Quantum Feynman–Kac perturbations

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1. A group of \(*\)-automorphisms

Fix a von Neumann algebra \( \mathcal{A} \subseteq \mathcal{B}(h) \). Let \( \alpha = (\alpha_t : t \in \mathbb{R}) \) be an ultraweakly continuous group of \(*\)-automorphisms of \( \mathcal{A} \) and let \( \delta \) be its ultraweak generator.

Example

Let \( \mathcal{A} = L^\infty(\mathbb{R}) \subseteq \mathcal{B}(L^2(\mathbb{R})) \), where

\[
(fg)(t) := f(t)g(t) \quad (t \in \mathbb{R})
\]

if \( f \in L^\infty(\mathbb{R}) \) and \( g \in L^2(\mathbb{R}) \). Then \( \mathcal{A}_* = \{ \phi_h : h \in L^1(\mathbb{R}) \} \), where

\[
\phi_h : f \mapsto \int_{\mathbb{R}} h(t)f(t)\,dt.
\]

Define \( \alpha \) by setting

\[
(\alpha_t f)(s) := f(s + t).
\]

The generator \( \delta \) is such that

\[
\delta f = \operatorname{uwl}\lim_{t \to 0} \frac{\alpha_t f - f}{t} = f'
\]

for all

\[
f \in \text{dom}(\delta) = \{ g \in \mathcal{A} : g' \in \mathcal{A} \}.
\]
2. Gaussian subordination

Given $\alpha$, *Gaussian subordination* may be used to construct a uw-continuous semigroup $P^0$ on $\mathcal{A}$ with ultraweak pre-generator $\frac{1}{2}\delta^2$.

If $(B_t : t \geq 0)$ is a standard Brownian motion and $\mathbb{P}$ is Wiener measure then

$$j_t : \mathcal{A} \to L^\infty(\mathbb{P}; \mathcal{A}) \subseteq \mathcal{B}(L^2(\mathbb{P}; h)); \ a \mapsto \alpha_{B_t}(a) \quad (t \geq 0)$$

is a $*$-homomorphism such that

$$j_t(x) = x + \int_0^t j_s(\delta x) \, dB_s + \frac{1}{2} \int_0^t j_s(\delta^2 x) \, ds$$

(1)

strongly on $L^2(\mathbb{P}; h) = h \otimes L^2(\mathbb{P})$, for all $x \in \text{dom}(\delta^2)$.

To see this, let

$$G(\xi; a) := \alpha_\xi(a)u \otimes f \quad (\xi \in \mathbb{R}, \ a \in \mathcal{A}, \ u \in h, \ f \in L^2(\mathbb{P})),$$

note that

$$\frac{\partial G}{\partial \xi}(\xi; x) = G(\xi; \delta x) \quad (x \in \text{dom}(\delta))$$

and apply the functional Itô formula.
2. Gaussian subordination

**Theorem 1**

If

\[ \epsilon : h \to L^2(\mathbb{P}; h); \ u \mapsto u \otimes 1_\Omega \]

then

\[ \epsilon^* : L^2(\mathbb{P}; h) \to h; \ F \mapsto \mathbb{E}[F] \]

and setting

\[ P_0^t (a) := \epsilon^* j_t (a) \epsilon \quad (t \geq 0, \ a \in A) \]

gives an uw-continuous CPC semigroup \( P^0 \) on \( A \) with generator as claimed.

Note that

\[ P_0^t (a)u = \epsilon^* (\alpha_{B_t} (a)u \otimes 1_\Omega) = \mathbb{E}[\alpha_{B_t} (a)u] \quad (t \geq 0, \ a \in A, \ u \in h), \]

so (1) shows the generator is an extension of \( \frac{1}{2} \delta^2 \).

