Markov semigroups and non-commutative Feynman–Kac formulae

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1. A group of \( \ast \)-automorphisms

Fix a von Neumann algebra \( \mathcal{A} \subseteq \mathcal{B}(\mathcal{H}) \). Let \( \alpha = (\alpha_t : t \in \mathbb{R}) \) be an ultraweakly continuous group of \( \ast \)-automorphisms of \( \mathcal{A} \) and let \( \delta \) be its ultraweak generator.

Example

Let \( \mathcal{A} = L^\infty(\mathbb{R}) \subseteq \mathcal{B}(L^2(\mathbb{R})) \), where
\[
(fg)(t) := f(t)g(t) \quad (t \in \mathbb{R})
\]
if \( f \in L^\infty(\mathbb{R}) \) and \( g \in L^2(\mathbb{R}) \). Then \( \mathcal{A}_\ast = \{\phi_h : h \in L^1(\mathbb{R})\} \), where
\[
\phi_h : f \mapsto \int_{\mathbb{R}} h(t)f(t) \, dt.
\]

Define \( \alpha \) by setting
\[
(\alpha_tf)(s) := f(s + t).
\]
Then
\[
\alpha_0 = \iota_{\mathcal{A}}, \quad \alpha_s \circ \alpha_t = \alpha_{s+t}
\]
and
\[
\alpha_t(f^\ast) = (\alpha_tf)^\ast.
\]
1. A group of ∗-automorphisms

Furthermore,

\[ \lim_{t \to 0} \phi_h(\alpha_t f - f) = \lim_{t \to 0} \int_{\mathbb{R}} h(s)(f(s + t) - f(s)) \, ds \]

\[ = \lim_{t \to 0} \int_{\mathbb{R}} f(s)(h(s - t) - h(s)) \, ds = 0, \]

so \( t \mapsto \alpha_t f \) is uw continuous.

Similarly,

\[ \phi_h(\alpha_t(f_\lambda - f)) = \int_{\mathbb{R}} h(s)(f_\lambda - f)(s + t) \, ds \]

\[ = \int_{\mathbb{R}} h(s - t)(f_\lambda - f)(s) \, ds \]

\[ = \phi_h(\cdot - t)(f_\lambda - f) \]

\[ \to 0 \]

if \( f_\lambda \xrightarrow{uw} f \). Hence \( f \mapsto \alpha_t f \) is uw continuous.
1. A group of $\ast$-automorphisms

The generator $\delta$ is such that

$$\delta f = \text{uwlim}_{t \to 0} \frac{\alpha_t f - f}{t} = f'$$

for all

$$f \in \text{dom}(\delta) = \{ g \in A : g' \in A \}.$$ 

One inclusion holds by the DCT and MVT, since

$$\left| \frac{f(s + t) - f(s)}{t} - f'(s) \right| = |f'(c) - f'(s)| \leq 2\|f\|_{\infty}.$$ 

The other holds because

$$f_\lambda \xrightarrow{\text{uw}} f \implies f_\lambda \to f \text{ almost everywhere.}$$
2. Gaussian subordination

Given $\alpha$, Gaussian subordination may be used to construct a uw-continuous semigroup $P^0$ on $\mathcal{A}$ with ultraweak pre-generator $\frac{1}{2}\delta^2$.

If $(B_t : t \geq 0)$ is a standard Brownian motion and $\mathbb{P}$ is Wiener measure then

$$j_t : \mathcal{A} \to L^\infty(\mathbb{P}; \mathcal{A}) \subseteq \mathcal{B}(L^2(\mathbb{P}; h)); \ a \mapsto \alpha_{B_t}(a) \quad (t \geq 0)$$

is a $\ast$-homomorphism such that

$$j_t(x) = x + \int_0^t j_s(\delta x) \, dB_s + \frac{1}{2} \int_0^t j_s(\delta^2 x) \, ds \quad (1)$$

strongly on $L^2(\mathbb{P}; h) = h \otimes L^2(\mathbb{P})$, for all $x \in \text{dom}(\delta^2)$.

