The construction of quantum dynamical semigroups by way of non-commutative Markov processes

Alexander Belton
(Joint work with Stephen Wills, University College Cork)

Department of Mathematics and Statistics
Lancaster University
United Kingdom

a.belton@lancaster.ac.uk

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Classical Markov semigroups

- Markov processes
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Markov processes

**Definition**

A *Markov process* with *state space* $S$ is a collection of $S$-valued random variables $(X_t)_{t \geq 0}$ on a common probability space such that

$$
\mathbb{E}\left[f(X_{s+t}) \mid \sigma(X_r : 0 \leq r \leq s)\right] = \mathbb{E}\left[f(X_{s+t}) \mid X_s\right] \quad (s, t \geq 0)
$$

for all $f \in L^\infty(S)$.

Such a Markov process is *time homogeneous* if

$$
\mathbb{E}\left[f(X_{s+t}) \mid X_s = x\right] = \mathbb{E}\left[f(X_t) \mid X_0 = x\right] \quad (s, t \geq 0, \ x \in S)
$$

for all $f \in L^\infty(S)$. 
Markov semigroups

Given a time-homogeneous Markov process, setting

$$(T_t f)(x) = \mathbb{E}[f(X_t) | X_0 = x] \quad (t \geq 0, \ x \in S)$$

defines a Markov semigroup on $L^\infty(S)$.

Definition

A Markov semigroup on $L^\infty(S)$ is a family $(T_t)_{t \geq 0}$ such that

1. $T_t : L^\infty(S) \to L^\infty(S)$ is a linear operator for all $t \geq 0$,
2. $T_s \circ T_t = T_{s+t}$ for all $s, t \geq 0$ and $T_0 = I$ (semigroup),
3. $\|T_t\| \leq 1$ for all $t \geq 0$ (contraction),
4. $T_t f \geq 0$ whenever $f \geq 0$, for all $t \geq 0$ (positive).

If $T_t 1 = 1$ for all $t \geq 0$ then $T$ is conservative.
**Feller semigroups**

**Definition**

Suppose the state space $S$ is a locally compact Hausdorff space. The Markov semigroup $T$ is *Feller* if

$$T_t(C_0(S)) \subseteq C_0(S) \quad (t \geq 0)$$

and

$$\|T_t f - f\|_{\infty} \to 0 \quad \text{as} \quad t \to 0 \quad (f \in C_0(S)).$$

Every sufficiently well-behaved time-homogeneous Markov process is Feller: Brownian motion, Poisson process, Lévy processes, . . .

**Theorem 1**

*If the state space $S$ is separable then every Feller semigroup gives rise to a time-homogeneous Markov process.*
Infinitesimal generators

**Definition**

Let \( T \) be a \( C_0 \) semigroup on a Banach space \( E \). Its *infinitesimal generator* is the linear operator \( \tau \) in \( E \) with domain

\[
\text{dom } \tau = \left\{ f \in E : \lim_{t \to 0} t^{-1}(T_t f - f) \text{ exists} \right\}
\]

and action

\[
\tau f = \lim_{t \to 0} t^{-1}(T_t f - f).
\]

The operator \( \tau \) is closed and densely defined.

If \( T \) comes from a Markov process \( X \) then

\[
\mathbb{E} \left[ f(X_{t+h}) - f(X_t) \mid X_t \right] = (T_h f - f)(X_t) = h(\tau f)(X_t) + o(h),
\]

so \( \tau \) describes the change in \( X \) over an infinitesimal time interval.
Theorem 2 (Lumer–Phillips)

A closed, densely defined operator $\tau$ in the Banach space $E$ generates a strongly continuous contraction semigroup on $E$ if and only if

$$\text{im}(\lambda I - \tau) = E \quad \text{for some } \lambda > 0$$

and (dissipativity)

$$\|(\lambda I - \tau)x\| \geq \lambda \|x\| \quad \text{for all } \lambda > 0 \text{ and } x \in \text{dom } \tau.$$
The Hille–Yosida–Ray theorem

**Definition**

Let $S$ be a locally compact Hausdorff space. A linear operator $\tau$ in $C_0(S)$ satisfies the **positive maximum principle** if whenever $f \in \text{dom } \tau$ and $s_0 \in S$ are such that $\sup_{s \in S} f(s) = f(s_0) \geq 0$ then $(\tau f)(s_0) \leq 0$.

