Quantum Markov semigroups
and quantum stochastic flows
– construction and perturbation

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Structure of the talk

Construction
- The classical theory
- Non-commutative probability
- Solving a quantum stochastic differential equation [BW]
- Examples [BW]

Perturbation
- The classical theory
- Quantum Feynman–Kac on von Neumann algebras [BLS]
- Quantum Feynman–Kac on $C^*$ algebras [BW]
Construction
Classical Markov semigroups

- Markov processes
- Markov semigroups
- Infinitesimal generators

Diagram:
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  Markov processes
  /      \
 /        \
Markov semigroups   Infinitesimal generators
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Alexander Belton (Lancaster University)
Markov processes

Definition

A Markov process with state space $S$ is a collection of $S$-valued random variables $(X_t)_{t \geq 0}$ on a common probability space such that

$$
\mathbb{E}\left[f(X_{s+t}) \mid \sigma(X_r : 0 \leq r \leq s)\right] = \mathbb{E}\left[f(X_{s+t}) \mid X_s\right] \quad (s, t \geq 0)
$$

for all $f \in L^\infty(S)$.

Such a Markov process is time homogeneous if

$$
\mathbb{E}\left[f(X_{s+t}) \mid X_s = x\right] = \mathbb{E}\left[f(X_t) \mid X_0 = x\right] \quad (s, t \geq 0, \ x \in S)
$$

for all $f \in L^\infty(S)$. 
Markov semigroups

Given a time-homogeneous Markov process, setting

$$(T_t f)(x) = \mathbb{E}[f(X_t) | X_0 = x] \quad (t \geq 0, \ x \in S)$$

defines a *Markov semigroup* on $L^\infty(S)$.

**Definition**

A *Markov semigroup* on $L^\infty(S)$ is a family $(T_t)_{t \geq 0}$ such that

1. $T_t : L^\infty(S) \rightarrow L^\infty(S)$ is a linear operator for all $t \geq 0$,  
2. $T_s \circ T_t = T_{s+t}$ for all $s, t \geq 0$ and $T_0 = I$ (semigroup),  
3. $\|T_t\| \leq 1$ for all $t \geq 0$ (contraction),  
4. $T_t f \geq 0$ whenever $f \geq 0$, for all $t \geq 0$ (positive).

If $T_t 1 = 1$ for all $t \geq 0$ then $T$ is *conservative*. 
**Feller semigroups**

**Definition**

Suppose the state space $S$ is a locally compact Hausdorff space. The Markov semigroup $T$ is *Feller* if

$$T_t(C_0(S)) \subseteq C_0(S) \quad (t \geq 0)$$

and

$$\|T_t f - f\|_\infty \to 0 \text{ as } t \to 0 \quad (f \in C_0(S)).$$

Every sufficiently well-behaved time-homogeneous Markov process is Feller: Brownian motion, Poisson process, Lévy processes, . . .

**Theorem 1**

*If the state space $S$ is separable then every Feller semigroup gives rise to a time-homogeneous Markov process.*
Infinitesimal generators

**Definition**

Let $T$ be a $C_0$ semigroup on a Banach space $E$. Its *infinitesimal generator* is the linear operator $\tau$ in $E$ with domain

$$\text{dom } \tau = \left\{ f \in E : \lim_{t \to 0} t^{-1} (T_t f - f) \text{ exists} \right\}$$

and action

$$\tau f = \lim_{t \to 0} t^{-1} (T_t f - f).$$

The operator $\tau$ is closed and densely defined.

If $T$ comes from a Markov process $X$ then

$$\mathbb{E} [ f(X_{t+h}) - f(X_t) | X_t ] = (T_h f - f)(X_t) = h(\tau f)(X_t) + o(h),$$

so $\tau$ describes the change in $X$ over an infinitesimal time interval.
Theorem 2 (Lumer–Phillips)

A closed, densely defined operator $\tau$ in the Banach space $E$ generates a strongly continuous contraction semigroup on $E$ if and only if

$$\text{im}(\lambda I - \tau) = E \text{ for some } \lambda > 0$$

and (dissipativity)

$$\| (\lambda I - \tau)x \| \geq \lambda \| x \| \text{ for all } \lambda > 0 \text{ and } x \in \text{dom } \tau.$$
Theorem 3 (Hille–Yosida–Ray)

A closed, densely defined operator $\tau$ in $C_0(S)$ is the generator of a Feller semigroup on $C_0(S)$ if and only if

- $\lambda I - \tau : \text{dom } \tau \to X$ has bounded inverse for all $\lambda > 0$ and
- $\tau$ satisfies the positive maximum principle.

Definition

Let $S$ be a locally compact Hausdorff space. A linear operator $\tau$ in $C_0(S)$ satisfies the positive maximum principle if whenever $f \in \text{dom } \tau$ and $s_0 \in S$ are such that

$$\sup_{s \in S} f(s) = f(s_0) \geq 0$$

then $(\tau f)(s_0) \leq 0$. 
Quantum Markov semigroups

Quantum Markov processes

Quantum Feller semigroups

Infinitesimal generators
Quantum Feller semigroups

**Theorem 4**

Every commutative $C^*$ algebra is isometrically isomorphic to $C_0(S)$, where $S$ is a locally compact Hausdorff space.

**Definition**

A quantum Feller semigroup on the $C^*$ algebra $A$ is a family $(T_t)_{t \geq 0}$ such that

1. $T_t : A \to A$ is a linear operator for all $t \geq 0$,
2. $T_s \circ T_t = T_{s+t}$ for all $s, t \geq 0$ and $T_0 = I$,
3. $\|T_t x - x\| \to 0$ as $t \to 0$ for all $x \in A$,
4. $\|T_t\| \leq 1$ for all $t \geq 0$,
5. $(T_t a_{ij}) \in M_n(A)_+$ whenever $(a_{ij}) \in M_n(A)_+$, for all $n \geq 1$ and $t \geq 0$.

If $A$ is unital and $T_t 1 = 1$ for all $t \geq 0$ then $T$ is conservative.
From generators to semigroups

Problem

Given a densely defined operator $\tau$ on the $C^*$ algebra $A$, show that $\tau$ is closable and its closure that generates a quantum Feller semigroup $T$.

- It can be very difficult to verify the conditions of the Lumer–Phillips theorem.
- Is there a non-commutative version of the positive maximum principle?

Strategy

Rather than construct the semigroup $T$ directly, we shall instead construct a quantum Markov process which is a dilation of $T$. 
If $X : \Omega \to S$ is a classical $S$-valued random variable then

$$\mathcal{A} \to \mathcal{M}; \ f \mapsto f \circ X$$

is a unital $\ast$-homomorphism, where $\mathcal{A} = C_0(S)$ and $\mathcal{M} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. A non-commutative random variable is a unital $\ast$-homomorphism

$$j : \mathcal{A} \to \mathcal{M}$$

from a unital $C^*$ algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ to a von Neumann algebra $\mathcal{M}$. Classical stochastic calculus has a universal sample space: càdlàg functions.