To see this, let

$$G(\xi; a) := \alpha_\xi(a) u \otimes f \quad (\xi \in \mathbb{R}, \ a \in \mathcal{A}, \ u \in h, \ f \in L^2(\mathbb{P})), $$

note that

$$\frac{\partial G}{\partial \xi}(\xi; x) = G(\xi; \delta x) \quad (x \in \text{dom}(\delta))$$

and apply Itô’s formula.
2. Gaussian subordination

Theorem 1

If

\[ \epsilon : h \rightarrow L^2(\mathbb{P}; h); \quad u \mapsto u \otimes 1_\Omega \]

then

\[ \epsilon^* : L^2(\mathbb{P}; h) \rightarrow h; \quad F \mapsto \mathbb{E}[F] \]

and setting

\[ P^0_t(a) := \epsilon^* j_t(a) \epsilon \quad (t \geq 0, \ a \in \mathcal{A}) \]

gives an uw-continuous CPC semigroup \( P^0 \) on \( \mathcal{A} \) with generator as claimed.

Note that

\[ P^0_t(a)u = \epsilon^* (\alpha_{B_t}(a)u \otimes 1_\Omega) = \mathbb{E}[\alpha_{B_t}(a)u] \quad (t \geq 0, \ a \in \mathcal{A}, \ u \in h), \]

so (1) shows the generator is an extension of \( \frac{1}{2}\delta^2 \).

For the translation group \( \alpha \) acting on \( L^\infty(\mathbb{R}) \),

\[ P^0_t(f)(s) = \int_\Omega f(s + \omega(t)) \, d\mathbb{P}(\omega). \]

The key to proving the semigroup property is the cocycle structure of the flow \( j \).
3. Cocycle structure

The shift semigroup

Let

\[ \theta_s : \Omega \to \Omega; \]

\[ (\theta_s\omega)(t) := \omega(s + t) - \omega(s) \]

and note that

\[ \theta_s \circ \theta_r = \theta_{r+s} \quad (r, s \geq 0). \]

If

\[ \mathbb{P}_s := \mathbb{P}|_{\mathcal{F}_s}, \quad \text{where} \quad \mathcal{F}_s := \sigma(B_{s+t} - B_s : t \geq 0) \]

then

\[ \sigma_s : L^\infty(\mathbb{P}; \mathcal{A}) \to L^\infty(\mathbb{P}_s; \mathcal{A}); \]

\[ F \mapsto F \circ \theta_s \]

is the shift semigroup on \( L^\infty(\mathbb{P}; \mathcal{A}). \)
3. Cocycle structure

The cocycle property

Note that

$$j_{s+t}(a)(\omega) = \alpha_{\omega(s+t)}(a) = \alpha_{\omega(s)}\left(\alpha_{\omega(s+t)-\omega(s)}(a)\right)$$

$$= \alpha_{\omega(s)}\left(\alpha_{(\theta_s\omega)(t)}(a)\right)$$

$$= \alpha_{\omega(s)}\left((\sigma_s\alpha_{B_t}(a))(\omega)\right) = (\hat{j}_s \circ \sigma_s \circ j_t)(a)(\omega),$$

where

$$\hat{j}_s : L^\infty(\mathbb{P}_s; \mathcal{A}) \to L^\infty(\mathbb{P}; \mathcal{A});$$

$$a \otimes f \mapsto \alpha_{B_s}(a) \otimes f \quad (a \in \mathcal{A}, \ f \in L^\infty(\mathbb{P}_s)).$$

Hence the flow $j$ is a cocycle for the shift semigroup:

$$j_{s+t} = \hat{j}_s \circ \sigma_s \circ j_t \quad (s, t \geq 0).$$

Equivalently, $(J_t := \hat{j}_t \circ \sigma_t : t \geq 0)$ is a semigroup on $L^\infty(\mathbb{P}; \mathcal{A})$. 
3. Cocycle structure

