \tag{7.2b}$$

and

$$|\eta_2(z)| \lesssim \omega(z)\omega(-z)^{\lambda_2}\ , \tag{7.3a}$$

$$|\eta_1(z)| \lesssim \omega(z)\omega(-z)^{\lambda_1}\ ; \tag{7.3b}$$

and

$$\sup_{z\in\mathbb{Z}}\left|\frac{\eta_1(z)\eta_2(-z)-\eta_1(-z)\eta_2(z)}{\omega(z)\omega(-z)}\right| = +\infty \tag{7.4}$$

Then, if we define

$$\boxed{g(x) = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \end{pmatrix}}$$

we find that g satisfies conditions (7.1b) *and* (7.1a) *of Proposition 7.2.4.*

Remark (Notation). In this lemma and those to follow, we freely use the relation $\lesssim$ to compare $\mathbb{R}_+$-valued functions on the same domain. This notation is commonplace in certain branches of analysis, and its precise meaning is as follows: let f, M be functions $X \to \mathbb{R}_+$ for any set X; then we say that $f \lesssim M$, or $f(x) \lesssim M(x)$ $(x \in X)$, if and only if there exists a constant $C > 0$ such that $f(x) \leq CM(x)$ for all $x \in X$.

In words, the expression "$f \lesssim M$" means that "f is majorized by M up to some constant factor".

Proof of Lemma 7.2.5.

$$\big(g(a) - g(a+b) + g(b)\big) \wedge g(x) = \begin{cases} \left(\eta_1(a) - \eta_1(a+b) + \eta_1(b)\right) \eta_2(x) \\ -\big(\eta_2(a) - \eta_2(a+b) + \eta_2(b)\big)\eta_1(x) \end{cases} \tag{7.5}$$

By conditions (7.2a) and (7.3a) the first term on the RHS of Equation (7.5) is bounded in modulus by

$$\begin{aligned} & C_1 \frac{\omega(a)\omega(b)}{\omega(a+b)^{\lambda_2}} \cdot \omega(x)\omega(-x)^{\lambda_2} \\ = \quad & \omega(a)\omega(b)\omega(x) \cdot \left(\frac{\omega(-x)}{\omega(a+b)}\right)^{\lambda_2} \\ \leq \quad & C_1\omega(a)\omega(b)\omega(x) \cdot \omega(-a-b-x)^{\lambda_2} && \text{(since } \omega \text{ is submultiplicative} \\ & && \text{and } \lambda_2 \geq 0) \\ \leq \quad & C_1\Omega(a,b,x) && \text{(since } \omega \geq 1 \text{ and } \lambda_2 \leq 1) \end{aligned}$$

for some absolute constant C_1. Similarly, using conditions (7.2b) and (7.3b), we see that the second term on the RHS of Equation (7.5) is bounded in modulus by $C_2\Omega(a,b,x)$, for some absolute constant C_2. Thus g satisfies the boundedness condition (7.1b).

It remains only to note that

$$\sup_{x,y\in\mathbb{Z}} \frac{|g(x) \wedge g(y)|}{\Omega(x,y)} \geq \sup_{z\in\mathbb{Z}} \frac{|g(z) \wedge g(-z)|}{\omega(z)\omega(-z)}$$

and condition (7.4) forces the RHS to be infinite; hence g satisfies the unboundedness condition (7.1a). □

7.3 Constructing cocycles: key estimates and final example

We now specialise to the case $\omega(x) = (1 + |x|)^\alpha$ where $\alpha > 0$. The rest of this section now consists of easy but tedious estimates: the important point to bear in mind is that we are merely seeking to find functions $\eta_1, \eta_2 : \mathbb{Z} \to \mathbb{R}$ and parameters $\lambda_1, \lambda_2 \in [0, 1]$ which satisfy the conditions of our technical lemma 7.2.5. By restricting ourselves to weights of the form $(1 + |\_\_|)^\alpha$ we can break up this task into easy steps.

Lemma 7.3.1. *Fix $\rho > 0$. Let*

$$g_1(u, v) := \frac{u}{|u|}|u|^\rho - \frac{u+v}{|u+v|}|u+v|^\rho + \frac{v}{|v|}|v|^\rho$$

and

$$g_2(u, v) := |u|^\rho - |u+v|^\rho + |v|^\rho$$

Then g_1, g_2 are continuous functions $\mathbb{R}^2 \to \mathbb{R}$ (hence locally bounded) and

$$|g_i(u, v)| \lesssim \begin{cases} \min(|u|, |v|)^\rho & \text{if } \rho \leq 1 \\ \min(|u|, |v|) \max(|u|, |v|)^{\rho - 1} & \text{if } \rho \geq 1 \end{cases}$$

for $i = 1, 2$.

Proof. This is just basic calculus. We introduce auxiliary functions

$$k_1 : r \mapsto 1 - (1+r)|1+r|^{\rho-1} + r|r|^{\rho-1} \quad \text{and} \quad k_2 : r \mapsto 1 - |1+r|^\rho + |r|^\rho$$

and claim that

$$\begin{aligned} &\text{if } \rho \leq 1 \;\; k_1, k_2 \text{ are bounded on } \mathbb{R}; \\ &\text{if } \rho \geq 1 \;\; \max\Big(|k_1(r)|, |k_2(r)|\Big) \lesssim 1 + |r|^{\rho-1}\,. \end{aligned} \tag{7.6}$$

Assume for the moment that the claim holds. A little thought shows that

$$g_2(u, v) = |v|^\rho k_2(u/v) = |u|^\rho k_2(v/u)$$

and a little more thought shows that

$$g_1(u, v) = v|v|^{\rho-1} k_1(u/v) = u|u|^{\rho-1} k_1(v/u)\,.$$

Applying the claim with $r = u/v$ and $r = v/u$ then yields the following:

- if $\rho \leq 1$ then for some constant C

$$|g_i(u,v)| = \begin{cases} |v|^\rho |k_i(u/v)| \leq C|v|^\rho \\ |u|^\rho |k_i(v/u)| \leq C|u|^\rho \end{cases} \qquad (i = 1,2)$$

so that $|g_1(u,v)|, |g_2(u,v)| \leq C\min(|u|,|v|)^\rho$, as required;

- if $\rho \geq 1$ then for some constant C'

$$|g_i(u,v)| = \begin{cases} |v|^\rho |k_i(u/v)| \leq C'|v|^\rho(1 + |u/v|^{\rho-1}) & = C'|v|(|v|^{\rho-1} + |u|^{\rho-1}) \\ |u|^\rho |k_i(v/u)| \leq C'|u|^\rho(1 + |v/u|^{\rho-1}) & = C'|u|(|u|^{\rho-1} + |v|^{\rho-1}) \end{cases}$$

so that

$$|g_i(u,v)| \leq C' \min(|u|,|v|)(|u|^{\rho-1} + |v|)^{\rho-1}) \leq 2C' \min(|u|,|v|)\max(|u|,|v|)^{\rho-1}$$

as required.