**Theorem 3 (Hille–Yosida–Ray)**

A closed, densely defined operator $\tau$ in $C_0(S)$ is the generator of a Feller semigroup on $C_0(S)$ if and only if

- $\lambda I - \tau : \text{dom } \tau \to X$ has bounded inverse for all $\lambda > 0$ and
- $\tau$ satisfies the positive maximum principle.
Quantum Markov processes

Quantum Feller semigroups

Infinitesimal generators
Quantum Feller semigroups

Theorem 4

Every commutative $C^*$ algebra is isometrically isomorphic to $C_0(S)$, where $S$ is a locally compact Hausdorff space.

Definition

A quantum Feller semigroup on the $C^*$ algebra $\mathcal{A}$ is a family $(T_t)_{t \geq 0}$ such that

1. $T_t : \mathcal{A} \rightarrow \mathcal{A}$ is a linear operator for all $t \geq 0$,
2. $T_s \circ T_t = T_{s+t}$ for all $s, t \geq 0$ and $T_0 = I$,
3. $\|T_t x - x\| \rightarrow 0$ as $t \rightarrow 0$ for all $x \in \mathcal{A}$,
4. $\|T_t\| \leq 1$ for all $t \geq 0$,
5. $(T_t a_{ij}) \in M_n(\mathcal{A})_+$ whenever $(a_{ij}) \in M_n(\mathcal{A})_+$, for all $n \geq 1$ and $t \geq 0$.

If $\mathcal{A}$ is unital and $T_t 1 = 1$ for all $t \geq 0$ then $T$ is conservative.
For simplicity, henceforth we consider only unital $C^*$ algebras.

If $X : \Omega \to E$ is a classical $S$-valued random variable then

$$j_X : \mathcal{A} \to \mathcal{B}; \ f \mapsto f \circ X$$

is a unital $\ast$-homomorphism, where $\mathcal{A} = C_0(S)$ and $\mathcal{B} = L^\infty(\Omega, \mathcal{A}, \mathbb{P})$.

**Definition**

A *non-commutative random variable* is a unital $\ast$-homomorphism $j_\bullet$ between $C^*$ algebras.

A family $(j_t : \mathcal{A} \to \mathcal{B})_{t \geq 0}$ of non-commutative random variables is a *dilation* of the quantum Feller semigroup $T$ on $\mathcal{A}$ if there exists a conditional expectation $\mathbb{E} : \mathcal{B} \to \mathcal{A}$ such that $T_t = \mathbb{E} \circ j_t$ for all $t \geq 0$. 
Problem
Given a densely defined operator $\tau$ on the $C^*$ algebra $A$, show that $\tau$ is closable and its closure that generates a quantum Feller semigroup $T$.

- It can be very difficult to verify the conditions of the Lumer–Phillips theorem.
- Is there a non-commutative version of the positive maximum principle?

Strategy
Rather than construct the semigroup $T$ directly, we shall instead construct a quantum Markov process which is a dilation of $T$. 
Fock space

Let $\mathcal{F} := \Gamma_+(L^2(\mathbb{R}_+; k))$ be Boson Fock space over $L^2(\mathbb{R}_+; k)$, where $k$ is a fixed complex Hilbert space. Then $\mathcal{F}$ has dense subspace

$$\mathcal{E} := \text{lin}\{\varepsilon(f) : f \in L^2(\mathbb{R}_+; k)\}$$

spanned by exponential vectors. Recall that $\langle \varepsilon(f), \varepsilon(g) \rangle = \exp\langle f, g \rangle$.