For the translation group \( \alpha \) acting on \( L^\infty (\mathbb{R}) \),

\[ P_0^t (f)(s) = \int_\Omega f(s + \omega(t)) \, d\mathbb{P}(\omega). \]

The key to proving the semigroup property is the *cocycle structure* of the flow \( j \).
3. Cocycle structure

The shift semigroup

Let

\[ \theta_s : \Omega \to \Omega; \]
\[ (\theta_s \omega)(t) := \omega(s + t) - \omega(s) \]

and note that

\[ \theta_s \circ \theta_r = \theta_{r+s} \quad (r, s \geq 0). \]

If

\[ P_s := P_{|F_s}, \quad \text{where} \quad F_s := \sigma(B_{s+t} - B_s : t \geq 0) \]

then

\[ \sigma_s : L^\infty(\mathbb{P}; \mathcal{A}) \to L^\infty(P_s; \mathcal{A}); \]
\[ F \mapsto F \circ \theta_s \]

is the shift semigroup on \( L^\infty(\mathbb{P}; \mathcal{A}). \)
3. Cocycle structure

The cocycle property

Note that

\[ j_{s+t}(a)(\omega) = \alpha_{\omega(s+t)}(a) = \alpha_{\omega(s)}(\alpha_{\omega(s+t)-\omega(s)}(a)) = \alpha_{\omega(s)}(\alpha_{(\theta_s \omega)(t)}(a)) = \alpha_{\omega(s)}(\sigma_s \alpha_{B_t}(a))(\omega) = (\hat{j}_s \circ \sigma_s \circ j_t)(a)(\omega), \]

where

\[ \hat{j}_s : L^\infty(\mathbb{P}_s; \mathcal{A}) \to L^\infty(\mathbb{P}; \mathcal{A}); \]

\[ a \otimes f \mapsto \alpha_{B_s}(a) \otimes f \quad (a \in \mathcal{A}, \ f \in L^\infty(\mathbb{P}_s)) . \]

Hence the flow \( j \) is a cocycle for the shift semigroup:

\[ j_{s+t} = \hat{j}_s \circ \sigma_s \circ j_t \quad (s, t \geq 0). \]

Equivalently, \( (J_t := \hat{j}_t \circ \sigma_t : t \geq 0) \) is a semigroup on \( L^\infty(\mathbb{P}; \mathcal{A}) \).
3. Cocycle structure

Proof that $P^0$ is a semigroup

By the cocycle property:

$$P^0_{s+t}(a) = e^* j_{s+t}(a) e$$

$$= e^* J_s(j_t(a)) e$$

$$= e^* j_s(e^* j_t(a) e) e$$

$$= P^0_s(P^0_t(a));$$

for (2), let $u, v \in h$ and note that

$$\langle u, e^* J_s(a \otimes f) e v \rangle = \langle u \otimes 1_\Omega, \alpha_{B_s}(a) v \otimes f \circ \theta_s \rangle$$

$$= \langle u \otimes 1_\Omega, \alpha_{B_s}(a) v \rangle \mathbb{E}[f \circ \theta_s]$$

$$= \langle u, e^* j_s(a \mathbb{E}[f]) e v \rangle$$

$$= \langle u, e^* j_s(e^* (a \otimes f) e) e v \rangle,$$

since $f \circ \theta_s \in L^\infty(\mathbb{P}_s)$ is independent of $B_s$ and $\mathbb{E}[f \circ \theta_s] = \mathbb{E}[f]$. 
4. Feynman–Kac perturbations

A measurable family of operators
\[ m = (m_t : t \geq 0) \subseteq L^\infty(\mathbb{P}; \mathcal{A}) \]
is a \emph{J cocycle} if
\[ m_{s+t} = J_s(m_t)m_s \quad (s, t \geq 0). \]

The process \( m \) is \emph{adapted} if \( m_t \) is \( \mathcal{F}_t \) measurable for all \( t \geq 0 \), where
\[ \mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t). \]

\textbf{Theorem 2}

If \( m \) and \( n \) are \( J \) cocycles then setting
\[ P_t(a) := \epsilon^* m_t^* j_t(a) n_t \epsilon \]
defines a \( u \)-continuous CB semigroup \( P \) on \( \mathcal{A} \), which is CP if \( m_t = n_t \) for all \( t \).
4. Feynman–Kac perturbations