It therefore only remains to prove the claim (7.6).

Proof of claim: Recall that k_1 and k_2 are defined by

$$k_1(r) = 1 - (1+r)|1+r|^{\rho-1} + r|r|^{\rho-1} \quad \text{and} \quad k_2(r) = 1 - |1+r|^\rho + |r|^\rho\ .$$

Since k_1 and k_2 are continuous on $\mathbb{R}$, they are both bounded on the closed interval $[-2,1]$, by some common constant C_ρ say.

Moreover, given $t \geq 1$ the mean value theorem yields

$$0 < (1+t)^\rho - t^\rho \leq \sup_{1 \leq s \leq t} \rho s^{\rho-1} = \rho\max(1, t^{\rho-1}). \tag{7.7}$$

We use this to bound k_1, k_2 outside the interval $[-2,1]$ as follows:

- when $r \geq 1$ we apply (7.7) with $t = r$ and deduce that

$$\begin{aligned} &\text{if } \rho \leq 1 \quad |k_1(r)| = |k_2(r)| &= \left|1 - (1+r)^\rho - r^\rho\right| \leq 1 + \rho; \\ &\text{if } \rho \geq 1 \quad |k_1(r)| = |k_2(r)| &= \left|1 - (1+r)^\rho - r^\rho\right| \leq 1 + \rho|r|^{\rho-1} \quad ; \end{aligned}$$

- when $r \leq -2$, let $t = -r - 1$ and observe that

$$\begin{aligned} k_1(r) &= 1 - (1+r)|1+r|^{\rho-1} + r|r|^{\rho-1} &= 1 + t^\rho - (1+t)^\rho \\ k_2(r) &= 1 - |1+r|^\rho + |r|^\rho &= 1 - t^\rho + (1+t)^\rho \end{aligned}$$

so that

$$|k_i(r)| \leq 1 + |(1+t)^\rho - t^\rho| \qquad \text{for } i = 1,2;$$

hence, applying (7.7), we deduce that

$$\begin{aligned} &\text{if } \rho \leq 1 \quad |k_i(r)| \leq \quad 1+\rho \\ &\text{if } \rho \geq 1 \quad |k_i(r)| \leq \quad 1+\rho t^{\rho-1} \leq 1+\rho|r|^{\rho-1} \end{aligned} \qquad (i=1,2).$$

Therefore, when $\rho \leq 1$ the estimates above combine to give

$$|k_i(r)| \leq \max(C_\rho, 1+\rho)$$

and when $\rho \geq 1$ they combine to give

$$|k_i(r)| \leq \max(C_\rho, 1+\rho|r|^{\rho-1}) \leq \max(C_\rho, 1+\rho)\left(1+|r|^{\rho-1}\right)$$

for $i=1,2$. We have thus established that the bounds claimed in (7.6) are correct, and this concludes the proof of the lemma. □

Proposition 7.3.2. *Given $\lambda_1, \lambda_2 \in [0,1]$ and $\gamma_1, \gamma_2 \geq 0$, take*

$$\eta_1(x) := x|x|^{\gamma_1 - 1} \quad ; \quad \eta_2(x) := |x|^{\gamma_2} \qquad (x \in \mathbb{R})$$

Fix some $\alpha > 0$ and let $\omega = (1+|\_\_|)^\alpha$. Then we can choose λ_i, γ_i such that $g(x) = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \end{pmatrix}$ satisfies all the conditions from Proposition 7.2.4.

More precisely, we have the following: conditions (7.3a), (7.3b) are satisfied if

$$\gamma_i \leq (1+\lambda_i)\alpha \qquad (i=1,2) \tag{7.8}$$

and condition (7.4) is satisfied if

$$\gamma_1 + \gamma_2 > 2\alpha\ . \tag{7.9}$$

Moreover:

- *in the case where $0 < \alpha \leq 1$, conditions (7.2a) and (7.2b) are satisfied if we ensure that*

$$\begin{aligned} \alpha \leq \gamma_1 &\leq (2-\lambda_2)\alpha \\ \alpha \leq \gamma_2 &\leq (2-\lambda_1)\alpha \end{aligned} \tag{7.10}$$

- *in the case where $\alpha \geq 1$, conditions (7.2a) and (7.2b) are satisfied if we ensure that*

$$\begin{aligned} \alpha \leq \gamma_1 &\leq (1-\lambda_2)\alpha + 1 \\ \alpha \leq \gamma_2 &\leq (1-\lambda_1)\alpha + 1 \end{aligned} \tag{7.11}$$

Finally, observe that we can choose λ_i, γ_i *satisfying the constraints* (7.8)-(7.11) *as follows:*

- *when* $0 < \alpha \leq 1$*, take* $\lambda_1 = \lambda_2 = \frac{1}{2}$ *and* $\gamma_1 = \gamma_2 = \frac{3}{2}\alpha$*;*
- *when* $\alpha \geq 1$*, take* $\lambda_1 = \lambda_2 = \frac{1}{2\alpha}$ *and* $\gamma_1 = \gamma_2 = \alpha + \frac{1}{2}$*.*

Proof. First let us observe that the values for λ_i, γ_i which are specified at the end of Proposition 7.3.2 do indeed satisfy the constraints (7.8)-(7.11). It therefore suffices to show how these constraints will ensure that η_1, η_2 satisfy conditions (7.2a)-(7.4) of Lemma 7.2.5.