Proof that $P^0$ is a semigroup

By the cocycle property:

$$P^0_{s+t}(a) = \epsilon^* j_{s+t}(a) \epsilon$$

$$= \epsilon^* J_s(j_t(a)) \epsilon$$

$$= \epsilon^* j_s(\epsilon^* j_t(a) \epsilon) \epsilon$$

$$= P^0_s(P^0_t(a));$$

for (2), let $u, v \in h$ and note that

$$\langle u, \epsilon^* J_s(a \otimes f) \epsilon v \rangle = \langle u \otimes 1_\Omega, \alpha_{B_s}(a) v \otimes f \circ \theta_s \rangle$$

$$= \langle u \otimes 1_\Omega, \alpha_{B_s}(a) v \rangle \mathbb{E}[f \circ \theta_s]$$

$$= \langle u, \epsilon^* j_s(a \mathbb{E}[f]) \epsilon v \rangle$$

$$= \langle u, \epsilon^* j_s(\epsilon^*(a \otimes f) \epsilon) \epsilon v \rangle,$$

since $f \circ \theta_s \in L^\infty(\mathbb{P}_s)$ is independent of $B_s$ and $\mathbb{E}[f \circ \theta_s] = \mathbb{E}[f]$. 
4. Feynman–Kac perturbations

A family of operators
\[ m = (m_t : t \geq 0) \subseteq L^\infty(\mathbb{P}; \mathcal{A}) \]

is a \textit{J cocycle} if
\[ m_{s+t} = J_s(m_t)m_s \quad (s, t \geq 0). \]

The process \( m \) is \textit{adapted} if \( m_t \) is \( \mathcal{F}_t \) measurable for all \( t \geq 0 \), where
\[ \mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t). \]

\textbf{Theorem 2}

If \( m \) and \( n \) are \( J \) cocycles then setting
\[ P_t(a) := \epsilon^* m_t^* J_t(a) n_t \epsilon \]

defines a uw-continuous CB semigroup \( P \) on \( \mathcal{A} \), which is CP if \( m_t = n_t \) for all \( t \).
4. Feynman–Kac perturbations

Proof of Theorem 2

The proof is similar to that for $P^0$:

$$P_{s+t}(a) = \epsilon^* m_{s+t}^* J_{s+t}(a) n_{s+t} \epsilon$$

$$= \epsilon^* m_s^* J_s (m_t^*) J_s (j_t(a)) J_s (n_t) n_s \epsilon$$

$$= \epsilon^* m_s^* \mathbb{E}_s (J_s (m_t^* j_t(a) n_t)) n_s \epsilon$$

$$= \epsilon^* m_s^* J_s (\epsilon^* m_t j_t(a) n_t \epsilon) n_s \epsilon$$

$$= P_s (P_t(a)), \tag{3}$$

where

$$\mathbb{E}_s : L^\infty (\mathbb{P}; A) \rightarrow L^\infty (\mathbb{P}_s; A); \ a \otimes f \mapsto a \otimes \mathbb{E}[f | \mathcal{F}_s];$$

for (3), note that

$$\mathbb{E}_s (J_s (a \otimes f)) = \mathbb{E}_s (\alpha_{B_s}(a) \otimes f \circ \theta_s) = \alpha_{B_s}(a) \mathbb{E}[f] = j_s (\epsilon^* (a \otimes f) \epsilon).$$
4. Feynman–Kac perturbations

L–S perturbation

Given \( b = b^* \in \mathcal{A} \), Lindsay and Sinha constructed an adapted process \( m^b \) in \( L^\infty(\mathbb{P}; \mathcal{A}) \) such that

\[
m^b_t = I + \int_0^t j_s(b) m^b_s \, dB_s \quad (t \geq 0)
\]

strongly on \( L^2(\mathbb{P}; h) \), and proved it unique.

If \( \alpha \) is unitarily implemented, they showed that the exponential martingale \( m^b \) satisfies the \( J \)-cocycle identity

\[
m^b_{s+t} = J_s(m^b_t) m^b_s \quad (s, t \geq 0)
\]

and the semigroup \( P^b \), where

\[
P^b_t(a) = \epsilon^* j_t(a) m^b_t \epsilon,
\]

has generator which extends

\[
\frac{1}{2} \delta^2 + \rho_b \delta : \text{dom}(\delta^2) \to \mathcal{A}; \ x \mapsto \frac{1}{2} \delta^2(x) + \delta(x) b.
\]
4. Feynman–Kac perturbations

Proof of (5)