In quantum stochastic calculus, $\mathcal{B}(\mathcal{F})$ plays this rôle, where $\mathcal{F}$ is Boson Fock space over $L^2(\mathbb{R}_+; k)$, and

$$\mathcal{A} \otimes \mathcal{m} \mathcal{B}(\mathcal{F}) \subseteq \mathcal{M} := \mathcal{B}(\mathcal{H} \otimes \mathcal{F}).$$
Non-commutative probability

Adaptedness

\[ L^2(\mathbb{R}_+) \cong L^2[0, t) \oplus L^2[t, \infty) \]

\[ \implies \mathcal{F} \cong \mathcal{F}_t \otimes \mathcal{F}_{[t]}; \quad \varepsilon(f) \cong \varepsilon(f|_{[0,t)}) \otimes \varepsilon(f|_{[t,\infty)}), \]

where \( \mathcal{E} := \text{lin}\{\varepsilon(f) : f \in L^2(\mathbb{R}_+; k)\} \), the linear span of the exponential vectors (such that \( \langle \varepsilon(f), \varepsilon(g) \rangle = \exp\langle f, g \rangle \)) is dense in \( \mathcal{F} \).

A quantum flow is a family of unital \( \ast \)-homomorphisms

\[ (j_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{m} \mathcal{B}(\mathcal{F}))_{t \geq 0} \]

which is adapted:

\[ j_t(a) = j_t(a) \otimes 1_{[t]} \in \mathcal{A} \otimes \mathcal{m} \mathcal{B}(\mathcal{F}_t) \otimes \mathbb{C}1_{[t]} \quad (t \geq 0, a \in \mathcal{A}). \]
Dilation

A quantum flow \((j_t : \mathcal{A} \to \mathcal{M})_{t \geq 0}\) is a \textit{dilation} of the quantum Feller semigroup \(T\) on \(\mathcal{A}\) if there exists a conditional expectation \(\mathbb{E} : \mathcal{M} \to \mathcal{A}\) such that \(T_t = \mathbb{E} \circ j_t\) for all \(t \geq 0\).

Markovianity

In this framework, Markovianity is given by a cocycle property:

\[
j_{s+t} = \hat{j}_s \circ \sigma_s \circ j_t \quad (s, t \geq 0),
\]

where

\[
\sigma_s : \mathcal{A} \otimes \mathcal{m} \mathcal{B}(\mathcal{F}) \xrightarrow{\sim} \mathcal{A} \otimes \mathcal{m} \mathcal{B}(\mathcal{F}_s)
\]

is the shift (CCR flow) and

\[
\hat{j}_s = j_s \otimes \mathcal{m} \text{id}_{\mathcal{F}_s} : \mathcal{A} \otimes \mathcal{m} \mathcal{B}(\mathcal{F}_s) \to \mathcal{A} \otimes \mathcal{m} \mathcal{B}(\mathcal{F}).
\]
Cocycles produce semigroups on $\mathcal{A}$

Let $\mathbb{E}_\Omega : \mathcal{M} \to \mathcal{A}$ be such that

$$\langle u, \mathbb{E}_\Omega(X)v \rangle = \langle u \otimes \varepsilon(0), Xv \otimes \varepsilon(0) \rangle \quad (u, v \in \mathfrak{h}, \ X \in \mathcal{M}).$$

Then $\mathbb{E}_\Omega \circ j$ is a semigroup on $\mathcal{A}$ if the quantum flow $j$ is a Markovian cocycle.

If, further, $t \mapsto \mathbb{E}_\Omega \circ j_t$ is norm continuous then the quantum flow $j$ is a Feller cocycle and $\mathbb{E}_\Omega \circ j$ is a quantum Feller semigroup.
Let $A_0 \subseteq A \subseteq \mathcal{B}(h)$ be a norm-dense $*$-subalgebra of $A$ which contains $1 = l_h$.

**Definition**

A family of linear operators $(X_t)_{t \geq 0}$ in $h \otimes \mathcal{F}$ with domains including $h \circ \mathcal{E}$ is an *adapted operator process* if

$$\langle u \in (f), X_t v \in (g) \rangle = \langle u \in (1_{[0,t]} f), X_t v \in (1_{[0,t]} g) \rangle \langle \in (1_{[t,\infty]} f), \in (1_{[t,\infty]} g) \rangle$$

for all $u, v \in h$, $f, g \in L^2(\mathbb{R}_+; k)$ and $t \geq 0$.

An *adapted mapping process on $A_0$* is a family of linear maps

$$\left( j_t : A_0 \rightarrow \mathcal{L}(h \circ \mathcal{E}; h \otimes \mathcal{F}) \right)_{t \geq 0}$$

such that $(j_t(x))_{t \geq 0}$ is an adapted operator process for all $x \in A_0$. 
A quantum stochastic differential equation

The generator

Let $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ be a linear map, where $\mathcal{B} = \mathcal{B}(\hat{k})$ and $\hat{k} := \mathbb{C} \oplus k$. Distinguish the unit vector $\omega := (1, 0) \in \hat{k}$ and let $\hat{x} := (1, x)$ for all $x \in k$. For all $z, w \in \hat{k}$, let $\phi^z_w : \mathcal{A}_0 \to \mathcal{A}_0$ be the linear map such that

$$\langle u, \phi^z_w(x)v \rangle = \langle u \otimes z, \phi(x)v \otimes w \rangle \quad (u, v \in h, \ x \in \mathcal{A}_0).$$

Definition 5

An adapted mapping process $j$ on $\mathcal{A}_0$ satisfies the QSDE

$$j_0(x) = x \otimes I_\mathcal{F}, \quad dj_t(x) = (\widetilde{j}_t \circ \phi)(x) d\Lambda_t \quad (x \in \mathcal{A}_0) \quad (1)$$

if and only if

$$\langle u \varepsilon(f), (j_t(x) - x \otimes I_\mathcal{F})v \varepsilon(g) \rangle = \int_0^t \langle u \varepsilon(f), j_s \left( \phi_{\frac{f(s)}{g(s)}(x)} \right) v \varepsilon(g) \rangle \, ds$$

for all $x \in \mathcal{A}_0$ and all $t \geq 0$, $u, v \in h$ and $f, g \in L^2(\mathbb{R}_+; k)$. 
Theorem 6

*If the quantum flow \( j \) satisfies the QSDE (1) then \( j \) is a Feller cocycle and the generator of the Feller semigroup \( \mathbb{E}_\Omega \circ j \) is an extension of \( \phi_\omega \).*

Theorem 7

*Let the quantum flow \( j \) satisfy (1). Then \( \phi : \mathcal{A}_0 \to \mathcal{A}_0 \odot \mathcal{B} \) is such that*

- \( \phi \) is \(*\)-linear,
- \( \phi(1) = 0 \) and
- \( \phi(xy) = \phi(x)(y \otimes I_k) + (x \otimes I_k)\phi(y) + \phi(x)\Delta\phi(y) \) for all \( x, y \in \mathcal{A}_0 \),

where \( \Delta := I_h \otimes P_k = \begin{bmatrix} 0 & 0 \\ 0 & I_h \otimes_k \end{bmatrix} \in \mathcal{A}_0 \odot \mathcal{B}(\hat{k}) \) and \( P_k := |\omega\rangle\langle\omega|_{\perp} \in \mathcal{B}(\hat{k}) \) is the orthogonal projection onto \( k \subset \hat{k} \).