Armed with the key technical estimates in Lemma 7.3.1 we need only do some book-keeping. The reader is recommended to have a separate copy of the requisite formulas from Lemma 7.2.5 at hand, for ease of reference.

We first note that if γ_1, γ_2 satisfy the upper bounds of constraint (7.8), then

$$|\eta_i(z)| = |z|^{\gamma_i} \leq (1+|z|)^{\gamma_i} \leq (1+|z|)^{(1+\lambda_i)\alpha} = \omega(z)\omega(-z)^{\lambda_i} \qquad (i=1,2) \ ,$$

and so conditions (7.3a),(7.3b) are satisfied. Secondly, the lower bound on $\gamma_1 + \gamma_2$ in constraint (7.9) ensures that

$$\sup_{z\in\mathbb{Z}} \left| \frac{\eta_1(z)\eta_2(-z) - \eta_1(-z)\eta_2(z)}{\omega(z)\omega(-z)} \right| = \sup_{z\in\mathbb{Z}} \frac{2|z|^{\gamma_1+\gamma_2}}{(1+|z|)^{2\alpha}} = +\infty$$

and thus condition (7.4) is satisfied. Therefore, it remains only to verify that conditions (7.2a) and (7.2b) can be met.

<u>The case $0 < \alpha \leq 1$.</u>

Given $a, b \in \mathbb{R}$ let us abbreviate $\min(|a|,|b|)$ to m and $\max(|a|,|b|)$ to M. We know from our key lemma 7.3.1 that

$$|\eta_i(a) - \eta_i(a+b) + \eta_i(b)| \lesssim \max\left(m^{\gamma_i},\ mM^{\gamma_i-1}\right) \quad (i=1,2) \ ;$$

moreover, since $1-\alpha \geq 0$ and $m \leq M$,

$$mM^{\gamma_i-1} = m^\alpha \left(\frac{m}{M}\right)^{1-\alpha} M^{\gamma_i-\alpha} \leq m^\alpha M^{\gamma_i-\alpha},$$

which gives

$$\max\left(m^{\gamma_i}, mM^{\gamma_i-1}\right) \leq m^\alpha M^{\gamma_i-\alpha}$$

(since $\gamma_i \geq \alpha$). Therefore, conditions (7.2a) and (7.2b) will follow if we can prove that

$$m^\alpha M^{\gamma_i-\alpha} \lesssim \frac{(1+m)^\alpha(1+M)^\alpha}{(1+|a+b|)^{\alpha\lambda_{3-i}}} \qquad (i=1,2) \tag{7.12}$$

Equation (7.10) gives us

$$(1+|a+b|)^{\lambda_{3-i}\alpha} \leq (1+|a+b|)^{2\alpha-\gamma_i} \leq (1+2M)^{2\alpha-\gamma_i} \qquad (i=1,2)$$

and therefore

$$\frac{(1+m)^\alpha(1+M)^\alpha}{(1+|a+b|)^{\alpha\lambda_{3-i}}} \geq \frac{(1+m)^\alpha(1+M)^\alpha}{(1+2M)^{2\alpha-\gamma_i}} \sim (1+m)^\alpha(1+M)^{\gamma_i-\alpha} .$$

This shows that the required inequality (7.12) holds, hence conditions (7.2a) and (7.2b) are satisfied.

The case $\alpha \geq 1$.

Given $a, b \in \mathbb{R}$ let us abbreviate $\min(|a|,|b|)$ to m and $\max(|a|,|b|)$ to M. We know from our key lemma 7.3.1 that

$$|\eta_i(a) - \eta_i(a+b) + \eta_i(b)| \lesssim mM^{\gamma_i-1} \quad (i=1,2)$$

(recall that $\gamma_i \geq \alpha \geq 1$). Thus, conditions (7.2a) and (7.2b) will follow if we can prove that

$$mM^{\gamma_i-1} \lesssim \frac{(1+m)^\alpha(1+M)^\alpha}{(1+|a+b|)^{\alpha\lambda_{3-i}}} \qquad (i=1,2) \tag{7.13}$$

Equation (7.11) gives us

$$(1+|a+b|)^{\alpha\lambda_{3-i}} \leq (1+|a+b|)^{\alpha+1-\gamma_i} \leq (1+2M)^{\alpha+1-\gamma_i} \qquad (i=1,2)$$

and therefore

$$\frac{(1+m)^\alpha(1+M)^\alpha}{(1+|a+b|)^{\alpha\lambda_{3-i}}} \geq \frac{(1+m)^\alpha(1+M)^\alpha}{(1+2M)^{\alpha+1-\gamma_i}} \sim (1+m)^\alpha(1+M)^{\gamma_i-1} ;$$

since $\alpha \geq 1$, this shows that the required inequality (7.13) holds. Hence conditions (7.2a) and (7.2b) are satisfied.

□

Proof of Theorem 7.1.2. Let $\alpha > 0$. By the preceding proposition we can find $g : \mathbb{Z} \to \mathbb{R}^2$ which satisfies all the conditions from Proposition 7.2.4. Proposition 7.2.4 then furnishes us with a continuous, simplicial 3-cocycle on A_α which is in the middle Hodge summand and is not a coboundary. Hence $\mathcal{H}^{2,1}(A_\alpha, {A_\alpha}') \neq 0$, as required. □

Conclusions

We have seen in the course of this thesis that calculating Hochschild cohomology groups of Banach algebras remains a rather difficult task. The technique illustrated in Chapter 7 gives a method for trying to prove negative results (i.e. that certain higher cohomology groups are non-zero). In contrast, Chapter 5 offers some hope that the cohomology of ℓ^1-semigroup algebras will prove a more tractable area of future research, at least for simplicial homology and cohomology.

At a more abstract level, we have seen that *commutative* simplicially trivial Banach algebras “tend” to have vanishing cohomology for all symmetric coefficients. More generally, if A is a commutative Banach algebra whose simplicial Harrison cochain complex is *split exact* in degrees 2 and above – for example, the results of [12] and [13] show that we could take $A = \ell^1(\mathbb{Z}_+^k)$ – then we may express Harrison cohomology of A with symmetric coefficents M in terms of $\operatorname{Ext}^*_A(\_\_, M)$. This suggests further work to determine if there are any other non-amenable examples of such A.