This is another exercise in stochastic integration: if $x \in \text{dom}(\delta^2)$ then

$$ j_t(x)m^b_t = \left( x + \int_0^t j_s(\delta x) \, dB_s + \int_0^t j_s\left(\frac{1}{2}\delta^2 x\right) \, ds \right) \left( 1 + \int_0^t j_s(b)m^b_s \, dB_s \right) $$

$$ = x + \int_0^t \ldots \, dB_s + \int_0^t j_s\left(\frac{1}{2}\delta^2 x\right) \, ds $$

$$ + \int_0^t j_s(\delta x)j_s(b)m^b_s \, ds + \int_0^t j_s\left(\frac{1}{2}\delta^2 x\right)(m^b_s - I) \, ds $$

$$ = x + \int_0^t \ldots \, dB_s + \int_0^t j_s\left(\frac{1}{2}\delta^2 x + \rho_b\delta x\right)m^b_s \, ds $$

and Itô integrals have zero expectation.
4. Feynman–Kac perturbations

**B–P perturbation**

Bahn and Park noted that such a L–S semigroup will not, in general, be positive or even real (i.e., $\ast$-preserving).

They investigated a more symmetric perturbation, using a $J$ cocycle $n^b$ such that

$$n_t^b = I + \int_0^t j_s(b) \mathbb{E}[n_s^b | \mathcal{F}_s] \, dB_s - \frac{1}{2} \int_0^t j_s(b^2) \mathbb{E}[n_s^b | \mathcal{F}_s] \, ds. \quad (6)$$

In this case, letting

$$Q_t^b(a) := \epsilon^* (n_t^b)^* j_t(a) n_t^b \epsilon$$

gives a CP semigroup $Q^b$ on $\mathcal{A}$, contractive if $b = b^*$, with generator extending

$$\frac{1}{2} \delta^2 + \lambda_b \delta + \rho_b \delta + \lambda_b \rho_b - \frac{1}{2} \lambda_b^2 - \frac{1}{2} \rho_b^2,$$

where $\lambda_c : a \mapsto ca$ and $\rho_c : a \mapsto ac$. 
5. Fock space

If $\Gamma = \Gamma(L^2(\mathbb{R}^+))$ is Boson Fock space over $L^2(\mathbb{R}^+)$ then

$$L^2(P) \cong \Gamma \quad \text{and} \quad \Gamma \cong \Gamma_t \otimes \Gamma_{[t]}$$

where

$$\Gamma_{[t]} = \Gamma(L^2[0, t]) \cong L^2(P_{[t]}) \quad \text{and} \quad \Gamma_{[t]} = \Gamma(L^2[t, \infty]) \cong L^2(P_{[t]}).$$

If $\mathcal{H}$ is a complex Hilbert space then $\Gamma(\mathcal{H})$ is a complex Hilbert space with total set of linearly independent exponential vectors $\{\varepsilon(f) : f \in \mathcal{H}\}$ such that

$$\langle \varepsilon(f), \varepsilon(g) \rangle = \exp \langle f, g \rangle.$$

Then

$$\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2) \cong \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2); \ \varepsilon(f \oplus g) \leftrightarrow \varepsilon(f) \otimes \varepsilon(g)$$

and

$$\Gamma \cong L^2(P); \ \varepsilon(f) \leftrightarrow \zeta(f),$$

where the stochastic exponential $\zeta(f)$ satisfies the SDE

$$\zeta(f)_0 = 1_\Omega, \quad d\zeta(f)_t = f(t)\zeta(f)_t \, dB_t.$$
6. Quantum flows

A quantum flow is a uw-continuous family \( j \) of \(*\)-homomorphisms

\[
j_t : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}(\Gamma) \quad (t \geq 0)
\]

which are

- all adapted or all vacuum adapted, so that
  \[
j_t(a) = j_t[1](a) \otimes 1_t \quad \text{or} \quad j_t(a) = j_t[1](a) \otimes |\Omega_t\rangle\langle\Omega_t|,
\]
  where \( j_t[1](a) \in \mathcal{A} \otimes \mathcal{B}(\Gamma_t) \) and \( \Omega_t := \varepsilon(0) \in \Gamma_t \) is the vacuum,
- unital, in the sense that \( j_t[1](1) = 1 \).

