Translation-finite sets and weakly compact derivations

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(from the ice age to the dole age, there is but one concern. . .

Joint work with M. J. Heath (IST, Lisbon).
(cf. arXiv 0811.4432)
Some sets are bigger than others

Translation-finite subsets of $\mathbb{N}$

- $\{n^2 : n \in \mathbb{N}\}$
- $\bigcup_{m=1}^{\infty} \{3^m + jm : 0 \leq j \leq \lfloor 2 \cdot 3^m/m \rfloor - 1\}$
  \[= \{3, 4, \ldots, 8, 9, 11, \ldots, 25, 27, 30, \ldots, 78, 81, 85, \ldots\}\]

Non-TF subsets of $\mathbb{N}$

- $\{3n + 1 : n \in \mathbb{N}\}$
- $\{2^n(n+1) + j : n \in \mathbb{N}, 0 \leq j \leq n - 1\}$
  \[= \{4, 64, 65, 4096, 4097, 4098, 1048576, 1048577, \ldots\}\]
1. Getting started

Let $A$ be a Banach algebra and $M$ a Banach $A$-bimodule (such as $A$ itself, or $A^*$).

**Definition**

A *derivation from $A$ to $M$* is a linear map $D : A \to M$ which satisfies

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad \text{for all } a, b \in A.$$  

For this talk, $A$ will be the algebra $\ell^1(\mathbb{Z}_+)$ equipped with the convolution product. We’ll identify this with the algebra of absolutely convergent power series in one variable.

**Notation**

- Point masses in $\ell^1(\mathbb{Z}_+)$ will be denoted by $z^n$.
- $\text{Der}(\ell^1(\mathbb{Z}_+))$ denotes the space of all bounded derivations from $\ell^1(\mathbb{Z}_+)$ to its dual.
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- $\text{Der}(\ell^1(\mathbb{Z}_+))$ denotes the space of all bounded derivations from $\ell^1(\mathbb{Z}_+)$ to its dual.
2. Parametrizing $\text{Der}(\ell^1(\mathbb{Z}_+))$

The ‘symbol’ of a derivation

Given $D \in \text{Der}(\ell^1(\mathbb{Z}_+))$, let $\psi = D(\_)(1) \in \ell^\infty(\mathbb{N})$. Then

$$D(z^j)(z^k) = \frac{j}{j+k} \psi_{j+k} \quad \text{for all } j, k \in \mathbb{Z}_+. \quad (1)$$

In particular, the range of $D$ is contained in $c_0(\mathbb{N})$.

Producing derivations

Conversely: given any $\psi \in \ell^\infty(\mathbb{N})$, the formula

$$D_\psi(z^j)(z^k) := \frac{j}{j+k} \psi_{j+k} \quad (j, k \in \mathbb{Z}_+) \quad (2)$$

defines a bounded derivation $D_\psi : \ell^1(\mathbb{Z}_+) \to \ell^\infty(\mathbb{Z}_+)$. The linear maps $D \mapsto D(\_)(1)$ and $\psi \mapsto D_\psi$ are mutually inverse and contractive.
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The linear maps $D \mapsto D(_{\text{}})(1)$ and $\psi \mapsto D_\psi$ are mutually inverse and contractive.
3. A bounded derivation that is not weakly compact

To ease notation: if $X \subset \mathbb{N}$, we write $D_X$ instead of $D_{(1_X)}$. That is,

$$D_X(z^j)(z^k) = \frac{j}{j+k}1_{j+k \in X} \quad (j, k \in \mathbb{Z}_+) .$$

**Example**

Let $X = \{3n + 1 : n \in \mathbb{N}\}$. Then $D_X$ is not weakly compact.

The result follows quickly by observing that

$$D_X(z^{3j})(z^{3k+1}) = \frac{3j}{3j + 3k + 1}$$

and then appealing to Grothendieck’s double-limit criterion.

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A (more) direct proof

Recall that \( D_X(z^j)(z^k) = \frac{j}{j + k} 1_{j + k \equiv 1 \mod 3} \quad (j, k \in \mathbb{Z}_+) \).

Proof without resorting to the GDLC.

If \( D_X \) were weakly compact then (by Eberlein-Šmulian) the sequence 
\( (D_X(z^n))_{n \geq 1} \) would have a subsequence converging weakly, and hence pointwise, to some \( \psi \in c_0(\mathbb{N}) \).
Furthermore, by passing to a further subsequence if necessary, we may assume that this sequence is indexed by \( n_1 < n_2 < \ldots \) where all the \( n_j \) are congruent modulo 3.

Now choose \( r \) such that \( n_j + r \equiv 1 \mod 3 \). We have

\[
D_X(z^{n_j})(z^{3k+r}) = \frac{n_j}{n_j + 3k + r}
\]

for all \( j, k \);

and so \( \psi_{3k+r} = \lim_j D_X(z^{n_j})(z^{3k+r}) = 1 \) for all \( k \). But this contradicts the assumption that \( \psi \in c_0(\mathbb{N}) \).
If $X \subseteq \mathbb{N}$ and $m \in \mathbb{Z}_+$, let $X - m$ denote \( \{ n \in \mathbb{N} : m + n \in X \} \).