The notion of smoothness for commutative Banach algebras, which was investigated in Chapter 6, may provide an interesting weakening of the notion of amenability in the commutative setting. More examples seem to be needed before one can make further conjectures.

Appendix A

Normalising over a contractible subalgebra

In this section we provide a full proof of the following known result (which was stated in Section 1.4).

Theorem 1.4.6 Let A be a unital Banach algebra and K a finite-dimensional, unital subalgebra.

Suppose that K is contractible, with diagonal Δ, and let X be a Banach A-bimodule. Then there exists a chain map $\alpha : \mathcal{C}_*(A, X) \to \mathcal{C}_*(A, X)$ with the following properties:

(a) each α_n is K-normalised, i.e. factors through the quotient map $\mathcal{C}_n(A, X) \to \mathcal{C}_n^K(A, X)$;

(b) there exists a chain homotopy from id to α, given by bounded linear maps $t_n : \mathcal{C}_n(A, X) \to \mathcal{C}_{n+1}(A, X)$ satisfying $\mathsf{d}_n t_n + t_{n-1}\mathsf{d}_{n-1} = \mathsf{id}_n - \alpha_n$ for all n;

(c) the norm of each t_n is bounded above by some constant depending only on n and $\|\Delta\|$.

Consequently, for each n the canonical map $\mathcal{H}_n(A, X) \to \mathcal{H}_n^K(A, X)$ is an isomorphism of seminormed spaces.

Remark. The hypothesis that K is finite-dimensional can be omitted, but it makes certain estimates a little cleaner. In any case, there are as yet no known examples of infinite-dimensional contractible Banach algebras, so for the purposes of this thesis we lose nothing by restricting attention to the finite-dimensional case.

I would like to thank Dr. Michael White for pointing out that the standard inductive proof of Theorem 1.4.6 without norm control is underpinned by an argument about comparing A-biprojective resolutions, and suggesting that this viewpoint might provide a clearer route through the calculations.

We induce the maps α and t from maps on the two-sided bar resolution of A (thus in some sense the module X plays no role in the main proof). Recall that the two-sided bar resolution of A is the complex

$$0 \leftarrow A \xleftarrow{\mathsf{b}_{-1}} \beta_0 \xleftarrow{\mathsf{b}_0} \beta_1 \xleftarrow{\mathsf{b}_1} \beta_2 \xleftarrow{\mathsf{b}_2} \dots \tag{A.1}$$

where $\beta_n := A\widehat{\otimes}\left(A^{\widehat{\otimes}n}\right)\widehat{\otimes}A$ and

$$\mathsf{b}_n(a_0\otimes\dots a_{n+1}) := \sum_{j=0}^{n+1}(-1)^j a_0\otimes\dots\otimes a_j a_{j+1}\otimes\dots\otimes a_{n+2} \qquad (n \geq -1)\,.$$

The complex (A.1) is split in Ban, via the contractive, linear splitting maps

$$b_0\otimes b_1\otimes\dots\otimes b_n \longmapsto 1_A\otimes b_0\otimes b_1\otimes\dots\otimes b_n$$

and thus $0 \leftarrow A \leftarrow \beta_\bullet$ is an admissible resolution of A by A-biprojective modules.

Lemma A.1. *For each $n \geq 0$ define $f_n : \beta_n \to \beta_n$ by*

$$\begin{aligned} &f_n(a_0\otimes\dots\otimes a_{n+1}) \\ &\quad = \sum_{i(1)\geq 1}\dots\sum_{i(n+1)\geq 1} a_0 u_{i(1)}\otimes v_{i(1)}a_1 u_{i(2)}\otimes\dots\otimes v_{i(n)}a_n u_{i(n+1)}\otimes v_{i(n+1)}a_{n+1} \end{aligned} \tag{A.2}$$

and let $f_{-1} : A \to A$ be the identity map. Then f is a chain map, and $\|f_n\| \leq \|\Delta\|^{n+1}$ for all n.

Proof. The proof that f is a chain map is a direct computation: it is perhaps easiest to first observe that f commutes with each face map, and then use linearity to deduce that it commutes with the boundary map b. The norm estimate on f_n is trivial. ☐

The following result is entirely standard homological algebra, except that we give norm estimates.

Lemma A.2 (Homotopy). *Let h be any chain map as shown below:*

$$\begin{array}{ccccccccc} 0 \leftarrow & A & \xleftarrow{\mathsf{b}_{-1}} & \beta_0 & \xleftarrow{\mathsf{b}_0} & \beta_1 & \xleftarrow{\mathsf{b}_1} & \dots \\ & \downarrow 0 & & \downarrow h_0 & & \downarrow h_1 & & \\ 0 \leftarrow & A & \xleftarrow{\mathsf{b}_{-1}} & \beta_0 & \xleftarrow{\mathsf{b}_0} & \beta_1 & \xleftarrow{\mathsf{b}_1} & \dots \end{array}$$

Let $C_n := \sum_{j=0}^{n} \frac{1}{(j+1)!}\|h_j\|$. *Then there exist* A*-bimodule maps* $s_n : \beta_n \to \beta_{n+1}$ *which satisfy*

$$s_{n-1}\mathsf{b}_{n-1} + \mathsf{b}_n s_n = h_n$$

and $\|s_n\| \leq C_n(n+1)!$.

Proof. Since $\mathsf{b}_{-1}h_0 = 0$, h_0 takes values in $\ker(\mathsf{b}_{-1})$. As b_0 is an admissible surjection onto $\ker(\mathsf{b}_{-1})$ and the A-bimodule β_0 is free, we may construct an A-bimodule map $s_0 : \beta_0 \to \beta_1$ such that

$$\mathsf{b}_0 s_0 = h_0 \quad \text{and} \quad \|s_0\| \leq \|h_0\| = C_0.$$

Note that therefore $\mathsf{b}_0 s_0 \mathsf{b}_0 = \mathsf{b}_0 h_0 = \mathsf{b}_0 h_1$.