Question

*When are these necessary conditions sufficient for the solution of (1) to be a quantum flow?*
**Flow generators**

**Lemma 8**

The map \( \phi : A_0 \to A_0 \otimes B \) is a flow generator if and only if

\[
\phi(x) = \begin{bmatrix}
\tau(x) & \delta^\dagger(x) \\
\delta(x) & \pi(x) - x \otimes I_k
\end{bmatrix}
\]

for all \( x \in A_0 \), \hspace{1cm} (2)

where

- \( \pi : A_0 \to A_0 \otimes B(k) \) is a unital \(*\)-homomorphism,
- \( \delta : A_0 \to A_0 \otimes B(\mathbb{C}; k) \) is a \( \pi \)-derivation, i.e., a linear map such that
  \[
  \delta(xy) = \delta(x)y + \pi(x)\delta(y) \quad (x, y \in A_0)
  \]
- \( \delta^\dagger : A_0 \to A_0 \otimes B(k; \mathbb{C}) \) is such that \( \delta^\dagger(x) = \delta(x^*)^* \) for all \( x \in A_0 \),
- \( \tau : A_0 \to A_0 \) is \(*\)-linear and such that
  \[
  \tau(xy) - \tau(x)y - x\tau(y) = \delta^\dagger(x)\delta(y) \quad (x, y \in A_0).
  \]
Quantum Wiener integrals

**Theorem 9**

For all $n \in \mathbb{N}$ and $T \in \mathcal{B}(h \otimes \hat{k} \otimes^n)$ there exists a family $(\Lambda^n_t(T))_{t \geq 0}$ of linear operators in $h \otimes \mathcal{F}$, with domains including $h \otimes \mathcal{E}$, that is adapted and such that

$$\langle u \varepsilon(f), \Lambda^n_t(T)v \varepsilon(g) \rangle = \int_{D_n(t)} \langle u \otimes \hat{f} \otimes^n(t), Tv \otimes \hat{g} \otimes^n(t) \rangle \, dt \langle \varepsilon(f), \varepsilon(g) \rangle$$

for all $u, v \in h$, $f, g \in L^2(\mathbb{R}_+; k)$ and $t \geq 0$, where the simplex

$$D_n(t) := \{ t := (t_1, \ldots, t_n) \in [0, t]^n : t_1 < \cdots < t_n \}$$

and

$$\hat{f} \otimes^n(t) := \hat{f}(t_1) \otimes \cdots \otimes \hat{f}(t_n) \quad \text{et cetera.}$$

We include $n = 0$ by setting $\Lambda^0_t(T) := T \otimes I_{\mathcal{F}}$ for all $t \geq 0$. 
Proposition 10

If \( n \in \mathbb{Z}_+ \), \( t \geq 0 \), \( T \in \mathcal{B}(h \otimes \widehat{k}^n) \) and \( f \in L^2(\mathbb{R}_+; k) \) then

\[
\| \Lambda^n_t(T) u \epsilon(f) \| \leq \frac{K^n_{f,t}}{\sqrt{n!}} \| T \| \| u \epsilon(f) \| ,
\]

where \( K_{f,t} := \sqrt{(2 + 4\| f \|^2)(t + \| f \|^2)} \).
Definition

Given a flow generator $\phi$, the family of linear maps

$$\phi_n : A_0 \to A_0 \otimes B \otimes n$$

is defined by setting

$$\phi_0 := \text{id}_{A_0} \quad \text{and} \quad \phi_{n+1} := (\phi_n \otimes \text{id}_B) \circ \phi \quad \text{for all } n \in \mathbb{Z}_+, \quad$$

where $\text{id}_{A_0}$ is the identity map on $A_0$ and similarly for $\text{id}_B$.

Let

$$A_\phi := \{ x \in A_0 : \exists C_x, M_x > 0 \text{ with } \|\phi_n(x)\| \leq C_x M_x^n \forall n \in \mathbb{Z}_+ \}$$

be those elements of $A_0$ for which $(\phi_n(x))_{n \geq 0}$ has polynomial growth.
**Theorem 11**

If \( x \in \mathcal{A}_\phi \) then the series

\[
j_t(x) := \sum_{n=0}^{\infty} \Lambda_t^n(\phi_n(x))
\]

is strongly absolutely convergent on \( \mathfrak{h} \otimes \mathcal{E} \) for all \( t \geq 0 \). The family of operators \( (j_t(x))_{t \geq 0} \) is an adapted mapping process on \( \mathcal{A}_\phi \) which satisfies the QSDE (1) on \( \mathcal{A}_\phi \).

**Questions**

1. When does \( \mathcal{A}_\phi = \mathcal{A}_0 \)?
2. When does the adapted mapping process \( j \) given by Theorem 11 extend to a quantum flow?
If $A_\phi = A_0$ then the adapted mapping process $j$ given by Theorem 11 extends to a quantum flow as long as $A$ is sufficiently well behaved.

In practice it will be difficult to find directly constants $C_x$ and $M_x$ such that $\|\phi_n(x)\| \leq C_x M_x^n$ for all $x \in A_0$.

Key to both: the higher-order Itô formula.

Two easy cases when $A_\phi = A_0$

If $k$ is finite dimensional and $\phi$ is bounded then so is each $\phi_n$, with

$$\|\phi_n\| \leq \left(\dim \hat{k}\right)^{n-1} \|\phi\|^n \quad (n \in \mathbb{N}).$$

If $\phi$ is completely bounded then so is each $\phi_n$, with

$$\|\phi_n\| \leq \|\phi\|_{cb}^n \quad (n \in \mathbb{Z}_+).$$
The higher-order Itô formula

**Notation**

Let $\alpha \subseteq \{1, \ldots, n\}$, with elements arranged in increasing order and cardinality $|\alpha|$. The unital $\ast$-homomorphism

$$\mathcal{A}_0 \otimes \mathcal{B}^{|\alpha|} \to \mathcal{A}_0 \otimes \mathcal{B}^n; \quad T \mapsto T(n, \alpha)$$

is defined by linear extension of the map

$$A \otimes B_1 \otimes \cdots \otimes B_{|\alpha|} \mapsto A \otimes C_1 \otimes \cdots \otimes C_n,$$

where

$$C_i := \begin{cases} B_j & \text{if } i \text{ is the } j\text{th element of } \alpha, \\ \mathbb{I}_k & \text{if } i \text{ is not an element of } \alpha. \end{cases}$$

**Example**

$$(A \otimes B_1 \otimes B_2 \otimes B_3)(5, \{1, 3, 4\}) = A \otimes B_1 \otimes \mathbb{I}_k \otimes B_2 \otimes B_3 \otimes \mathbb{I}_k.$$
More notation

For all \( n \in \mathbb{Z}_+ \) and \( \alpha \subseteq \{1, \ldots, n\} \), let

\[
\phi_{|\alpha|}(x; n, \alpha) := \left(\phi_{|\alpha|}(x)\right)(n, \alpha) \quad \text{for all } x \in A_0.
\]

Also let

\[
\Delta(n, \alpha) := (I_h \otimes P_{k_{|\alpha|}})(n, \alpha),
\]

so that \( P_k \) acts on the components of \( \hat{k} \otimes n \) which have indices in \( \alpha \) and \( I_{\hat{k}} \) acts on the others.