**Definition**

Let $X \subseteq \mathbb{N}$. We say $X$ is translation-finite (TF for short) if, for each sequence $n_1 < n_2 < \ldots$ in $\mathbb{Z}_+$, there exists $m$ such that $\bigcap_{i=1}^{m} (X - n_i)$ is at most finite.

**Remarks**

- Every finite set is TF.
- The set $\{ n^2 : n \in \mathbb{N} \}$ is TF.
- If $X$ contains an infinite arithmetic progression, then it is non-TF.
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5. Characterizing the weakly compact derivations

**Theorem (C. + Heath)**

Let $\psi \in \ell^\infty(\mathbb{N})$. For each $\varepsilon > 0$, let $S_\varepsilon := \{ n \in \mathbb{N} : |\psi_n| \geq \varepsilon \}$. Then $D_\psi$ is weakly compact if and only if the sets $S_\varepsilon$ are TF for all $\varepsilon > 0$.

**Corollary**

Let $X \subseteq \mathbb{N}$. Then the derivation $D_X$ is weakly compact if and only if $X$ is TF.

The proof of the theorem is direct, using only the equivalence of weak compactness and sequential weak compactness (via Eberlein-Šmulian or a special case thereof), together with ‘diagonal subsequence’ arguments.
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It turns out that translation-finite sets had been previously defined by Ruppert (Semigroup Forum, 1985), for a somewhat different purpose.

**Theorem (Ruppert)**

Let $S \subseteq \mathbb{N}$. Then the following are equivalent:

(i) every bounded function supported on $S$ belongs to $\text{WAP}(\mathbb{N})$;

(ii) $S$ is translation finite.

We have been unable to deduce our result from Ruppert’s in a “soft” way.

**Remark**

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*Ruppert’s result immediately implies that the union of two TF sets is TF. It is an interesting exercise to prove this directly from the definition...*
7. Density of TF-sets

Finer classification of TF-sets, if such exists, seems hard to find in the literature, and there seems to be no ‘structure theory’ for these sets. We can at least show that Banach density seems not to be useful here.

Theorem (C.+Heath, *ibid.*)

Let $X \subseteq \mathbb{N}$ be a subset which has strictly positive Banach density, i.e.

$$\lim_{n \to \infty} \sup_{m \in \mathbb{Z}_+} n^{-1} |X \cap \{m+1, \ldots, m+n\}| > 0.$$  \hspace{1cm} (3)

Then $X$ is not TF.

Idea of the proof

The density hypothesis easily yields a sequence $(V_n)$ of subsets of $\mathbb{N}$, with $|V_n| \to \infty$, such that

- for each $n$, there are infinitely many $s \in \mathbb{N}$ such that $V_n \subseteq X - s$.

The hard part is to arrange things so that $V_n \subseteq V_{n+1}$ for all $n$. (Once this is achieved, it is easy to see that $X$ cannot be TF.)
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8. Special kinds of TF sets

The following definition goes back to work of Rudin, Ramirez and others.

**Definition**

Let $X \subseteq \mathbb{Z}$. We say $X$ is a **$T$-set** if for every distinct $m, n \in \mathbb{Z}$, the intersection $(X - m) \cap (X - n)$ is at most finite.

In fact, $X$ is a $T$-set if and only if “the size of gaps between entries tends to infinity”.

It is immediate that if $X \subseteq \mathbb{N}$ is a $T$-set, it is TF. A little more work shows that finite unions of $T$-sets are also TF. However, there exist TF sets which are not of this form (Chou, *Trans. AMS*, 1990).

**Question**

Is there a functional-analytic interpretation of (the closure of) \( \{D_\psi : \text{supp}(\psi) \text{ is a } T\text{-set}\} \)?
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It is immediate that if $X \subseteq \mathbb{N}$ is a T-set, it is TF. A little more work shows that finite unions of T-sets are also TF. However, there exist TF sets which are not of this form (Chou, *Trans. AMS*, 1990).

**Question**

Is there a functional-analytic interpretation of (the closure of) \( \{D_\psi : \text{supp}(\psi) \text{ is a T-set}\} \)?
Given $1 \leq p < \infty$ and $X \subseteq \mathbb{N}$, when is $D_X$ $p$-summing? (Works for $X = \{\text{powers of 2}\}$, but fails for the $T$-set $X = \{\text{square numbers}\}$.)

Perturbation questions: if $T : \ell^1(\mathbb{Z}_+) \to \ell^\infty(\mathbb{Z}_+)$ is weakly compact and

$$\sup_{j,k,l} |T(z^k)(z^{l+j}) - T(z^{j+k})(z^l) + T(z^j)(z^{k+l})| \leq 1,$$

does there exist a weakly compact derivation $D \in \text{Der}(\ell^1(\mathbb{Z}_+))$ such that $\|T - D\|$ is “small”?