Now let $m \geq 0$ and assume that there exists s_m such that

(i) $\mathsf{b}_m s_m \mathsf{b}_m = \mathsf{b}_m h_{m+1}$

(ii) $\|s_m\| \leq C_m(m+1)!$

Then $h_{m+1} - s_m\mathsf{b}_m$ takes values in $\ker(\mathsf{b}_m)$. As b_{m+1} is an admissible surjection onto $\ker(\mathsf{b}_m)$ and β_{m+1} is a free A-bimodule, there exists an A-bimodule map $s_{m+1} : \beta_{m+1} \to \beta_{m+2}$ which satisfies

$$\mathsf{b}_{m+1} s_{m+1} = h_{m+1} - s_m \mathsf{b}_m$$

and $\|s_{m+1}\| \leq \|h_{m+1} - s_m\mathsf{b}_m\|$. By our inductive hypothesis $\|s_m\| \leq C_m(m+1)!$, so

$$\begin{aligned}\|s_{m+1}\| &\leq \|h_{m+1}\| + C_m\|\mathsf{b}_m\|m! \\ &\leq \|h_{m+1}\| + C_m(m+2)! = C_{m+1}(m+2)!\end{aligned}$$

Finally, since

$$\mathsf{b}_{m+1}s_{m+1}\mathsf{b}_{m+1} = (h_{m+1} - s_m\mathsf{b}_m)\,\mathsf{b}_{m+1} = h_{m+1}\mathsf{b}_{m+1}$$

our inductive step is complete. □

Applying the lemma to the chain map $h = \mathsf{id} - f$ immediately yields the following.

Corollary A.3. *Let* f *be the chain map from Lemma A.1. Then there exist* A*-bimodule maps* $\tau_n : \beta_n \to \beta_{n+1}$ *such that*

(i) $\tau_{-1} = 0$

(ii) $\mathsf{b}_{n-1}\tau_{n-1} + \tau_n \mathsf{b}_n = \mathsf{id} - f_n$ *for all* $n \geq 0$

(iii) $\|\tau_n\| \leq (n+1)!(e + \|\Delta\| e^{\|\Delta\|})$ *for all* $n \geq 0$.

We now have everything in place to prove Theorem 1.4.6. Given an A-bimodule X, the complex $X \mathop{\widehat{\otimes}}_{A^e} \beta_*$ is isometrically isomorphic to the Hochschild chain complex $\mathcal{C}_*(A,X)$; under this isomorphism $\mathsf{id}_X \mathop{\widehat{\otimes}}_{A^e} \mathsf{b}_n$ is identified with the Hochschild boundary map d_n, and the map $\mathsf{id}_X \mathop{\widehat{\otimes}}_{A^e} f_n$ is identified with the bounded linear map $\alpha_n : \mathcal{C}_n(A,X) \to \mathcal{C}_n(A,X)$ that is given by the formula

$$\alpha_n(x \otimes a_1 \otimes \ldots \otimes a_n) := \sum_{i(1)\geq 1} \ldots \sum_{i(n+1)\geq 1} v_{i(n+1)} x u_{i(1)} \otimes v_{i(1)} a_1 u_{i(2)} \otimes \ldots \otimes v_{i(n)} a_n u_{i(n+1)} \ .$$

If we let t_n be the map corresponding to $\mathsf{id}_X \mathop{\widehat{\otimes}}_{A^e} \tau_n$, then

$$\mathsf{d}_n t_n + t_{n-1} \mathsf{d}_{n-1} = \mathsf{id} - \alpha_n$$

and $\|t_n\| \leq \|\tau_n\| \leq (n+1)!(e + \|\Delta\| e^{\|\Delta\|})$. Finally, since $\sum_{i\geq} x u_i \otimes v_i = \sum_{i\geq 1} u_i \otimes v_i x$ for all $x \in K$, it is straightforward to check that α factors through the quotient chain map $\mathcal{C}_*(A,X) \to \mathcal{C}_*^K(A,X)$, and this completes the proof of Theorem 1.4.6.

Appendix B

Biflatness implies simplicial triviality

Here we give a direct proof that a biflat Banach algebra is simplicially trivial.

It is convenient to use the following lemma (which is sometimes taken as an alternative *definition* of biflatness).

Lemma B.1 ([19, Exercise VII.2.8]). *Let A be a biflat Banach algebra. Then there exists a bounded net (s_ν) of A-bimodule maps $A \to A\widehat{\otimes}A$, such that $\pi s_\nu(a)$ converges weakly to a for all $a \in A$.*

Let A be a biflat Banach algebra, and let $(s_\nu)_{\nu\in\Lambda}$ be the net of A-bimodule maps $A \to A\widehat{\otimes}A$ that is provided by Lemma B.1. We shall use the net (s_ν) to construct a splitting homotopy for the cochain complex $\mathcal{C}^*(A, A')$.

We identify the space $\mathcal{C}^n(A, A')$ with the dual space $\left(A^{\widehat{\otimes} n+1}\right)'$, so that the Hochschild coboundary map takes the form

$$\delta\psi(a_0\otimes a_1\otimes \dots a_n) = \begin{cases} \displaystyle\sum_{j=0}^{n-1}(-1)^j\psi(a_0\otimes \dots \otimes a_j a_{j+1}\otimes \dots \otimes a_n) \\ \qquad +(-1)^n\psi(a_n a_0\otimes a_1\otimes \dots a_{n-1})x \end{cases}$$

Define $\alpha_\nu : \mathcal{C}^{n+1}(A, A') \to \mathcal{C}^n(A, A')$ by

$$\alpha_\nu\psi(a_0\otimes \dots \otimes a_n) := \psi(s_\nu(a_0)\otimes a_1 \dots \otimes a_n)$$

and let α be a w^*-cluster point of the net $(\alpha_\nu)_{\nu\in\Lambda}$. Then α is a bounded linear map and we need only show that $\alpha\delta + \delta\alpha = \mathsf{id}$.