Moreover, \( j \) is required to satisfy the cocycle equation

\[
j_{s+t} = \hat{j}_s \circ \sigma_s \circ j_t \quad (s, t \geq 0),
\]

where

\[
\hat{j}_s := j_s[1] \otimes 1_B(\Gamma_s) : \mathcal{A} \otimes \mathcal{B}(\Gamma_s) \rightarrow \mathcal{A} \otimes \mathcal{B}(\Gamma)
\]

and

\[
\sigma_s : \mathcal{A} \otimes \mathcal{B}(\Gamma) \rightarrow \mathcal{A} \otimes \mathcal{B}(\Gamma_s).
\]
6. Quantum flows

The flow $j$ is assumed to satisfy a quantum stochastic differential equation of the form

$$
dj_t(x) = j_t(\psi_x^0(x)) \, dA_t + j_t(\psi^x(x)) \, d\Lambda_t + j_t(\psi_0^x(x)) \, dA_t^\dagger + j_t(\psi_0^0(x)) \, dt \quad (7)$$

for all $x \in A_0 \subseteq A$, where the structure maps

$$\psi_x^0, \quad \psi_x^x, \quad \psi_0^x, \quad \psi_0^0 : A_0 \to A.$$

The QSDE (7) generalises the equation (1), to which it reduces when $j$ is adapted and

$$A_0 = \text{dom}(\delta^2), \quad \psi_x^x = 0, \quad \psi_x^0 = \psi_0^x = \delta|_{A_0} \quad \text{and} \quad \psi_0^0 = \frac{1}{2}\delta^2.$$

From (7), the flow $j$ has Markov semigroup $P^0$ such that

$$\langle u, P_t^0(x)v \rangle = \langle u \Omega, j_t(x)v \Omega \rangle = \langle u, v \rangle + \int_0^t \langle u, j_s(\psi_0^0(x))v \rangle \, ds$$

for all $t \geq 0$ and $x \in A_0$. Hence the generator of $P^0$ extends $\psi_0^0$. 
7. Quantum stochastic integration

Let $F$, $G$ and $H$ be processes of operators in $h \otimes \Gamma$, with domains containing $h \otimes \mathcal{E}$, which are adapted (or vacuum adapted) and locally square-integrable, i.e.,

$$\int_0^t \| F_s u \varepsilon (f) \|^2 + \| G_s u \varepsilon (f) \|^2 + \| H_s u \varepsilon (f) \|^2 \, ds < \infty \quad (t \geq 0).$$

There exists a unique process, the \textit{quantum stochastic integral}

$$M_t := \int_0^t F_s \, dA_s + G_s \, d\Lambda_s + H_s \, dA_s^\dagger \quad (t \geq 0)$$

which has these properties and satisfies

$$\langle u \varepsilon (f), M_t v \varepsilon (g) \rangle = \int_0^t \langle u \varepsilon (f), (g(s) F_s + \overline{f(s)} g(s) G_s + \overline{f(s)} H_s) v \varepsilon (g) \rangle \, ds$$

for all $t \geq 0$.

In short,

$$dM_t = F_t \, dA_t + G_t \, d\Lambda_t + H_t \, dA_t^\dagger.$$
7. Quantum stochastic integration

If $B$ is classical Brownian motion then there exists an isometric isomorphism $U_W : L^2(\Omega_W) \rightarrow \Gamma$ such that

$$U_W^*(A_t + A_t^\dagger)U_W = B_t \quad (t \geq 0),$$

where $B_t$ acts by multiplication on $L^2(\Omega_W) = L^2(\mathbb{P})$.

