An introduction to the chaotic-representation property for normal martingales

Alexander Belton, Lancaster University

Séminaire d’analyse fonctionnelle
Laboratoire de mathématiques de Besançon
Université de Franche-Comté
17th April 2012

1. Normal martingales and the CRP
2. Structure equations
3. Kurtz’s theorem
4. A chaotic Kabanov formula
5. A comparison argument
6. The strong Markov property
7. References
1. Normal martingales and the CRP

Definition

An \((\mathcal{F}_t)_{t \geq 0}\)-martingale \((Z_t)_{t \geq 0}\) is normal if \((Z^2_t - t)_{t \geq 0}\) is also an \((\mathcal{F}_t)_{t \geq 0}\)-martingale.

Equivalently, a process \((Z_t)_{t \geq 0}\) is a normal martingale if

- \(Z_t \in L^2(\mathcal{F}_t)\) for all \(t \geq 0\),
- \(\mathbb{E}[Z_t - Z_s | \mathcal{F}_s] = 0\) for all \(t > s \geq 0\) and
- \(\mathbb{E}[(Z_t - Z_s)^2 | \mathcal{F}_s] = t - s\) for all \(t > s \geq 0\).

Assumptions

- The martingale \(Z\) lies in the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and the \(\sigma\)-algebra \(\mathcal{F}\) is generated by \(Z\).
- The initial value \(Z_0 = 0\).
1. Normal martingales and the CRP

Examples

- Standard Brownian motion \((W_t)_{t \geq 0}\).
- The compensated Poisson process \((N_t - t)_{t \geq 0}\) with unit jumps and intensity 1.
- Azéma’s (first) martingale \((\sqrt{2} \mathbb{E}[W_t | G_t])_{t \geq 0}\), where the \(\sigma\)-field

\[ G_t := \sigma(\text{sign}(W_s) : 0 \leq s \leq t). \]

It may be shown that \(\mathbb{E}[W_t | G_t] = \text{sign}(W_t) \sqrt{t - g_t}\) for all \(t \geq 0\), where

\[ g_t := \sup\{s \in [0, t] : W_s = 0\} \]

is the last exit time from 0 before \(t\).
1. Normal martingales and the CRP

**Chaos space**

Let $\Delta_0 = \{\emptyset\}$ and $\Delta_n := \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n : 0 < t_1 < \cdots < t_n\}$.

For all $n \geq 0$ there exists a linear isometry

$$I_n : L^2(\Delta_n) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$$

such that $I_0(f) = f(\emptyset)$ and $I_n(f) = (Z_{b_1} - Z_{a_1}) \cdots (Z_{b_n} - Z_{a_n})$ if

$$f : (t_1, \ldots, t_n) \mapsto 1_{a_1 < t_1 \leq b_1} \cdots 1_{a_n < t_n \leq b_n}.$$

The random variable $I_n(f)$ is the multiple Wiener integral

$$\int_{\Delta_n} f(t_1, \ldots, t_n) \, dZ_{t_1} \cdots dZ_{t_n}.$$

**Theorem**

Let $\Delta_n(t) := \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n : 0 < t_1 < \cdots < t_n \leq t\}$. Then

$$\mathbb{E}[I_n(f)|\mathcal{F}_t] = I_n(1_{\Delta_n(t)}f) \quad \text{for all } t \geq 0$$

and $(I_n((1_{\Delta_n(t)}f))_{t \geq 0}$ is an $L^2$-martingale.
1. Normal martingales and the CRP

Theorem

Let \( \Xi_n := I_n(L^2(\Delta_n)) \) for all \( n \geq 0 \).

The closed subspaces \( \Xi_m \) and \( \Xi_n \) are orthogonal if \( m \neq n \). Hence \( \Xi := \bigoplus_{n \geq 0} \Xi_n \) is a closed subspace of \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \).

The chaotic-representation property

A normal martingale has the chaotic-representation property (the CRP) if \( \Xi = L^2(\Omega, \mathcal{F}, \mathbb{P}) \).

Examples

- Standard Brownian motion.
- The compensated Poisson process.
- Azéma’s martingale.
- The parabolic martingale \( (P_t)_{t \geq 0} \), which is such that
  \[
  \mathbb{P}(P_t = \sqrt{t}) = \mathbb{P}(P_t = -\sqrt{t}) = \frac{1}{2} \quad (t \geq 0)
  \]
  and the jump times are distributed as a Poisson point process with intensity \( dt/(4t) \).
1. Normal martingales and the CRP

Guichardet space

Every chaos space is naturally isomorphic to Boson Fock space over $L^2(\mathbb{R}_+)$, via the mapping

$$\sum_{n \geq 0} l_n(f_n) \leftrightarrow (f_0(\emptyset), f_1, f_2, \ldots) \in \mathbb{C} \oplus \bigoplus_{n \geq 1} L^2(\Delta_n).$$

A pleasant way of viewing this Fock space has been given by Guichardet.