The algebra

Let the unital $C^*$ algebra $\mathcal{A}$ be faithfully represented as bounded operators on the Hilbert space $h$, and let $\mathcal{A}_0 \subseteq \mathcal{A} \subseteq B(h)$ be a norm-dense $*$-subalgebra of $\mathcal{A}$ which contains 1.
**Quantum flows**

**Definition**

A family of linear operators \((X_t)_{t \geq 0}\) in \(h \otimes \mathcal{F}\) with domains including \(h \otimes \mathcal{E}\) is **adapted** if

\[
\langle u \varepsilon(f), X_t v \varepsilon(g) \rangle = \langle u \varepsilon(1_{[0,t]}f), X_t v \varepsilon(1_{[0,t]}g) \rangle \langle \varepsilon(1_{[t,\infty]}f), \varepsilon(1_{[t,\infty]}g) \rangle
\]

for all \(u, v \in h, f, g \in L^2(\mathbb{R}_+; k)\) and \(t \geq 0\).

An **adapted mapping process on** \(\mathcal{A}_0\) is a family of linear maps

\[
(j_t : \mathcal{A}_0 \to \mathcal{L}(h \otimes \mathcal{E}; h \otimes \mathcal{F}))_{t \geq 0}
\]

such that \((j_t(x))_{t \geq 0}\) is an adapted process for all \(x \in \mathcal{A}_0\).

A **quantum flow** is an adapted mapping process \(j\) on \(\mathcal{A}\) composed of unital *-homomorphisms.
Feller cocycles

Definition

For a quantum flow $j$, let

$$\langle u, j_t[f, g](a)v \rangle = \langle u\varepsilon(1_{[0,t]}f), j_t(a)v \varepsilon(1_{[0,t]}g) \rangle \quad (u, v \in h).$$

Then $j$ is a Feller cocycle if

- $j_0[0,0](a) = a$,
- $j_t[f, g](a) \in A$,
- $t \mapsto j_t[f, g](a)$ is norm continuous,
- $j_{s+t}[f, g] = j_s[f, g] \circ j_t[f(\cdot + s), g(\cdot + s)]$

for all $s, t \geq 0, f, g \in L^2(\mathbb{R}_+; k)$ and $a \in A$.

Theorem 5

If the quantum flow $j$ is a Feller cocycle then setting $T_t : a \mapsto j_t[0,0](a)$ gives a quantum Feller semigroup $T$ on $A$. 
A quantum stochastic differential equation

The generator

Let \( \phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B} \) be a linear map, where \( \mathcal{B} = \mathcal{B}(\hat{k}) \) and \( \hat{k} := \mathbb{C} \oplus k \).

Distinguish the unit vector \( \omega := (1, 0) \in \hat{k} \).

For all \( z, w \in \hat{k} \), let \( \phi^z_w : \mathcal{A}_0 \to \mathcal{A}_0 \) be the linear map such that

\[
\langle u, \phi^z_w(x)v \rangle = \langle u \otimes z, \phi(x)v \otimes w \rangle \quad (u, v \in h, x \in \mathcal{A}_0).
\]

Theorem 6

Suppose the quantum flow \( j \) is such that

\[
\langle u \varepsilon(f), (j_t(x) - x \otimes 1_F)\varepsilon(g) \rangle = \int_0^t \langle u \varepsilon(f), j_s \left( \frac{f(s)}{g(s)}(x) \right) \varepsilon(g) \rangle \, ds 
\]

for all \( x \in \mathcal{A}_0 \) and all \( t \geq 0 \), \( u, v \in h \) and \( f, g \in L^2(\mathbb{R}^+; k) \). Then \( j \) is a Feller cocycle and the generator of the corresponding quantum Feller semigroup is an extension of \( \phi^\omega \)
If the quantum flow $j$ satisfies (1) on $A_0$ then the map $\phi : A_0 \to A_0 \otimes B$ is such that