Theorem 12

Let \( \phi \) be a flow generator. For all \( n \in \mathbb{Z}_+ \) and \( x, y \in A_0 \),

\[
\phi_n(xy) = \sum_{\alpha \cup \beta = \{1, \ldots, n\}} \phi_{|\alpha|}(x; n, \alpha) \Delta(n, \alpha \cap \beta) \phi_{|\beta|}(y; n, \beta),
\]

where the summation is taken over all \( \alpha \) and \( \beta \) with union \( \{1, \ldots, n\} \).
Two corollaries

**Corollary 13**

If $\phi : A_0 \to A_0 \otimes \mathcal{B}$ is a flow generator then $A_{\phi}$ is a unital $\ast$-subalgebra of $A_0$, which is equal to $A_0$ if $A_{\phi}$ contains a $\ast$-generating set for $A_0$.

**Corollary 14**

Let $\phi : A_0 \to A_0 \otimes \mathcal{B}$ be a flow generator and let $j$ be the adapted mapping process on $A_{\phi}$ given by Theorem 11. If $x, y \in A_{\phi}$ then $x^\ast y \in A_{\phi}$, with

$$\langle j_t(x) u \varepsilon(f), j_t(y) v \varepsilon(g) \rangle = \langle u \varepsilon(f), j_t(x^\ast y) v \varepsilon(g) \rangle$$

for all $u, v \in h$ and $f, g \in L^2(\mathbb{R}_+; k)$. 

The first dilation theorem

**Theorem 15**

Let \( \phi : A_0 \to A_0 \circledast B \) be a flow generator and suppose \( A_0 \) contains its square roots: for all non-negative \( x \in A_0 \), the square root \( x^{1/2} \) lies in \( A_0 \).

If \( A_\phi = A_0 \) then there exists a quantum flow \( \bar{j} \) such that

\[
\bar{j}_t(x) = j_t(x) \text{ on } h \circledast \mathcal{E} \quad (x \in A_0),
\]

where \( j \) is the adapted mapping process given by Theorem 11.

**Remark**

If \( \mathcal{A} \) is an AF algebra, i.e., the norm closure of an increasing sequence of finite-dimensional \( * \)-subalgebras, then its local algebra \( A_0 \), the union of these subalgebras, contains its square roots.
The second dilation theorem

Theorem 16

Let $\mathcal{A}$ be the universal $C^*$ algebra generated by isometries $\{s_i : i \in \mathbb{I}\}$, and let $\mathcal{A}_0$ be the $\ast$-algebra generated by $\{s_i : i \in \mathbb{I}\}$.

If $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ is a flow generator such that $\mathcal{A}_{\phi} = \mathcal{A}_0$ then there exists a quantum flow $\bar{j}$ such that

$$\bar{j}_t(x) = j_t(x) \text{ on } h \otimes \mathcal{E} \quad (x \in \mathcal{A}_0),$$

where $j$ is the adapted mapping process given by Theorem 11.
The group algebra

Let $G$ is a discrete group and set $\mathcal{A} = C_0(G) \oplus \mathbb{C}1 \subseteq B(\ell^2(G))$, where $x \in C_0(G)$ acts on $\ell^2(G)$ by multiplication.

Let $\mathcal{A}_0 = \text{lin}\{1, \, e_g : g \in G\}$, where $e_g(h) := 1_{g=h}$ for all $h \in G$.

Permitted moves

Let $H$ be a non-empty finite subset of $G \setminus \{e\}$ and let the Hilbert space $k$ have orthonormal basis $\{f_h : h \in H\}$; the maps

$$\lambda_h : G \rightarrow G; \ g \mapsto hg \quad (h \in H)$$

correspond to the permitted moves in the random walk to be constructed on $G$. 
Lemma 17

Given a transition function

\[ t : H \times G \to \mathbb{C}; \ (h, g) \mapsto t_h(g), \]

the map \( \phi : A_0 \to A_0 \otimes B \) such that

\[
x \mapsto \left[ \sum_{h \in H} |t_h|^2(x \circ \lambda_h - x) \sum_{h \in H} \overline{t_h} (x \circ \lambda_h - x) \otimes \langle f_h \rangle \right] \left[ \sum_{h \in H} t_h(x \circ \lambda_h - x) \otimes |f_h\rangle \sum_{h \in H} (x \circ \lambda_h - x) \otimes |f_h\rangle \langle f_h | \right]
\]

is a flow generator with \( \phi_n(e_g) \) equal to

\[
\sum_{h_1 \in H \cup \{e\}} \cdots \sum_{h_n \in H \cup \{e\}} e_{h_n^{-1} \cdots h_1^{-1}g} \otimes m_{h_n}(h_n^{-1} \cdots h_1^{-1}g) \otimes \cdots \otimes m_{h_1}(h_1^{-1}g)
\]

for all \( n \in \mathbb{N} \) and \( g \in G \).
For all $g \in G$ and $h \in H$, let

$$m_e(g) := \begin{bmatrix}
- \sum_{h \in H} |t_h(g)|^2 & - \sum_{h \in H} t_h(g) \langle f_h \rangle \\
- \sum_{h \in H} t_h(g) \langle f_h \rangle & -I_k
\end{bmatrix}$$

and

$$m_h(g) := \begin{bmatrix}
|t_h(g)|^2 & \overline{t_h(g)} \langle f_h \rangle \\
t_h(g) \langle f_h \rangle & \langle f_h \rangle \langle f_h \rangle
\end{bmatrix}.$$ 

Then

$$\|m_e(g)\| = 1 + \sum_{h \in H} |t_h(g)|^2$$

and

$$\|m_h(g)\| = 1 + |t_h(g)|^2.$$
A sufficient condition for $A_\phi = A_0$

If

$$M_g := \lim_{n \to \infty} \sup \left\{ |t_h(h_n^{-1} \cdots h_1^{-1}g)| : h_1, \ldots, h_n \in H \cup \{e\}, \ h \in H \right\} < \infty$$

(3)

then

$$\|\phi_n(e_g)\| \leq (1 + |H| + 2|H|M_g^2)^n \quad (n \in \mathbb{Z}_+),$$

where $|H|$ denotes the cardinality of $H$.