Let $\phi \in \mathcal{C}^n(A, A')$. Clearly, for any $a_0, \ldots, a_n \in A$,

$$(\alpha\delta\phi + \delta\alpha\phi)(a_0 \otimes \ldots \otimes a_n) = \lim_{\nu \in \Lambda} (\alpha_\nu \delta\phi + \delta\alpha_\nu\phi)(a_0 \otimes \ldots \otimes a_n)$$

Let $\nu \in \Lambda$ and let $a_0, \ldots, a_n \in A$: when we expand out terms in the sum

$$(\alpha_\nu \delta\phi + \delta\alpha_\nu\phi)(a_0 \otimes \ldots \otimes a_n)$$

most of them cancel and we have

$$\begin{aligned}
&(\alpha_\nu \delta\phi + \delta\alpha_\nu\phi)(a_0 \otimes \ldots \otimes a_n) \\
&= \begin{cases} \phi(\pi s_\nu(a_0) \otimes a_1 \ldots \otimes a_n) \\ - \phi(s_\nu(a_0) \cdot a_1 \otimes a_2 \otimes \ldots \otimes a_n) \\ + (-1)^{n+1} \phi(a_n \cdot s_\nu(a_0) \otimes a_1 \otimes \ldots \otimes a_{n-1}) \\ + \phi(s_\nu(a_0 a_1) \otimes a_2 \otimes \ldots \otimes a_n) \\ + (-1)^n \phi(s_\nu(a_n a_0) \otimes a_1 \otimes \ldots \otimes a_{n-1} \end{cases} \\
&= \phi(\pi s_\nu(a_0) \otimes a_1 \ldots \otimes a_n)
\end{aligned}$$

Since α is a w^*-cluster point of $(\alpha_\nu)_{\nu \in \Lambda}$,

$$\liminf_{\nu \in \Lambda} |(\alpha_\nu \delta\phi + \delta\alpha_\nu\phi)(a_0 \otimes \ldots \otimes a_n) - (\alpha\delta\phi + \delta\alpha\phi)(a_0 \otimes \ldots \otimes a_n)| = 0$$

and so

$$(\alpha\delta\phi + \delta\alpha\phi)(a_0 \otimes \ldots \otimes a_n) = \lim_{\nu \in \Lambda} \phi(\pi s_\nu(a_0) \otimes a_1 \ldots \otimes a_n) = \phi(a_0 \otimes \ldots \otimes a_n).$$

Hence $\alpha\delta\phi + \delta\alpha\phi = \phi$, as required.

Appendix C

Another proof that $\mathcal{H}_1(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+))$ is an ℓ^1-space

Let A denote the Banach algebra $\ell^1(\mathbb{Z}_+)$ with convolution multiplication.

Lemma 4.1.1 Let $q : \mathcal{C}_1(A, A) \to \ell^1(\mathbb{N})$ be the bounded linear map defined by $q(1 \otimes 1) = 0$ and

$$q(z^k \otimes z^l) = \frac{l}{k+l} z^{k+l} \quad (k, l \in \mathbb{Z}_+; k + l \geq 1).$$

Then q is surjective and $\ker(q) = \mathcal{B}_1(A, A)$.

The proof of this result in [13, Propn 7.3] is somewhat fiddly. We present a slightly more streamlined approach (due to the author).

A proof of Lemma 4.1.1

Let $q : \mathcal{C}_1(A, A) \to \ell^1(\mathbb{N})$ be defined as above, and let

$$\mathsf{B}(z^N) := 1 \otimes z^N$$

$$\mathsf{S}(z^{N-j} \otimes z^j) := \begin{cases} 1 \otimes z^j \otimes z^{N-j} + z^j \otimes z^j \otimes z^{N-2j} & \text{if } 0 \leq j \leq N/2 \\ 1 \otimes z^j \otimes z^{N-j} - z^{N-j} \otimes z^{2j-N} \otimes z^{N-j} & \text{if } N/2 \leq j \leq N \end{cases}$$

$$\mathsf{H}(z^{N-j} \otimes z^j) := \begin{cases} 2z^{N-j} \otimes z^j + z^{2j} \otimes z^{N-2j} & \text{if } 0 \leq j \leq N/2 \\ 2z^{N-j} \otimes z^j - z^{2N-2j} \otimes z^{2j-N} & \text{if } N/2 \leq j \leq N \end{cases}$$

Proposition C.1. *The maps q, B, H fit into a diagram*

$$\begin{array}{ccccc} \ell^1(\mathbb{N}) & \underset{\mathsf{B}}{\overset{q}{\rightleftarrows}} & \mathcal{C}_1(A,A) & \overset{\mathsf{d}}{\longleftarrow} & \mathcal{C}_2(A,A) \\ & & \mathsf{H}\big\uparrow & \overset{\mathsf{S}}{\nearrow} & \\ & & \mathcal{C}_1(A,A) & & \end{array}$$

where $q\mathsf{B} = \mathsf{id}$ and

$$(\mathsf{id} - \mathsf{B}q)\mathsf{H} = \mathsf{dS}$$

(Here d denotes the Hochschild boundary operator.)

Proof. This follows by direct computation on elementary tensors. ☐

Now observe that since $\|2\mathsf{id} - \mathsf{H}\| \leq 1$, H is *invertible* as a bounded linear operator on the Banach space $\mathcal{C}_1(A, A)$. Hence $\mathsf{id} - \mathsf{B}q = \mathsf{dSH}^{-1}$ and the complex

$$\ell^1(\mathbb{N}) \xleftarrow{\ q\ } \mathcal{C}_1(A,A) \xleftarrow{\ \mathsf{d}\ } \mathcal{C}_2(A,A)$$

is thus split exact in Ban. ☐

The argument just given may seem slightly mysterious, as we have provided no explanation of how one might come up with the maps B and S. In fact the construction above was discovered while considering the dual problem of proving that $\mathcal{H}^2(A, A')$ is a Banach space. To help motivate the proof just given, we shall give this argument for $\mathcal{H}^2$ below.

An easy proof that $\mathcal{H}^2(A, A')$ is Banach

We note that the argument to follow differs from the original one given in [6], although a comparison between the two shows them to be based on similar principles.

Lemma C.2. *Fix $N \in \mathbb{N}$. Let $\varphi : \{0, 1, \dots, N\} \to \mathbb{C}$ be a function that is 1-almost additive, in the sense that whenever $0 \leq j, 0 \leq k$ and $j + k \leq N$*

$$|\varphi(j) - \varphi(j + k) + \varphi(k)| \leq 1 \quad . \tag{C.1}$$

Suppose furthermore that $\varphi(N) = 0$. Then $|\varphi(n)| \leq 2$ for all $n \in \{0, 1, \dots, N\}$.