Similarly, if $P$ is a compensated Poisson process (with intensity 1 and unit jumps) then there exists an isometric isomorphism $U_P : L^2(\Omega_P) \rightarrow \Gamma$ such that

$$U_P^*(A_t + \Lambda_t + A_t^\dagger)U_P = P_t \quad (t \geq 0).$$

The classical Itô table is generalised by the quantum one:

\[
\begin{array}{c|cccc}
  & dP_t & dB_t & dt & \\
\hline
dP_t & dP_t & 0 & 0 & \\
dB_t & 0 & dt & 0 & \\
dt & 0 & 0 & 0 & \\
\end{array}
\]

becomes

\[
\begin{array}{c|cccc}
  & dA_t & d\Lambda_t & dA_t^\dagger & dt \\
\hline
dA_t & 0 & dA_t & dt & 0 \\
d\Lambda_t & 0 & d\Lambda_t & dA_t^\dagger & 0 \\
dA_t^\dagger & 0 & 0 & 0 & 0 \\
dt & 0 & 0 & 0 & 0 \\
\end{array}
\]

For example,

$$(dB_t)^2 = (dA_t + dA_t^\dagger)^2 = (dA_t)^2 + dA_t dA_t^\dagger + dA_t^\dagger dA_t + (dA_t^\dagger)^2 = dt.$$
8. Unitary perturbation

Previous authors (Evans and Hudson, Bradshaw, Das and Sinha) have examined perturbations of quantum flows given by conjugation with a unitary process.

This work focused on the situation where the structure maps of the flow $j$ are elements of $\mathcal{B}(\mathcal{A})$, in which case the Markov semigroup is uniformly continuous.

If $h = h^* \in \mathcal{A}$ and $k \in \mathcal{A}$ then there exists a unitary process $U$ such that

$$U_0 = I$$

and

$$dU_t = j_t(-k^*)U_t \, dA_t + j_t(k)U_t \, dA_t^\dagger + j_t(-ih - \frac{1}{2}k^*k)U_t \, dt.$$ 

The process $U$ is a $J$ cocycle and the Markov semigroup of the perturbed flow

$$(a \mapsto U_t^*j_t(a)U_t : t \geq 0)$$

has generator

$$\psi_0 + \rho_k \psi_0^\times + \lambda_k^* \psi_0^\times + \rho_k \lambda_k^* \psi^\times + i[h, \cdot] - \frac{1}{2}\{k^*k, \cdot\},$$

where $[\cdot, \cdot]$ is the commutator and $\{\cdot, \cdot\}$ the anticommutator.
9. Vacuum perturbation

Let \( c = (c_0, c_\times) \in \mathcal{A} \times \mathcal{A} \). There exists a unique process \( M^c \) such that \( M^c - I \) is vacuum adapted and satisfies the QSDE

\[
d(M^c - I)_t = j_t(c_\times)M^c_t dA^\dagger_t + j_t(c_0)M^c_t dt.
\]

This is a generalisation of the B–P equation (6).

Furthermore, \( M^c \) is a \( J \) cocycle: for all \( s, t \geq 0 \),

\[
M^c_{s+t} = J_s(M^c_t)M^c_s.
\]

To establish this, an identity of the form

\[
\left( \int_s^t F_r d\Xi_r \right) G_s = \int_s^t F_r G_s d\Xi_r
\]

is required, where \( \Xi_r \in \{A^\dagger_r, r\} \).

This identity is simple to establish for these integrators, but does not hold in the vacuum-adapted setting for annihilation or gauge integrals.
9. Vacuum perturbation

Let \( c = (c_0, c_x), d = (d_0, d_x) \in \mathcal{A} \times \mathcal{A} \).

There exists an uw-continuous semigroup \( P^{d,c} \) of CB maps on \( \mathcal{A} \) with

\[
\langle u, P^{d,c}_t(a)v \rangle = \langle u \Omega, (M^d_t)^* j_t(a) M^c_t v \Omega \rangle.
\]

If \( c = d \) then \( P^{d,c} \) is CP.

The ultraweak generator of \( P^{d,c} \) extends

\[
\psi^0_0 + \rho_{c_x} \psi^0_{c_x} + \lambda_{d_x^*} \psi^c_x + \rho_{c_x} \lambda_{d_x^*} \psi^c_x + \rho_{c_0} + \lambda_{d_0^*}.
\] (8)

This class includes both the L–S and the B–P examples.

It also includes those obtained by unitary conjugation; the latter give a version of (8), subject to the constraints that \( c_x = d_x = k \) and \( c_0 = d_0 = -ih - \frac{1}{2} k^* k \).
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