Hence $A_\phi = A_0$ if (3) holds for all $g \in G$. 
Examples

1. If $t$ is bounded then (3) holds for all $g \in G$.

2. If $G = (\mathbb{Z}, +)$, $H = \{\pm 1\}$ and the transition function $t$ is bounded, with $t_{+1}(g) = 0$ for all $g < 0$ and $t_{-1}(g) = 0$ for all $g \leq 0$, then the Feller semigroup $T$ which arises corresponds to the classical birth-death process with birth and death rates $|t_{+1}|^2$ and $|t_{-1}|^2$, respectively.

3. If $G = (\mathbb{Z}, +)$, $H = \{+1\}$ and $t_{+1} : g \mapsto 2^g$ then $M_g = 2^g$ and the condition (3) holds for all $g \in G$. Thus the construction applies to examples where the transition function $t$ is unbounded.
The symmetric quantum exclusion process

The CAR algebra

For a non-empty set $\mathbb{I}$, the CAR algebra is the unital $C^*$ algebra $\mathcal{A}$ with generators $\{b_i : i \in \mathbb{I}\}$, subject to the anti-commutation relations

\[ \{b_i, b_j\} = 0 \quad \text{and} \quad \{b_i, b_j^*\} = 1_{i=j} \quad (i, j \in \mathbb{I}). \]

Let $\mathcal{A}_0$ be the unital algebra generated by $\{b_i, b_i^* : i \in \mathbb{I}\}$.

Lemma 18

For each $x \in \mathcal{A}_0$ there exists a finite subset $\mathbb{I}_0 \subseteq \mathbb{I}$ such that $x$ lies in the finite-dimensional $*$-subalgebra

\[ \mathcal{A}_{\mathbb{I}_0} := \text{lin}\{b_{j_1}^* \cdots b_{j_q}^* b_{i_1} \cdots b_{i_p} : \text{distinct } i_1, \ldots, i_p \in \mathbb{I}_0, j_1, \ldots, j_q \in \mathbb{I}_0\}. \]

Consequently, $\mathcal{A}$ is an AF algebra and $\mathcal{A}_0$ contains its square roots.
The symmetric quantum exclusion process

Amplitudes

Let \( \{ \alpha_{i,j} : i, j \in \mathbb{I} \} \subseteq \mathbb{C} \) be a fixed collection of amplitudes, so that \((\mathbb{I}, \{ \alpha_{i,j} \})\) is a complex digraph. For all \( i \in \mathbb{I} \), let

\[
\text{supp}(i) := \{ j \in \mathbb{I} : \alpha_{i,j} \neq 0 \} \quad \text{and} \quad \text{supp}^+(i) := \text{supp}(i) \cup \{ i \}.
\]

Thus \( \text{supp}(i) \) is the set of sites with which site \( i \) interacts and \( |\text{supp}(i)| \) is the valency of the vertex \( i \). Suppose

\[
|\text{supp}(i)| < \infty \quad (i \in \mathbb{I}).
\]

The transport of a particle from site \( i \) to site \( j \) with amplitude \( \alpha_{i,j} \) is described by the operator

\[
t_{i,j} := \alpha_{i,j} b_j^* b_i.
\]
The symmetric quantum exclusion process

**Energies**

Let \( \{ \eta_i : i \in \mathbb{I} \} \subseteq \mathbb{R} \) be fixed. The total energy in the system is given by

\[
h := \sum_{i \in \mathbb{I}} \eta_i b_i^* b_i,
\]

where \( \eta_i \) gives the energy of a particle at site \( i \).

**Lemma 19**

**Setting**

\[
\tau(x) := i \sum_{i \in \mathbb{I}} \eta_i [b_i^* b_i, x] - \frac{1}{2} \sum_{i, j \in \mathbb{I}} \tau_{i, j}(x)
\]

defines a \(^*\)-linear map \( \tau : A_0 \rightarrow A_0 \), where

\[
\tau_{i, j}(x) := t_{i, j}^* [t_{i, j}, x] + [x, t_{i, j}^*] t_{i, j}.
\]
Lemma 20

Let $k$ be a Hilbert space with orthonormal basis $\{f_{i,j} : i, j \in \mathbb{I}\}$. Setting

$$\delta(x) := \sum_{i,j \in \mathbb{I}} [t_{i,j}, x] \otimes |f_{i,j}\rangle \quad (x \in A_0),$$

where

$$|f_{i,j}\rangle : \mathbb{C} \mapsto k; \quad \lambda \mapsto \lambda f_{i,j},$$

defines a linear map $\delta : A_0 \to A_0 \otimes \mathcal{B}(\mathbb{C}; k)$ such that

$$\delta(xy) = \delta(x)y + (x \otimes I_k)\delta(y)$$

and

$$\delta^\dagger(x)\delta(y) = \tau(xy) - \tau(x)y - x\tau(y) \quad (x, y \in A_0),$$

with $\tau$ defined as in Lemma 19. Hence

$$\phi : A_0 \to A_0 \otimes \mathcal{B}; \quad x \mapsto \begin{bmatrix} \tau(x) & \delta^\dagger(x) \\ \delta(x) & 0 \end{bmatrix}$$

is a flow generator.
The symmetric quantum exclusion process

Lemma 21

If the amplitudes satisfy the symmetry condition

$$|\alpha_{i,j}| = |\alpha_{j,i}| \quad \text{for all } i, j \in \mathbb{I}$$

(4)

then

$$\phi_n(b_{i_0}) = \sum_{i_1 \in \text{supp}^+(i_0)} \cdots \sum_{i_n \in \text{supp}^+(i_{n-1})} b_{i_n} \otimes B_{i_{n-1},i_n} \otimes \cdots \otimes B_{i_0,i_1}$$

for all $n \in \mathbb{N}$ and $i_0 \in \mathbb{I}$, where

$$B_{i,j} := 1_{j=i} \lambda_i |\omega\rangle \langle \omega | + |\omega\rangle \langle \alpha_{i,j} f_{i,j} | - |\alpha_{j,i} f_{j,i}\rangle \langle \omega | \quad (i, j \in \mathbb{I})$$

and

$$\lambda_i := -i\eta_i - \frac{1}{2} \sum_{j \in \text{supp}(i)} |\alpha_{j,i}|^2 \quad (i \in \mathbb{I}).$$
The symmetric quantum exclusion process