The idea used to prove this is very simple, and goes roughly as follows. Choose n maximising $|\varphi(n)|$. The condition (C.1) implies that $\varphi(N - n)$ is equal to $-\varphi(n)$ give or take some constant, and then applying the condition to the *difference* of n

and $N - n$ tells us that the modulus of $\varphi(|N - 2n|)$ is twice that of $\varphi(n)$, give or take some constant. Since n was chosen to maximise $|\varphi(n)|$ we deduce that $|\varphi(n)|$ is at least $2|\varphi(n)|$ give or take some constant.

It is not hard to make this precise. First we record a trivial geometric lemma.

Lemma C.3. *Let $w, z \in \mathbb{C}$ with $|w| \geq |z|$. Then $|w| \leq |2w - z|$.*

Proof. Immediate from the triangle inequality (or a picture). □

Proof of Lemma C.2. Choose n such that $|\varphi(n)| = \max_{0 \leq j \leq N} |\varphi(j)|$, and note that

$$|\varphi(n) + \varphi(N - n)| \leq 1$$

We have two cases to consider:

Case A: $0 \leq n \leq \frac{N}{2}$.

In this case $N - n \geq n$, and so

$$|\varphi(n) - \varphi(N - n) + \varphi(N - 2n)| \leq 1 \ ;$$

hence

$$|2\varphi(n) - \varphi(N - 2n)| \leq |\varphi(n) - \varphi(N - n) + \varphi(N - 2n)| + |\varphi(n) + \varphi(N - n)| \leq 2 \ .$$

Since $|\varphi(N - 2n)| \leq |\varphi(n)|$ we may apply Lemma C.3 and deduce that $|\varphi(n)| \leq 2$.

Case B: $\frac{N}{2} \leq n \leq N$.

In this case $n \geq N - n$, and so

$$|\varphi(N - n) - \varphi(n) + \varphi(2n - N)| \leq 1 \ ;$$

hence

$$|2\varphi(n) - \varphi(2n - N)| \leq |\varphi(n) - \varphi(N - n) - \varphi(2n - N)| + |\varphi(n) + \varphi(N - n)| \leq 2 \ .$$

Since $|\varphi(2n - N)| \leq |\varphi(n)|$ we may apply Lemma C.3 and deduce that $|\varphi(n)| \leq 2$. □

Corollary C.4 (Second cohomology is Banach). *Let $\psi \in \mathcal{C}^1(A, A')$ and suppose that $\|\delta\psi\| \leq 1$. Then there exists $\phi \in \mathcal{C}^1(A, A')$ such that $\delta\phi = \delta\psi$ and $\|\phi\| \leq 2$.*

Proof. Let $N \in \mathbb{N}$. The condition that $\|\delta\psi\| \leq 1$ implies that

$$\left|\psi(z^j)(z^{N-j}) - \psi(z^{j+k})(z^{N-j-k}) + \psi(z^k)(z^{N-k}\right| = |\delta\psi(z^j, z^k)(z^{N-j-k})| \leq 1$$

whenever $0 \leq j, 0 \leq k, j + k \leq N$.

For each N, let

$$\varphi_N(n) := \psi(z^n)(z^{N-n}) - \frac{n}{N}\psi(z^N)(1) \qquad (0 \leq n \leq N) \quad ;$$

then φ_N satisfies all the conditions of Lemma C.2, and therefore $|\varphi_N(n)| \leq 2$ for all $n \leq N$. It follows that if we set

$$\phi(z^j)(z^k) := \psi(z^j)(z^k) - \frac{j}{j+k}\psi(z^{j+k})(1)$$

then

$$\|\phi\| = \sup_{j,k} |\phi(z^j)(z^k)| = \sup_{j,k} |\varphi_{j+k}(j)| \leq 2$$

and a quick check shows that $\delta\phi = \delta\psi$ as required. □

An "algebraic" proof that $\mathcal{H}^2(A, A')$ is Banach

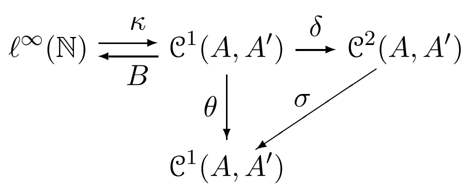
Retracing our way through the steps of the previous section, we can reformulate them "in a more algebraic fashion" as follows.

Let $B : \mathcal{C}^1(A, A') \to \ell^\infty(\mathbb{N})'$ be given by $B\psi(N) = \psi(N, 0)$; let $\kappa : \ell^\infty(\mathbb{N}) \to \mathcal{C}^1(A, A')$ be given by $\kappa\gamma(j, N-j) := \frac{j}{N}\gamma(N)$i, and let

$$\theta(j, N-j) := \begin{cases} 2\psi(j, N-j) + \psi(N-2j, 2j) & \text{if } 0 \leq j \leq N/2 \\ 2\psi(j, N-j) - \psi(2j-N, 2N-2j) & \text{if } N/2 \leq j \leq N \end{cases}$$

$$\sigma\chi(j, N-j) := \begin{cases} \chi(j, N-j, 0) + \chi(j, N-2j, j) & \text{if } 0 \leq j \leq N/2 \\ \chi(j, N-j, 0) - \chi(2j-N, N-j, N-j) & \text{if } N/2 \leq j \leq N \end{cases}$$

Proposition C.5. *These maps fit into a diagram*



where the top row is exact and

$$\theta(\mathsf{id} - \kappa B) = \sigma\delta$$

Proof. First note that

$$\delta\kappa\gamma(j, k, N-k) = \kappa\gamma(k, N-kj) - \kappa\gamma(j+k, N-j-k) + \kappa\gamma(j, N-j) = 0$$

for all j, k, N with $j + k \leq N$.