Definition

Let $P_n := \{ \sigma \subseteq \mathbb{R}_+ : |\sigma| = n \}$ for all $n \geq 0$, where $| \cdot |$ denotes cardinality, and note that

$$\iota_n : P_n \to \Delta_n; \{ t_1 < \cdots < t_n \} \mapsto (t_1, \ldots, t_n)$$

is a bijection.

Equip $P_n$ with the measure it inherits thus from $\mathbb{R}^n$ (for each $n \geq 1$) and let $P := \bigcup_{n \geq 0} P_n$ be the disjoint union of these (together with an atom $\emptyset$ with mass $1$). Then

$$U : \Xi \to L^2(P); \ (Uf)(\sigma) = (f_{|\sigma|} \circ \iota_{|\sigma|})(\sigma)$$

is an isomorphism, where $f = \sum_{n \geq 0} l_n(f_n)$. 
2. Structure equations

Quadratic variation

If $X$ is a semimartingale then there exists a process $(\lbrack X \rbrack_t)_{t \geq 0}$, its quadratic variation, such that

$$X_t^2 = X_0^2 + 2 \int_0^t X_s^- dX_s + [X]_t$$

$$= X_0^2 + 2 \int_{\Delta_2} 1_{0 < t_1 < t_2 \leq t} dX_{t_1} dX_{t_2} + [X]_t$$

for all $t \geq 0$.

If $0 = t_0 < \cdots < t_n = t$ then

$$X_t^2 - X_0^2 = \sum_{j=1}^n (X_{t_j}^2 - X_{t_{j-1}}^2)$$

$$= \sum_{j=1}^n (2X_{t_{j-1}} + X_{t_j} - X_{t_{j-1}})(X_{t_j} - X_{t_{j-1}})$$

$$= 2 \sum_{j=1}^n X_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) + \sum_{j=1}^n (X_{t_j} - X_{t_{j-1}})^2.$$
2. Structure equations

Quadratic variation

If $Z$ is a normal martingale then

$$I_2(1_{\Delta_2(t)}) = \int_{\Delta_2} 1_{0 < t_1 < t_2 \leq t} dZ_{t_1} dZ_{t_2} = L^2 - \lim \sum_{j=1}^{n} Z_{t_j} (Z_{t_j} - Z_{t_{j-1}})$$

and

$$[Z]_t = L^2 - \lim \sum_{j=1}^{n} (Z_{t_j} - Z_{t_{j-1}})^2,$$

where the limits are taken over the net of partitions of $[0, t]$ ordered by refinement.

Furthermore,

$$[Z]_t - t = Z_t^2 - t - 2I_2(1_{\Delta_2(t)}),$$

and the right-hand side is the linear combination of two martingales, so is a martingale. In good situations (such as when $Z$ has the CRP) we can find a predictable process $\Phi$ such that

$$d[Z]_t = \Phi_t dZ_t + dt.$$

What about the converse?
2. Structure equations

The structure equation

Let \( Z \) be a normal martingale and suppose there exist Borel-measurable functions \( \alpha : \mathbb{R}_+ \to \mathbb{R} \) and \( \beta : \mathbb{R}_+ \to \mathbb{R} \) such that the structure equation

\[
\mathrm{d}[Z]_t = (\alpha(t) + \beta(t)Z_{t-}) \, \mathrm{d}Z_t + \mathrm{d}t
\]

holds for all \( t \geq 0 \).

Examples

- If \( \alpha \equiv \beta \equiv 0 \) then \( Z \) is a standard Brownian motion (a result due to Lévy).
- If \( \alpha \equiv a \neq 0 \) and \( \beta \equiv 0 \) then
  \[
  Z_t = a(N_{t/a^2} - t/a^2) \quad \forall t \geq 0;
  \]
  this is the Poisson case.
- If \( \alpha \equiv 0 \) and \( \beta \equiv -1 \) then \( Z \) is Azéma’s martingale.
- If \( \alpha \equiv 0 \) and \( \beta \equiv -2 \) then \( Z_t^2 = t \) for all \( t \geq 0 \) and \( Z \) is the parabolic martingale.
3. Kurtz’s Theorem

Theorem

A normal martingale $Z$ has the CRP if and only if

$$\hat{Z}_t: \{F \in \Xi: Z_t F \in \Xi\} \to \Xi; \ F \mapsto Z_t F$$

is a self-adjoint operator in $\Xi$ for all $t \geq 0$.