Example

Suppose that the amplitudes satisfy the symmetry condition (4), and further that there are uniform bounds on the amplitudes, valencies and energies, so that

\[ M := \sup_{i,j \in \mathcal{I}} |\alpha_{i,j}|, \quad V := \sup_{i \in \mathcal{I}} |\text{supp}(i)| \quad \text{and} \quad H := \sup_{i \in \mathcal{I}} |\eta_i| \]

are all finite. Then

\[ |\lambda_i| \leq |\eta_i| + \frac{1}{2} VM^2 \quad \text{and} \quad \|B_{i,j}\| \leq |\lambda_i| + 2M \leq H + \frac{1}{2} VM^2 + 2M \]

for all \( i, j \in \mathcal{I} \). Hence

\[ \|\phi_n(b_i)\| \leq (V + 1)^n (H + \frac{1}{2} VM^2 + 2M)^n \quad (n \in \mathbb{Z}_+, i \in \mathcal{I}) \]

and \( A_\phi = A_0 \).
The universal rotation algebra

Definition

Let $\mathcal{A}$ be the *universal rotation algebra*: this is the universal $C^*$ algebra with unitary generators $U$, $V$ and $Z$ satisfying the relations

$$UV = ZVU, \quad UZ = ZU \quad \text{and} \quad VZ = ZV.$$ 

It is the group $C^*$ algebra corresponding to the discrete Heisenberg group

$$\Gamma := \langle u, v, z \mid uv = zvu, \ uz = zu, \ vz = zv \rangle.$$ 

A pair of derivations

Letting $\mathcal{A}_0$ denote the $*$-subalgebra generated by $U$, $V$ and $Z$, there are skew-adjoint derivations

$$\delta_1 : \mathcal{A}_0 \to \mathcal{A}_0; \ U^m V^n Z^p \mapsto mU^m V^n Z^p$$

and

$$\delta_2 : \mathcal{A}_0 \to \mathcal{A}_0; \ U^m V^n Z^p \mapsto nU^m V^n Z^p.$$
**Theorem 22**

Fix $c_1, c_2 \in \mathbb{C}$, let $\delta = c_1 \delta_1 + c_2 \delta_2$ and define the Bellissard map

$$\tau : A_0 \rightarrow A_0;$$

$$U^m V^n Z^p \mapsto - \left( \frac{1}{2} |c_1|^2 m^2 + \frac{1}{2} |c_2|^2 n^2 + \overline{c_1} c_2 mn + (\overline{c_1} c_2 - c_1 \overline{c_2}) p \right) U^m V^n Z^p.$$  

Then

$$\phi : A_0 \rightarrow A_0 \otimes B(\mathbb{C}^2); \quad x \mapsto \begin{bmatrix} \tau(x) & \delta^\dagger(x) \\ \delta(x) & 0 \end{bmatrix}$$

is a flow generator. Furthermore, $U, V, Z \in A_\phi$ and $A_\phi = A_0$.

**Remark**

If $\overline{c_1} c_2 = c_1 \overline{c_2}$ then this construction specialises to the non-commutative torus.
The non-commutative torus

**Definition**

Let $\mathcal{A}$ be the non-commutative torus with parameter $\lambda \in \mathbb{T}$, so that $\mathcal{A}$ is the universal $C^*$ algebra with unitary generators $U$ and $V$ subject to the relation

$$UV = \lambda VU,$$

and let

$$\mathcal{A}_0 := \langle U, V \rangle = \text{lin}\{U^mV^n : m, n \in \mathbb{Z}\}.$$

**An automorphism**

For each $(\mu, \nu) \in \mathbb{T}^2$, let $\pi_{\mu,\nu}$ be the automorphism of $\mathcal{A}$ such that

$$\pi_{\mu,\nu}(U^mV^n) = \mu^m\nu^nU^mV^n$$

for all $m, n \in \mathbb{Z}$.
The non-commutative torus

Theorem 23

Fix \((\mu, \nu) \in \mathbb{T}^2\) with \(\mu \neq 1\). There exists a flow generator

\[
\phi : A_0 \to A_0 \otimes B(\mathbb{C}^2); \quad x \mapsto \begin{bmatrix}
\tau(x) & -\mu \delta(x) \\
\delta(x) & \pi_{\mu, \nu}(x) - x
\end{bmatrix},
\]

where the \(\pi_{\mu, \nu}\)-derivation

\[
\delta : A_0 \to A_0; \quad U^m V^n \mapsto \frac{1 - \mu^m \nu^n}{1 - \mu} U^m V^n
\]

is such that \(\delta^\dagger = -\mu \delta\) and the map

\[
\tau := \frac{\mu}{1 - \mu} \delta : A_0 \to A_0; \quad U^m V^n \mapsto \frac{\mu(1 - \mu^m \nu^n)}{(1 - \mu)^2} U^m V^n.
\]

Furthermore, \(U, V \in A_\phi\) and so \(A_\phi = A_0\).
Perturbation
Itô’s formula for Brownian motion

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is twice continuously differentiable then

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) \, dB_s + \frac{1}{2} \int_0^t f''(B_s) \, ds \quad (t \geq 0).$$

Corollary

Define a Feller semigroup $(T_t)_{t \geq 0}$ on $C_0(\mathbb{R})$ by setting

$$(T_t f)(x) := \mathbb{E} [f(B_t) \mid B_0 = x] \quad (t \geq 0, \, f \in C_0(\mathbb{R}), \, x \in \mathbb{R}).$$

Then

$$\frac{1}{t} \left[ f(B_t) - f(B_0) \mid B_0 = x \right] \to \frac{1}{2} f''(x) \quad \text{as } t \to 0,$$

so the generator of this semigroup extends $f \mapsto \frac{1}{2} f''$. 

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Quantum semigroups and stochastic flows  
IMPAN, 25xi13  
48 / 67
Proposition

If \( v : \mathbb{R} \to \mathbb{R} \) is well behaved and \( Y_t := \exp\left( \int_0^t v(B_s) \, ds \right) \) then

\[
Y_t f(B_t) = f(B_0) + \int_0^t Y_s f'(B_s) \, dB_s \\
+ \int_0^t \left( v(B_s) Y_s f(B_s) + \frac{1}{2} Y_s f''(B_s) \right) \, ds.
\]

Proof

This follows from the classical Itô product formula:

\[
d(Y_t f(B_t)) = (dY_t)f(B_t) + Y_t d(f(B_t)) + dY_t d(f(B_t)).
\]
Theorem (F–K)

Let

\[(S_t f)(x) := \mathbb{E}\left[ \exp\left( \int_0^t \nu(B_s) \, ds \right) f(B_t) \mid B_0 = x \right].\]

Then \((S_t)_{t \geq 0}\) is a Feller semigroup on \(C_0(\mathbb{R})\) with generator which extends \(f \mapsto \frac{1}{2} f'' + \nu f\).
Introducing non-commutativity

Idea

There are two ways in which non-commutativity can appear:

1. quantum stochastic noises replace Brownian motion;

2. $C_0(\mathbb{R})$ is replaced by a general $C^*$ algebra.
Feynman–Kac perturbation (von Neumann)

The free flow

Until further notice, $\mathcal{A}$ is a von Neumann algebra, so $\otimes_m = \overline{\otimes}$, and $j$ is an ultraweakly continuous, normal quantum flow and a Feller cocycle.