- $\phi$ is $\ast$-linear,
- $\phi(1) = 0$ and
- $\phi(xy) = \phi(x)(y \otimes 1_k) + (x \otimes 1_k)\phi(y) + \phi(x)\Delta \phi(y)$ for all $x, y \in A_0$,

where

$$\Delta := 1_h \otimes P_k = \begin{bmatrix} 0 & 0 \\ 0 & 1_h \otimes k \end{bmatrix} \in A_0 \otimes B(\hat{k})$$

and $P_k := |\omega\rangle\langle \omega|^\perp \in B(\hat{k})$ is the orthogonal projection onto $k \subset \hat{k}$.

Question

When are these necessary conditions sufficient for the solution of (1) to be a quantum flow?
Lemma 8

The map \( \phi : A_0 \to A_0 \otimes B \) is a flow generator if and only if

\[
\phi(x) = \begin{bmatrix}
\tau(x) & \delta^\dagger(x) \\
\delta(x) & \pi(x) - x \otimes 1_k
\end{bmatrix}
\]

for all \( x \in A_0 \), (2)

where

- \( \pi : A_0 \to A_0 \otimes B(k) \) is a unital \(*\)-homomorphism,
- \( \delta : A_0 \to A_0 \otimes B(C; k) \) is a \( \pi \)-derivation, i.e., a linear map such that

  \[
  \delta(xy) = \delta(x)y + \pi(x)\delta(y) \quad (x, y \in A_0)
  \]

- \( \delta^\dagger : A_0 \to A_0 \otimes B(k; C) \) is such that \( \delta^\dagger(x) = \delta(x^\ast)^\ast \) for all \( x \in A_0 \),
- \( \tau : A_0 \to A_0 \) is \(*\)-linear and such that

  \[
  \tau(xy) - \tau(x)y - x\tau(y) = \delta^\dagger(x)\delta(y) \quad (x, y \in A_0).
  \]
For all \( n \in \mathbb{N} \) and \( T \in \mathcal{B}(h \otimes \bar{k} \otimes \hat{n}) \) there exists a family \( (\Lambda_t^n(T))_{t \geq 0} \) of linear operators in \( h \otimes \bar{F} \), with domains including \( h \otimes \mathcal{E} \), that is adapted and such that

\[
\langle u \varepsilon(f), \Lambda_t^n(T)v \varepsilon(g) \rangle = \int_{D_n(t)} \langle u \otimes \hat{f} \otimes^n(t), Tv \otimes \hat{g} \otimes^n(t) \rangle \, dt \langle \varepsilon(f), \varepsilon(g) \rangle
\]

for all \( u, v \in h, f, g \in L^2(\mathbb{R}_+; k) \) and \( t \geq 0 \), where the simplex

\[
D_n(t) := \{ t := (t_1, \ldots, t_n) \in [0, t]^n : t_1 < \cdots < t_n \}
\]

and

\[
\hat{f} \otimes^n(t) := \hat{f}(t_1) \otimes \cdots \otimes \hat{f}(t_n) \quad \text{et cetera.}
\]

We include \( n = 0 \) by setting \( \Lambda_t^0(T) := T \otimes 1_F \) for all \( t \geq 0 \).
An estimate for quantum Wiener integrals

Proposition 10

If \( n \in \mathbb{Z}_+ \), \( t \geq 0 \), \( T \in \mathcal{B}(h \otimes \hat{k} \otimes^n) \) and \( f \in L^2(\mathbb{R}_+; k) \) then

\[
\| \Lambda^n_T u_{\varepsilon}(f) \| \leq \frac{K_{f,t}^n}{\sqrt{n!}} \| T \| \| u_{\varepsilon}(f) \|,
\]

where \( K_{f,t} := \sqrt{(2 + 4\|f\|^2)(t + \|f\|^2)} \).
Definition

Given a flow generator \( \phi \), the family of linear maps

\[
\phi_n : A_0 \rightarrow A_0 \otimes B^\otimes n
\]

is defined by setting

\[
\phi_0 := \iota_{A_0} \quad \text{and} \quad \phi_{n+1} := (\phi_n \otimes \iota_B) \circ \phi \quad \text{for all } n \in \mathbb{Z}_+,
\]

where \( \iota_{A_0} \) is the identity map on \( A_0 \) and similarly for \( \iota_B \).