Proof of Theorem 2

The proof is similar to that for \( P^0 \):

\[
P_{s+t}(a) = \epsilon^* m_{s+t}^* J_{s+t}(a)n_{s+t}\epsilon
= \epsilon^* m_s^* J_{s}(m_{t}^*) J_{s}(j_t(a)) J_{s}(n_t) n_s\epsilon
= \epsilon^* m_{s}^* E_s\left(J_{s}(m_{t}^*) j_{t}(a) n_t\right) n_s \epsilon
= \epsilon^* m_{s}^* j_{s}(\epsilon^* m_{t}j_{t}(a) n_t \epsilon) n_s \epsilon
= P_s(P_t(a)),
\]

where

\[
E_s : L^\infty(\mathbb{P}; \mathcal{A}) \to L^\infty(\mathbb{P}_s; \mathcal{A}); \ a \otimes f \mapsto a \otimes E[f | \mathcal{F}_s];
\]

for (3), note that

\[
E_s(J_s(a \otimes f)) = E_s(\alpha_{B_s}(a) \otimes f \circ \theta_s) = \alpha_{B_s}(a) E[f] = j_s(\epsilon^*(a \otimes f) \epsilon).
\]
4. Feynman–Kac perturbations

L–S perturbation

Given $b = b^* \in \mathcal{A}$, Lindsay and Sinha constructed an adapted process $m^b$ in $L^\infty(\mathbb{P}; \mathcal{A})$ such that

$$m^b_t = I + \int_0^t j_s(b)m^b_s dB_s \quad (t \geq 0) \quad (4)$$

strongly on $L^2(\mathbb{P}; h)$, and proved it unique.

If $\alpha$ is unitarily implemented, they showed that the exponential martingale $m^b$ satisfies the $J$-cocycle identity

$$m^b_{s+t} = J_s(m^b_t)m^b_s \quad (s, t \geq 0)$$

and the semigroup $P^b$, where

$$P^b_t(a) = \epsilon^* j_t(a)m^b_t \epsilon,$$

has generator which extends

$$\frac{1}{2} \delta^2 + \rho_b \delta : \text{dom}(\delta^2) \to \mathcal{A}; \ x \mapsto \frac{1}{2} \delta^2(x) + \delta(x)b. \quad (5)$$
4. Feynman–Kac perturbations

Proof of (5)

This is another exercise in stochastic integration: if \( x \in \text{dom}(\delta^2) \) then

\[
j_t(x)m_t^b = \left(x + \int_0^t j_s(\delta x) dB_s + \int_0^t j_s\left(\frac{1}{2}\delta^2 x\right) ds\right) \left(1 + \int_0^t j_s(b)m_s^b dB_s\right)
\]

\[
= x + \int_0^t \ldots dB_s + \int_0^t j_s\left(\frac{1}{2}\delta^2 x\right) ds
\]

\[
+ \int_0^t j_s(\delta x)j_s(b)m_s^b ds + \int_0^t j_s\left(\frac{1}{2}\delta^2 x\right)(m_s^b - I) ds
\]

\[
= x + \int_0^t \ldots dB_s + \int_0^t j_s\left(\frac{1}{2}\delta^2 x + \rho_b \delta x\right)m_s^b ds
\]

and Itô integrals have zero expectation.
4. Feynman–Kac perturbations

B–P perturbation

Bahn and Park noted that such a L–S semigroup will not, in general, be positive or even real (i.e., *-preserving).

They investigated a more symmetric perturbation, using a $J$ cocycle $n^b$ such that

$$n_t^b = 1 + \int_0^t j_s(b) \mathbb{E}[n_s^b | \mathcal{F}_s] \, dB_s - \frac{1}{2} \int_0^t j_s(b^2) \mathbb{E}[n_s^b | \mathcal{F}_s] \, ds. \quad (6)$$

In this case, letting

$$Q_t^b(a) := \epsilon^*(n_t^b)^* j_t(a) n_t^b \epsilon$$

gives a CP semigroup $Q^b$ on $\mathcal{A}$, contractive if $b = b^*$, with generator extending

$$\frac{1}{2} \delta^2 + \lambda_b \delta + \rho_b \delta + \lambda_b \rho_b - \frac{1}{2} \lambda_b^2 - \frac{1}{2} \rho_b^2,$$

where $\lambda_c : a \mapsto ca$ and $\rho_c : a \mapsto ac$. 