Proof

If $\Xi = L^2(\Omega, \mathcal{F}, \mathbb{P})$ then self-adjointness is immediate.

Conversely, if $\hat{Z}_t$ is self adjoint for all $t \geq 0$ then

$$\exp(i\lambda \hat{Z}_t)F = \exp(i\lambda Z_t)F \quad \forall \lambda \in \mathbb{R}, \ F \in \Xi,$$

using Nelson’s idea of analytic vectors, and so

$$\exp(-i(\lambda_1 Z_{t_1} + \cdots + \lambda_n Z_{t_n})) \in \Xi$$

for all $n \geq 1$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $\{t_1, \ldots, t_n\} \in \mathbb{P}_n$. Now suppose $G \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is orthogonal to $\Xi$ and note that

$$A \mapsto \mathbb{E}[\mathbf{1}_{(Z_{t_1},\ldots,Z_{t_n}) \in A} G]$$

has Fourier transform

$$(\lambda_1, \ldots, \lambda_n) \mapsto \mathbb{E}[\exp(-i(\lambda_1 Z_{t_1} + \cdots + \lambda_n Z_{t_n})) G] \equiv 0.$$

$\square$
4. A chaotic Kabanov formula

**Theorem**

Suppose the normal martingale $Z$ satisfies (1). If $\alpha$ and $\beta$ are bounded on $[0, t]$ then

$$Z_t I_n(f) = I_{n-1}(f^-) + I_n(f^\circ) + I_{n+1}(f^+) \in \Xi$$

for all $n \geq 1$ and $f \in L^2(\Delta^n)$, where

$$f_t^-(t_1, \ldots, t_{n-1}) := \sum_{k=1}^{n-1} \int_{t_{k-1} \wedge t}^{t_k \wedge t} \prod_{l=k}^{n-1} \left( 1 + \beta_{t_l}(t_l) \right) f(t_1, \ldots, t_{k-1}, s, t_k, \ldots, t_{n-1}) \, ds$$

$$+ \int_{t_{n-1} \wedge t}^t f(t_1, \ldots, t_{n-1}, s) \, ds,$$

$$f_t^\circ(t_1, \ldots, t_n) := \sum_{k=1}^n \mathbf{1}_{t_k \in (0, t]} \alpha(t_k) \prod_{l=k+1}^{n} \left( 1 + \beta_{t_l}(t_l) \right) f(t_1, \ldots, t_n),$$

$$f_t^+(t_1, \ldots, t_{n+1}) := \sum_{k=1}^{n+1} \mathbf{1}_{t_k \in (0, t]} \prod_{l=k+1}^{n+1} \left( 1 + \beta_{t_l}(t_l) \right) f(t_1, \ldots, \hat{t}_k, \ldots, t_{n+1}),$$

with $t_0 := 0$, $\beta_{t_l}(s) := \mathbf{1}_{s \in (0, t]} \beta(s)$ and $(t_1, \ldots, \hat{t}_k, \ldots, t_{n+1})$ denoting the $n$-tuple obtained by removing $t_k$ from $(t_1, \ldots, t_{n+1})$. 
5. A comparison argument

Definition

A measurable function $f : P \to \mathbb{C}$ is a test vector if there exist $T, C, M \geq 0$ such that

$$|f(\sigma)| \leq 1_{\sigma \subseteq [0, T]}CM|\sigma|$$

for all $\sigma \in P$. The set $T$ of all such functions is a dense subspace of $L^2(P)$.

Theorem

If $Z$ is a normal martingale satisfying (1), with $\alpha$ and $\beta$ bounded on $[0, t]$, then $U\hat{Z}_tU^{-1}f \in T$ for all $f \in T$ and

$$
(U\hat{Z}_tU^{-1}f)(\sigma) = \int_0^t \prod_{r \in \sigma_{(s,t)}} (1 + \beta(r)) f(\sigma \cup s) \, ds
$$

$$
+ \sum_{s \in \sigma_{t]} \alpha(s) \prod_{r \in \sigma_{(s,t)}} (1 + \beta(r)) f(\sigma)
$$

$$
+ \sum_{s \in \sigma_{t]} \prod_{r \in \sigma_{(s,t)}} (1 + \beta(r)) f(\sigma \setminus s)
$$

for all $\sigma \in P$, where $\sigma_{t]} := \sigma \cap (0, t]$ and $\sigma_{(s,t]} := \sigma \cap (s, t]$. 
5. A comparison argument