Also, $\kappa B\psi(i)(N-i) = \frac{i}{N} B\psi(N) = \frac{i}{N}[\psi(N,0) - \psi(0,N)]$. Direct calculation now gives

$$\begin{aligned} \theta\kappa B\psi(j)(N-j) &= \begin{cases} 2\kappa B\psi(j, N-j) + \kappa B\psi(N-2j, 2j) & \text{if } 0 \leq j \leq N/2 \\ 2\kappa B\psi(j, N-j) - \kappa B\psi(2j-N, 2N-2j) & \text{if } N/2 \leq j \leq N \end{cases} \\ &= \psi(N,0) \end{aligned}$$

while

$$\begin{aligned} \sigma\delta\psi(j, N-j) &= \begin{cases} \delta\psi(j, N-j, 0) + \delta\psi(j, N-2j, j) & \text{if } 0 \leq j \leq N/2 \\ \delta\psi(j, N-j, 0) - \delta\psi(2j-N, N-j, N-j) & \text{if } N/2 \leq j \leq N \end{cases} \\ &= \begin{cases} 2\psi(j, N-j) - \psi(N,0) + \psi(N-2j, 2j) & \text{if } 0 \leq j \leq N/2 \\ 2\psi(N-j, j) - \psi(N,0) + \psi(2j-N, 2N-2j) & \text{if } N/2 \leq j \leq N \end{cases} \\ &= \tau(j, N-j) - \psi(N,0) \end{aligned}$$

Hence $\theta\kappa B\psi(j)(N-j) + \sigma\delta\psi(j, N-j) = \theta\psi(j, N-j)$ as required. ☐

"Algebraicised" proof that $\mathcal{H}^2(A, A')$ is Banach. Since $\|\theta - 2\mathsf{id}\| \leq 1$, θ is invertible, and $\mathsf{id} - \kappa B = \theta^{-1}\sigma\delta$. In particular $\delta = \delta(\mathsf{id} - \kappa B) = \delta\theta^{-1}\sigma\delta$ has closed range. ☐

Remark. Note that the diagram (C.5) is the dual of the diagram (C.1), which was used at the start of this appendix to obtain results for homology.

Bibliography

[1] J. M. Alonso *et al.* Notes on word hyperbolic groups. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 3–63. World Sci. Publishing, River Edge, NJ, 1991. Edited by H. Short.

[2] W. G. Bade, H. G. Dales, and Z. A. Lykova. Algebraic and strong splittings of extensions of Banach algebras. *Mem. Amer. Math. Soc.*, 137(656):viii+113, 1999.

[3] M. Barr. Harrison homology, Hochschild homology and triples. *J. Algebra*, 8:314–323, 1968.

[4] S. Bowling and J. Duncan. First order cohomology of Banach semigroup algebras. *Semigroup Forum*, 56(1):130–145, 1998.

[5] Y. Choi. Simplicial homology and Hochschild cohomology of Banach semilattice algebras. To appear *Glasgow Math. Journal* (2006).

[6] H. G. Dales and J. Duncan. Second-order cohomology groups of some semigroup algebras. In *Banach algebras '97 (Blaubeuren)*, pages 101–117. de Gruyter, Berlin, 1998.

[7] A. Defant and K. Floret. *Tensor norms and operator ideals*, volume 176 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1993.

[8] J. Duncan and I. Namioka. Amenability of inverse semigroups and their semigroup algebras. *Proc. Roy. Soc. Edinburgh Sect. A*, 80(3-4):309–321, 1978.

[9] M. Gerstenhaber. Developments from Barr's thesis. *J. Pure Appl. Algebra*, 143(1-3):205–220, 1999.

[10] M. Gerstenhaber and S. D. Schack. A Hodge-type decomposition for commutative algebra cohomology. *J. Pure Appl. Algebra*, 48(3):229–247, 1987.

[11] M. Gerstenhaber and S. D. Schack. The cohomology of presheaves of algebras. I. Presheaves over a partially ordered set. *Trans. Amer. Math. Soc.*, 310(1):135–165, 1988.

[12] F. Gourdeau, B. E. Johnson, and M. C. White. The cyclic and simplicial cohomology of $\ell^1(\mathbb{N})$. *Trans. Amer. Math. Soc.*, 357(12):5097–5113 (electronic), 2005.

[13] F. Gourdeau, Z. A. Lykova, and M. C. White. A Künneth formula in topological homology and its applications to the simplicial cohomology of $\ell^1(\mathbb{Z}_+^k)$. *Studia Math.*, 166(1):29–54, 2005.

[14] F. Gourdeau, A. Pourabbas, and M. C. White. Simplicial cohomology of some semigroup algebras. *In preparation.*

[15] R. I. Grigorchuk. Some results on bounded cohomology. In *Combinatorial and geometric group theory (Edinburgh, 1993)*, volume 204 of *London Math. Soc. Lecture Note Ser.*, pages 111–163. Cambridge Univ. Press, Cambridge, 1995.

[16] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[17] N. Grønbæk. Morita equivalence for Banach algebras. *J. Pure Appl. Algebra*, 99(2):183–219, 1995.

[18] N. Grønbæk and A. T.-M. Lau. On Hochschild cohomology of the augmentation ideal of a locally compact group. *Math. Proc. Cambridge Philos. Soc.*, 126(1):139–148, 1999.

[19] A. Y. Helemskii. *The homology of Banach and topological algebras*, volume 41 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1989.

[20] J. M. Howie. *Fundamentals of semigroup theory*, volume 12 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1995. Oxford Science Publications.

[21] B. E. Johnson. *Cohomology in Banach algebras*. American Mathematical Society, Providence, R.I., 1972.

[22] B. E. Johnson. Higher-dimensional weak amenability. *Studia Math.*, 123(2):117–134, 1997.

[23] J.-L. Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[24] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*. Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[25] S. Mac Lane. *Categories for the working mathematician (2nd ed.)*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[26] A. Pourabbas and M. C. White. The second bounded group cohomology of group algebras. *preprint*.

[27] J. J. Rotman. *An introduction to homological algebra*, volume 85 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979.

[28] V. Runde. A functorial approach to weak amenability for commutative Banach algebras. *Glasgow Math. J.*, 34(2):241–251, 1992.

[29] R. A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2002.

[30] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[31] M. C. White. Injective modules for uniform algebras. *Proc. London Math. Soc. (3)*, 73(1):155–184, 1996.