5. Fock space

If $\Gamma = \Gamma(\mathcal{L}^2(\mathbb{R}_+))$ is Boson Fock space over $\mathcal{L}^2(\mathbb{R}_+)$ then

$$\mathcal{L}^2(\mathbb{P}) \cong \Gamma \quad \text{and} \quad \Gamma \cong \Gamma_{[t]} \otimes \Gamma_{[t]}$$

where

$$\Gamma_{[t]} = \Gamma(\mathcal{L}^2[0, t]) \cong \mathcal{L}^2(\mathbb{P}_{[t]}) \quad \text{and} \quad \Gamma_{[t]} = \Gamma(\mathcal{L}^2[t, \infty]) \cong \mathcal{L}^2(\mathbb{P}_{[t]})$$

If $\mathcal{H}$ is a complex Hilbert space then $\Gamma(\mathcal{H})$ is a complex Hilbert space with total set of linearly independent exponential vectors $\{\varepsilon(f) : f \in \mathcal{H}\}$ such that

$$\langle \varepsilon(f), \varepsilon(g) \rangle = \exp \langle f, g \rangle.$$  

Then

$$\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2) \cong \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2); \ \varepsilon(f \oplus g) \leftrightarrow \varepsilon(f) \otimes \varepsilon(g)$$

and

$$\Gamma \cong \mathcal{L}^2(\mathbb{P}); \ \varepsilon(f) \leftrightarrow \mathfrak{z}(f),$$

where the stochastic exponential $\mathfrak{z}(f)$ satisfies the SDE

$$\mathfrak{z}(f)_0 = 1_\Omega, \quad d\mathfrak{z}(f)_t = f(t)\mathfrak{z}(f)_t dB_t.$$  

6. Quantum flows

A quantum flow is a \( \mathfrak{uw} \)-continuous family \( j \) of \( * \)-homomorphisms

\[
j_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma) \quad (t \geq 0)
\]

which are

- all adapted or all vacuum adapted, so that

\[
j_t(a) = j_t[a](a) \otimes l_{[t]} \quad \text{or} \quad j_t(a) = j_t[a](a) \otimes |\Omega_t\rangle\langle\Omega_t|,
\]

where \( j_t[a](a) \in \mathcal{A} \otimes \mathcal{B}(\Gamma_t) \) and \( \Omega_t := \varepsilon(0) \in \Gamma_t \) is the vacuum,

- unital, in the sense that \( j_t[1](1) = 1 \).

Moreover, \( j \) is required to satisfy the cocycle equation

\[
j_{s+t} = \hat{j}_s \circ \sigma_s \circ j_t \quad (s, t \geq 0),
\]

where

\[
\hat{j}_s := j_s \otimes l_{B(\Gamma_{[s]})} : \mathcal{A} \otimes \mathcal{B}(\Gamma_{[s]}) \to \mathcal{A} \otimes \mathcal{B}(\Gamma)
\]

and

\[
\sigma_s : \mathcal{A} \otimes \mathcal{B}(\Gamma) \to \mathcal{A} \otimes \mathcal{B}(\Gamma_{[s]}).
\]
6. Quantum flows

The flow $j$ is assumed to satisfy a quantum stochastic differential equation of the form

$$d j_t(x) = j_t(\psi^0_1(x)) \, dA_t + j_t(\psi^1_1(x)) \, d\Lambda_t + j_t(\psi^1_0(x)) \, dA_t^\dagger + j_t(\psi^0_0(x)) \, dt$$

(7)

for all $x \in \mathcal{A}_0 \subseteq \mathcal{A}$, where the structure maps

$$\psi^0_1, \quad \psi^1_1, \quad \psi^1_0, \quad \psi^0_0 : \mathcal{A}_0 \to \mathcal{A}.$$