Let

\[
\mathcal{A}_\phi := \{ x \in A_0 : \exists C_x, M_x > 0 \text{ with } \| \phi_n(x) \| \leq C_x M_x^n \forall n \in \mathbb{Z}_+ \}
\]

be those elements of \( A_0 \) for which \( (\phi_n(x))_{n \in \mathbb{Z}_+} \) has polynomial growth.
Solutions to the QSDE

Theorem 11

If \( x \in \mathcal{A}_\phi \) then the series

\[
j_t(x) := \sum_{n=0}^{\infty} \Lambda_t^n(\phi_n(x))
\]

is strongly absolutely convergent on \( h \otimes \mathcal{E} \) for all \( t \geq 0 \). The family of operators \((j_t(x))_{t \geq 0}\) is an adapted mapping process on \( \mathcal{A}_\phi \) which satisfies the QSDE (1) on \( \mathcal{A}_\phi \).

Questions

1. When does \( \mathcal{A}_\phi = \mathcal{A}_0 \)?
2. When does the adapted mapping process \( j \) given by Theorem 11 extend to a quantum flow?
If $\mathcal{A}_\phi = \mathcal{A}_0$ then the adapted mapping process $j$ given by Theorem 11 extends to a quantum flow as long as $\mathcal{A}$ is sufficiently well behaved.

In practice it will be difficult to find directly constants $C_x$ and $M_x$ such that $\|\phi_n(x)\| \leq C_x M_x^n$ for all $x \in \mathcal{A}_0$.

Key to both: the higher-order Itô formula.

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**Two cases when $\mathcal{A}_\phi = \mathcal{A}_0$**

If $k$ is finite dimensional and $\phi$ is bounded then so is each $\phi_n$, with

$$\|\phi_n\| \leq \left(\dim \hat{k}\right)^{n-1} \|\phi\|^n \quad (n \in \mathbb{N}).$$

If $\phi$ is completely bounded then so is each $\phi_n$, with

$$\|\phi_n\| \leq \|\phi\|_{cb}^n \quad (n \in \mathbb{Z}_+).$$
The higher-order Itô formula

Notation

Let $\alpha \subseteq \{1, \ldots, n\}$, with elements arranged in increasing order and cardinality $|\alpha|$. The unital $\ast$-homomorphism

$$\mathcal{A}_0 \otimes \mathcal{B}^{\otimes |\alpha|} \rightarrow \mathcal{A}_0 \otimes \mathcal{B}^{\otimes n}; \quad T \mapsto T(n, \alpha)$$

is defined by linear extension of the map

$$A \otimes B_1 \otimes \cdots \otimes B_{|\alpha|} \mapsto A \otimes C_1 \otimes \cdots \otimes C_n,$$

where

$$C_i := \begin{cases} B_j & \text{if } i \text{ is the } j\text{th element of } \alpha, \\ 1_k \hat{} & \text{if } i \text{ is not an element of } \alpha. \end{cases}$$

Example

$$(A \otimes B_1 \otimes B_2 \otimes B_3)(5, \{1, 3, 4\}) = A \otimes B_1 \otimes 1_k \hat{} \otimes B_2 \otimes B_3 \otimes 1_k.$$
More notation

For all \( n \in \mathbb{Z}_+ \) and \( \alpha \subseteq \{1, \ldots, n\} \), let

\[
\phi|_{\alpha}|(x; n, \alpha) := (\phi|_{\alpha}|(x))(n, \alpha) \quad \text{for all } x \in A_0.
\]

Also let

\[
\Delta(n, \alpha) := (1_h \otimes P_k^\otimes|\alpha|)(n, \alpha),
\]

so that \( P_k \) acts on the components of \( \hat{k} \otimes n \) which have indices in \( \alpha \) and \( 1_{\hat{k}} \) acts on the others.