Multipliers

A multiplier for $j$ is an adapted operator process $Y \subseteq \mathcal{A} \overline{\otimes} \mathcal{B}(\mathcal{F})$ such that

$$Y_0 = I_{h \otimes \mathcal{F}} \quad \text{and} \quad Y_{s+t} = J_s(Y_t)Y_s \quad (s, t \geq 0).$$

Theorem 24

If $Y$ and $Z$ are multipliers for $j$ and

$$k_t : \mathcal{A} \to \mathcal{A} \overline{\otimes} \mathcal{B}(\mathcal{F}); \quad a \mapsto Y_t^* j_t(a)Z_t \quad (t \geq 0)$$

then $k$ is a Feller cocycle and $\mathbb{E}_\Omega \circ k$ is a semigroup on $\mathcal{A}$.
Some questions

Generators

1. How is the generator of $E_{\Omega} \circ k$ related to that of $E_{\Omega} \circ j$?

2. How is the generator of $k$ related to that of $j$?

Existence

1. How do we produce multipliers?
The multiplier QSDE

**Theorem 25**

Let $F \in \mathcal{A} \otimes \mathcal{B}(\hat{k})$ be such that

$$q(F) := F + F^* + F\Delta F^* \leq 0,$$

where $\Delta := \begin{bmatrix} 0 & 0 \\ 0 & l_h \otimes k \end{bmatrix}$.

There exists a unique operator process $Y \subseteq \mathcal{A} \otimes_m \mathcal{B}(\mathcal{F})$ such that

$$Y_t = l_h \otimes \mathcal{F} + \int_0^t \tilde{j}_s(F) \tilde{Y}_s \, d\Lambda_s \quad (t \geq 0),$$

where $\tilde{j}_s := j_s \otimes_m \text{id}_{\mathcal{B}(\hat{k})}$ and $\tilde{Y}_s := Y_s \otimes l_h$.

Each $Y_t$ is a contraction, and is an isometry if and only if $q(F) = 0$.

The adapted operator process $Y$ is a multiplier for $j$ with generator $F$. 
Classical multipliers

**Feynman–Kac**

\[ Y_t = \exp\left( \int_0^t \nu(B_s) \, ds \right) \Rightarrow Y_t = 1 + \int_0^t \nu(B_s) Y_s \, ds = 1 + \int_0^t j_s(\nu) Y_s \, ds. \]

**Cameron–Martin**

\[ Y_t = \exp\left( B_t - \frac{1}{2} t \right) \Rightarrow Y_t = 1 + \int_0^t Y_s \, dB_s = 1 + \int_0^t \tilde{j}_s \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \tilde{Y}_s \, d\Lambda_s. \]

Then \( E_\Omega \circ k \) has generator \( f \mapsto \frac{1}{2} f'' + f' \).

C–M: Setting \( (T_t f)(x) := E[f(B_t + t)|B_0 = x] \) gives the same generator.

A QSDE for the free flow

Assumption

Suppose the free flow $j$ satisfies the QSDE (1) on $A_0 \subseteq A$, i.e.,

$$dj_t(x) = (\widetilde{j}_t \circ \phi)(x) d\Lambda_t \quad (x \in A_0),$$

where $\phi : A_0 \to A \otimes_m B$ is the generator of $j$.

Example: Brownian motion

If $j_t : f \mapsto f(B_t)$ then $A_0 \subseteq C^2(\mathbb{R})$ and

$$j_t(g) = g(B_t) = g(B_0) + \int_0^t g'(B_s) dB_s + \frac{1}{2} \int_0^t g''(B_s) ds$$

$$= j_0(g) + \int_0^t \widetilde{j}_s \left( \begin{bmatrix} \frac{1}{2} \delta^2(g) & \delta(g) \\ \delta(g) & 0 \end{bmatrix} \right) d\Lambda_s \quad (g \in A_0),$$

where $\delta : h \mapsto h'$. 
The flow generator

Structure of $\phi$

As in Lemma 8, the generator $\phi$ has the form

$$\phi = \begin{bmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \pi - \iota \end{bmatrix},$$

where

- $\pi : A_0 \to A \otimes_m B(k)$ is a unital $\ast$-homomorphism,
- $\iota : A_0 \to A_0 \odot B(k); x \mapsto x \otimes I_k$,
- $\delta : A_0 \to A \otimes_m B(\mathbb{C}; k)$ is a $\pi$-derivation: $\delta(xy) = \delta(x)y + \pi(x)\delta(y)$
- $\delta^\dagger : x \mapsto \delta(x^*)^*$
- $\mathcal{L}$ “generates” $\mathbb{E}_\Omega \circ j$, $\mathcal{L}(xy) - \mathcal{L}(x)y - x\mathcal{L}(y) = \delta^\dagger(x)\delta(y)$.

The Brownian case

$$(fg)'' - f''g - fg'' = 2f'g'.$$
Quantum Feynman–Kac perturbation

**Theorem 26**

If the multipliers $Y$ and $Z$ have generators $F$ and $G$ respectively then

$$dk_t(x) = (\tilde{k}_t \circ \psi)(x) \, d\Lambda_t \quad (x \in A_0),$$

where

$$\psi(x) = \phi(x) + F^* \left( \Delta \phi(x) + \iota(x) \right) + F^* \Delta \left( \phi(x) + \iota(x) \right) \Delta G + \left( \phi(x) \Delta + \iota(x) \right) G.$$

**Corollary**

The semigroup $\mathbb{E}_\Omega \circ k$ has generator which extends

$$x \mapsto L(x) + \ell_F^* \delta(x) + \ell_F^* \pi(x) \ell_G + \delta^\dagger(x) \ell_G + m^*_F x + x m_G,$$

where

$$F = \begin{bmatrix} m_F & * \\ \ell_F & * \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} m_G & * \\ \ell_G & * \end{bmatrix}.$$
Quantum Feynman–Kac perturbation – proof

Quantum Itô product formula

If \( X = \int F \, d\Lambda \) and \( Y = \int G \, d\Lambda \) then

\[
X_t Y_t = \int_0^t \tilde{X}_s \, dY_s + (d\tilde{X}_s) \tilde{Y}_s + dX_s \, dY_s
\]

\[
= \int_0^t \left( \tilde{X}_s G_s + F_s \tilde{Y}_s + F_s \Delta G_s \right) \, d\Lambda_s \quad (t \geq 0).
\]

Key step in proving of Theorem 26

\[
d(j_t(x) Z_t) = (j_t(x) \tilde{j}_t(G) \tilde{Z}_t + \tilde{j}_t(\phi(x)) \tilde{Z}_t + \tilde{j}_t(\phi(x)) \Delta \tilde{j}_t(G) \tilde{Z}_t) \, d\Lambda_t
\]

\[
= \tilde{j}_t (\nu(x) G + \phi(x) + \phi(x) \Delta G) \tilde{Z}_t \, d\Lambda_t.
\]
The matrix-space product

\[ A \otimes_m B(F) := \{ T \in B(h \otimes F) : T^x_y \in A \text{ for all } x, y \in F \}, \]

where \( T^x_y \in B(h) ; \) \( \langle u, T^x_y v \rangle = \langle u \otimes x, Tv \otimes y \rangle . \)