Corollary

Suppose \( t \geq 0 \) is such that \( c := \sup\{|\alpha(s)| : 0 \leq s \leq t\} < \infty \) and \( \beta(s) \in [-2, 0] \) for all \( s \in [0, t] \). If \( C_s = c(N_s/c^2 - s/c^2) \) for all \( s \geq 0 \), so that \( d[C]_s = c \, dC_s + ds \), then

\[
|U\hat{Z}_tU^{-1}f|(\sigma) \leq \int_0^t |f|(\sigma \cup s) \, ds + c|\sigma_t||f|(\sigma) + \sum_{s \in \sigma_t} |f|(\sigma \setminus s)
\]

\[
= (U\hat{C}_tU^{-1}|f|)(\sigma) \quad \forall \sigma \in \mathcal{P}
\]

for all \( f \in \mathcal{T} \).

Since \( U\hat{C}_tU^{-1}g \geq 0 \) if \( g \geq 0 \), it follows that

\[
|U\hat{Z}_t^nU^{-1}f| \leq (U\hat{C}_t^nU^{-1})|f| \quad \forall n \geq 0, \ f \in \mathcal{T};
\]

this, and the fact that \( \sum_{n=0}^{\infty} \mathbb{E}[N_t^n] \frac{z^n}{n!} \) has positive radius of convergence, implies that exponential vectors

\[
\varepsilon(f) : \mathcal{P} \to \mathbb{C}; \ \sigma \mapsto \prod_{t \in \sigma} f(t)
\]

corresponding to bounded functions with compact support are analytic vectors for \( U\hat{Z}_tU^{-1} \), which is therefore self adjoint.

[This technique was introduced in this context by Parthasarathy.]
5. A comparison argument

Theorem (Attal & ACRB)
If $\alpha$ is locally bounded and $\beta(t) \in [-2, 0]$ for all $t \geq 0$ then $Z$ has the CRP and is unique in law.

Proof
The CRP holds by the working above and Kurtz’s theorem.

Uniqueness in law holds because

$$(\lambda_1, \ldots, \lambda_n) \mapsto \mathbb{E}\left[\exp\left(i(\lambda_1 Z_{t_1} + \cdots + \lambda_n Z_{t_n})\right)\right]$$

$$= \left\langle 1, \exp(i\lambda_1 \hat{Z}_{t_1}) \cdots \exp(i\lambda_n \hat{Z}_{t_n})1 \right\rangle$$

and $\hat{Z}_t$ is determined by its action on $\mathcal{T}$, which depends only on $\alpha$ and $\beta$. \qed
6. The strong Markov property

**Theorem**

If $Z$ is a normal martingale satisfying (1), with $\alpha$ locally bounded and $\beta(t) \in [-2, 0]$ for all $t \geq 0$, then $(Z_t, t)_{t \geq 0}$ has the **strong Markov property**: if $T$ is a finite stopping time then

$$
\mathbb{E}[f(Z_{t+T}) | \mathcal{F}_T] = \mathbb{E}[f(Z_{t+T}) | \sigma(Z_T, T)] \quad \forall t \geq 0
$$

for any bounded, Borel-measurable function $f: \mathbb{R} \to \mathbb{R}$.

**Proof**

Note that $(M_t := Z_{t+T})$ is a normal martingale with respect to both $(\mathcal{F}_{t+T})_{t \geq 0}$ and also $(\mathcal{G}_t := \sigma(T, M_s : 0 \leq s \leq t))_{t \geq 0}$. It satisfies the structure equation

$$
d[M]_t = (A(t) + B(t)M_{t-}) dM_t + dt,
$$

where $A(t) := \alpha(t + T)$ and $B(t) := \beta(t + T)$.

All the previous working generalises to the case of $L^2(\mathcal{F}_0)$-valued coefficient functions, and the uniqueness-in-law result becomes uniqueness in law conditional on the initial $\sigma$-algebra, whence

$$
\mathcal{L}(Z_{t+T} | \mathcal{F}_T) = \mathcal{L}(Z_{t+T} | \mathcal{G}_0) = \mathcal{L}(Z_{t+T} | \sigma(T, Z_T)).
$$
7. References

Introduction


M. Émery

Existence

G. Taviot

Kurtz’s theorem

D. Kurtz
7. References

Parthasarathy’s technique

K.R. Parthasarathy

Today’s talk

S. Attal & A.C.R. Belton

http://dx.doi.org/10.1007/s00440-006-0052-z