Theorem 12

Let \( \phi \) be a flow generator. For all \( n \in \mathbb{Z}_+ \) and \( x, y \in A_0 \),

\[
\phi_n(xy) = \sum_{\alpha \cup \beta = \{1, \ldots, n\}} \phi|_{\alpha}|(x; n, \alpha) \Delta(n, \alpha \cap \beta) \phi|_{\beta}|(y; n, \beta),
\]

where the summation is taken over all \( \alpha \) and \( \beta \) with union \( \{1, \ldots n\} \).
Two corollaries

Corollary 13

If $\phi : A_0 \to A_0 \otimes B$ is a flow generator then $A_\phi$ is a unital $\ast$-subalgebra of $A_0$, which is equal to $A_0$ if $A_\phi$ contains a $\ast$-generating set for $A_0$.

Corollary 14

Let $\phi : A_0 \to A_0 \otimes B$ be a flow generator and let $j$ be the adapted mapping process on $A_\phi$ given by Theorem 11. If $x, y \in A_\phi$ then $x^* y \in A_\phi$, with

$$\langle j_t(x) u \varepsilon(f), j_t(y) v \varepsilon(g) \rangle = \langle u \varepsilon(f), j_t(x^* y) v \varepsilon(g) \rangle$$

for all $u, v \in h$ and $f, g \in L^2(\mathbb{R}_+; k)$. 
The first dilation theorem

**Theorem 15**

Let $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ be a flow generator and suppose $\mathcal{A}_0$ contains its square roots: for all non-negative $x \in \mathcal{A}_0$, the square root $x^{1/2}$ lies in $\mathcal{A}_0$.

If $\mathcal{A}_\phi = \mathcal{A}_0$ then there exists a quantum flow $\bar{j}$ such that

$$\bar{j}_t(x) = j_t(x) \text{ on } h \otimes \mathcal{E} \quad (x \in \mathcal{A}_0),$$

where $j$ is the adapted mapping process given by Theorem 11.

**Remarks**

- If $\mathcal{A} \cong C(S)$ for some compact Hausdorff space $S$ then $\mathcal{A}$ contains its square roots.

- If $\mathcal{A}$ is an AF algebra, i.e., the norm closure of an increasing sequence of finite-dimensional $*$-subalgebras, then its local algebra $\mathcal{A}_0$, the union of these subalgebras, contains its square roots.
The second dilation theorem

Theorem 16

Let $\mathcal{A}$ be the universal $C^*$ algebra generated by isometries $\{s_i : i \in I\}$, and let $\mathcal{A}_0$ be the $\ast$-algebra generated by $\{s_i : i \in I\}$.

If $\phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B}$ is a flow generator such that $\mathcal{A}_0 = \mathcal{A}_0$ then there exists a quantum flow $\overline{j}$ such that

$$\overline{j}_t(x) = j_t(x) \quad \text{on } \mathcal{h} \otimes \mathcal{E} \quad (x \in \mathcal{A}_0),$$

where $j$ is the adapted mapping process given by Theorem 11.
The group algebra

Let $G$ be a discrete group and set $\mathcal{A} = C_0(G) \oplus \mathbb{C}1 \subseteq B(\ell^2(G))$, where $x \in C_0(G)$ acts on $\ell^2(G)$ by multiplication.

Let $\mathcal{A}_0 = \text{span}\{1, e_g : g \in G\}$, where $e_g(h) := 1_{g=h}$ for all $h \in G$.