If \( k : A_1 \to A_2 \) is completely bounded then

\[ k \otimes_m \text{id}_B(H) : A_1 \otimes_m B(H) \to A_2 \otimes_m B(H); \ (k \otimes_m \text{id}_B(H))(T)^x_y = k(T^x_y). \]

Problems

Henceforth \( A \) is only a \( C^* \) algebra. Lack of ultraweak continuity/closure means

1. \( A \otimes_m B(F) \) is not an algebra \( \Rightarrow \) \( k_t(a) = Y^*_t j_t(a) Z_t \notin A \otimes_m B(F) , \)

2. \( \tilde{j}_t := j_t \otimes_m \text{id}_{B(k)} \) is not multiplicative \( \Rightarrow \) key step for Theorem 26 fails.
Theorem 27

A mapping process \((k_t : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{h} \otimes \mathcal{F}))_{t \geq 0}\) of completely bounded maps is a Feller cocycle if

1. \(\text{im } k_t \subseteq \mathcal{A} \otimes_{m} \mathcal{B}(\mathcal{F}) \quad (t \geq 0)\);
2. \(k_{s+t} = \hat{k}_s \circ \sigma_s \circ k_t \quad (s, t \geq 0)\);
3. \(\mathbb{E}_{\Omega} \circ k_0 = \text{id}_\mathcal{A}\).

A mapping process \(k\) is a Feller cocycle if and only if there exists a total subset \(T \subseteq k\) such that \(0 \in T\),

\[
 t \mapsto \mathcal{P}_t^{z,w} : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{h}); \quad a \mapsto k_t(a)_{\overline{1_{[0,t]}^z}}_{\overline{1_{[0,t]}^w}}
\]

is a semigroup on \(\mathcal{A}\) for all \(z, w \in T\),

whenever \(f\) and \(g\) are step functions subordinate to some partition \(\{0 = t_0 < t_1 < \cdots\}\),

\[
 k_t(a)_{\overline{1_{[0,t]}^f}}_{\overline{1_{[0,t]}^g}} = \mathcal{P}_{t-t_0}^{f(t_0),g(t_0)} \circ \cdots \circ \mathcal{P}_{t-t_n}^{f(t_n),g(t_n)} \quad (t \in [t_n, t_{n+1})).
\]
**Theorem 28**

Let $k$ be a mapping process with locally bounded norm and such that

$$k_0(x) = x \otimes I_F, \quad dk_t(x) = (\tilde{k}_t \circ \psi)(x) \, d\Lambda_t \quad \text{on } \mathcal{E}(T) \quad (x \in A_0)$$

for a norm-densely defined linear map $\psi : A_0 \to A \otimes \mathcal{B}(\hat{k})$ and a total set $0 \in T \subseteq k$, where

$$\mathcal{E}(T) := \text{lin}\{\varepsilon(f) : f \text{ is a } T\text{-valued, right-continuous step function}\}.$$  

If, for all $z, w \in T$, there exists a $C_0$-semigroup generator $\eta_{z,w}$ with a core $A_{z,w} \subseteq A_0$ such that

$$\psi(x)\hat{z}_w = \eta_{z,w}(x) \quad (x \in A_{z,w})$$

then $k$ is a Feller cocycle.
Existence of multipliers

Obstruction

Let $F \in \mathcal{A} \otimes_m \mathcal{B}(\hat{k})$. The construction of Theorem 25 applies, but the proof of contractivity fails as $\tilde{j}_t$ is not multiplicative.

Solution

Choose the generator $F$ such that $q(F) \leq 0$ and

$$\tilde{j}_t(F)^* \Delta \tilde{j}_t(F) = \tilde{j}_t(F^* \Delta F) \quad (t \geq 0).$$

(5)
A QSDE for the perturbed process

**Theorem 29**

If $Y$ and $Z$ are contractive multipliers with generators $F$ and $G$ respectively such that

$$\Delta F, \Delta G \in \{ T \in \mathcal{B}(h \otimes (\hat{k})) : T_y \in A \otimes \mathcal{B}(\mathbb{C}; \hat{k}) \text{ for all } y \in \hat{k} \}$$

then the mapping process

$$k_t : A \rightarrow \mathcal{B}(h \otimes \mathcal{F}); \quad a \mapsto Y_t^* j_t(a) Z_t \quad (t \geq 0)$$

is such that

$$k_0(x) = x \otimes 1_{\mathcal{F}}, \quad dk_t(x) = (\tilde{k}_t \circ \psi)(x) d\Lambda_t \quad \text{on } \mathcal{E}(T) \quad (x \in A_0)$$

where $\psi$ is as in Theorem 26.
Simplifying the generator

Bounded parts

Note that

\[
\psi(x) = \phi(x) + F^*(\Delta\phi(x) + \nu(x)) + F^*\Delta(\phi(x) + \nu(x))\Delta G
\]

\[
+ (\phi(x)\Delta + \nu(x))G
\]

\[
= (I_h \otimes k + \Delta F)^* \phi(x)(I_h \otimes k + \Delta G)
\]

\[
+ F^* \nu(x) + F^* \Delta \nu(x)\Delta G + \nu(x)G.
\]

Let \( \Delta F = \begin{bmatrix} 0 & 0 \\ \ell_F & w_F - I_h \otimes k \end{bmatrix} \) and \( \Delta G = \begin{bmatrix} 0 & 0 \\ \ell_G & w_G - I_h \otimes k \end{bmatrix} \). Then

\[
(I_h \otimes k + \Delta F)^* \phi(x)(I_h \otimes k + \Delta G)
\]

\[
= \begin{bmatrix} \mathcal{L}(x) + \ell_F^* \delta(x) + \delta^\dagger(x) \ell_G & \delta^\dagger(x) w_G \\ w_F^* \delta(x) & 0 \end{bmatrix}
\]

\[
+ \begin{bmatrix} \ell_F^* \\ w_F^* \end{bmatrix} (\pi - \iota)(x) \begin{bmatrix} \ell_G & w_G \end{bmatrix}.
\]
Applying Theorem 28

Conclusion

If, for all $z, w \in k$, there exist a subspace $A_{z,w} \subseteq A_0$ such that

$$A_{z,w} \to A; \quad x \mapsto \mathcal{L}(x) + \ell_F^* \delta(x) + \delta^\dagger(x) \ell_G + (w_F^* \delta(x))^z + (\delta^\dagger(x) w_G)_w$$

is closable with closure which generates a $C_0$ semigroup then $k$ is a Feller cocycle and $E_\Omega \circ k$ is a semigroup on $A$ with generator which extends

$$x \mapsto \mathcal{L}(x) + \ell_F^* \delta(x) + \ell_F^* \pi(x) \ell_G + \delta^\dagger(x) \ell_G + m_F^* x + x m_G.$$
### References

#### Construction


#### Perturbation