The QSDE (7) generalises the equation (1), to which it reduces when $j$ is adapted and

$$\mathcal{A}_0 = \text{dom}(\delta^2), \quad \psi^1_1 = 0, \quad \psi^1_0 = \psi^0_1 = \delta|_{\mathcal{A}_0} \quad \text{and} \quad \psi^0_0 = \frac{1}{2}\delta^2.$$

From (7), the flow $j$ has Markov semigroup $P^0$ such that

$$\langle u, P_t^0(x)v \rangle = \langle u, \Omega, j_t(x)v \, \Omega \rangle = \langle u, v \rangle + \int_0^t \langle u, j_s(\psi^0_0(x))v \rangle \, ds$$

for all $t \geq 0$ and $x \in \mathcal{A}_0$. Hence the generator of $P^0$ extends $\psi^0_0$. 
7. Unitary perturbation

Previous authors (Evans and Hudson, Bradshaw, Das and Sinha, ...) have examined perturbations of quantum flows given by unitary conjugation.

This work focused on the situation where the structure maps of the flow $j$ are elements of $\mathcal{B}(\mathcal{A})$, in which case the Markov semigroup is uniformly continuous.

If $h = h^* \in \mathcal{A}$ and $k \in \mathcal{A}$ then there exists a unitary process $U$ such that

$$U_0 = I$$

and

$$dU_t = j_t(-k^*)U_t \, dA_t + j_t(k)U_t \, dA_t^\dagger + j_t(-ih - \frac{1}{2}k^*k)U_t \, dt.$$ 

The process $U$ is a $J$ cocycle and the Markov semigroup of the perturbed flow

$$(a \mapsto U^*_t j_t(a) U_t : t \geq 0)$$

has generator

$$\psi_0^0 + \rho_k \psi_1^0 + \lambda_k^* \psi_0^1 + \rho_k \lambda_k^* \psi_1^1 + i[h, \cdot] - \frac{1}{2}\{k^*k, \cdot\},$$

where $[\cdot, \cdot]$ is the commutator and $\{\cdot, \cdot\}$ the anticommutator.
8. Vacuum perturbation

Let $c = (c_0, c_1) \in \mathcal{A} \times \mathcal{A}$. There exists a unique process $M^c$ such that $M^c - I$ is vacuum adapted and satisfies the QSDE

$$d(M^c - I)_t = j_t(c_1) M^c_t dA^\dagger_t + j_t(c_0) M^c_t dt.$$  

This is a generalisation of the B–P equation (6).

Furthermore, $M^c$ is a $J$ cocycle: for all $s, t \geq 0$,

$$M^c_{s+t} = J_s(M^c_t) M^c_s.$$  

To establish this, an identity of the form

$$\left( \int_s^t F_r d\Xi_r \right) G_s = \int_s^t F_r G_s d\Xi_r$$  

is required, where $\Xi_r \in \{A^\dagger_r, r\}$.

This identity is simple to establish for these integrators, but does not hold in the vacuum-adapted setting for annihilation or gauge integrals.
8. Vacuum perturbation

Let \( c = (c_0, c_1), \ d = (d_0, d_1) \in \mathcal{A} \times \mathcal{A} \).

There exists an uw-continuous semigroup \( P_{d,c}^t \) of CB maps on \( \mathcal{A} \) with
\[
\langle u, P_{t}^{d,c}(a)v \rangle = \langle u \Omega, (M_{t}^{d})^* j_t(a)M_{t}^{c} v \Omega \rangle.
\]
If \( c = d \) then \( P_{d,c}^t \) is CP.

The ultraweak generator of \( P_{d,c}^t \) extends
\[
\psi_0^0 + \rho c_1 \psi_1^0 + \lambda d_1^* \psi_0^1 + \rho c_1 \lambda d_1^* \psi_1^1 + \rho c_0 + \lambda d_0^*.
\]
This class includes both the L–S and the B–P examples.

It also includes those obtained by unitary conjugation; the latter give a version of (8), subject to the constraints that \( c_1 = d_1 = k \) and \( c_0 = d_0 = -ih - \frac{1}{2} k^*k \).
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