Permitted moves

Let $H$ be a non-empty finite subset of $G \setminus \{e\}$ and let the Hilbert space $k$ have orthonormal basis $\{f_h : h \in H\}$; the maps

$$\lambda_h : G \to G; \ g \mapsto hg \quad (h \in H)$$

correspond to the permitted moves in the random walk to be constructed on $G$. 
Random walks on discrete groups

One-step matrices

For all \( g \in G \) and \( h \in H \), let

\[
m_e(g) := \begin{bmatrix}
-\sum_{h \in H} |t_h(g)|^2 & -\sum_{h \in H} \overline{t_h(g)} \langle f_h \rangle \\
-\sum_{h \in H} t_h(g) |f_h\rangle & -1_k
\end{bmatrix}
\]

and

\[
m_h(g) := \begin{bmatrix}
|t_h(g)|^2 & \overline{t_h(g)} \langle f_h \rangle \\
t_h(g) |f_h\rangle & |f_h\rangle \langle f_h |$
\].

Then

\[
\|m_e(g)\| = 1 + \sum_{h \in H} |t_h(g)|^2
\]

and

\[
\|m_h(g)\| = 1 + |t_h(g)|^2.
\]
Lemma 17

Given a transition function

\[ t : H \times G \to \mathbb{C}; \ (h, g) \mapsto t_h(g), \]

the map \( \phi : \mathcal{A}_0 \to \mathcal{A}_0 \otimes \mathcal{B} \) such that

\[
\begin{align*}
\sum_{h \in H} |t_h|^2 (x \circ \lambda_h - x) & \quad \sum_{h \in H} \overline{t_h} (x \circ \lambda_h - x) \otimes \langle f_h | \\
\sum_{h \in H} t_h (x \circ \lambda_h - x) \otimes |f_h\rangle & \quad \sum_{h \in H} (x \circ \lambda_h - x) \otimes |f_h\rangle \langle f_h |
\end{align*}
\]

is a flow generator with \( \phi_n(e_g) \) equal to

\[
\sum_{h_1 \in H \cup \{e\}} \cdots \sum_{h_n \in H \cup \{e\}} e_{h_1^{-1} \cdots h_n^{-1} g} \otimes m_{h_n}(h_n^{-1} \cdots h_1^{-1} g) \otimes \cdots \otimes m_{h_1}(h_1^{-1} g)
\]

for all \( n \in \mathbb{N} \) and \( g \in G \).
A sufficient condition for $\mathcal{A}_\phi = \mathcal{A}_0$

If

$$M_g := \lim_{n \to \infty} \sup \{ |t_h(h_n^{-1} \cdots h_1^{-1}g)| : h_1, \ldots, h_n \in H \cup \{e\}, h \in H \} < \infty$$

(3)

then

$$\|\phi_n(e_g)\| \leq (1 + |H| + 2|H| M_g^2)^n \quad (n \in \mathbb{Z}_+),$$

where $|H|$ denotes the cardinality of $H$.

Hence $\mathcal{A}_\phi = \mathcal{A}_0$ if (3) holds for all $g \in G$. 
Examples

1. If $t$ is bounded then (3) holds for all $g \in G$.

2. If $G = (\mathbb{Z}, +)$, $H = \{\pm 1\}$ and the transition function $t$ is bounded, with $t_{+1}(g) = 0$ for all $g < 0$ and $t_{-1}(g) = 0$ for all $g \leq 0$, then the Markov semigroup $T$ which arises corresponds to the classical birth-death process with birth and death rates $|t_{+1}|^2$ and $|t_{-1}|^2$, respectively.

3. If $G = (\mathbb{Z}, +)$, $H = \{+1\}$ and $t_{+1}: g \mapsto 2^g$ then $M_g = 2^g$ and the condition (3) holds for all $g \in G$. Thus the construction applies to examples where the transition function $t$ is unbounded.
The CAR algebra

For a non-empty set $I$, the **CAR algebra** is the unital $C^*$ algebra $\mathcal{A}$ with generators $\{b_i : i \in I\}$, subject to the anti-commutation relations

$$\{b_i, b_j\} = 0 \quad \text{and} \quad \{b_i, b_j^*\} = 1_{i=j} \quad (i, j \in I).$$

Let $\mathcal{A}_0$ be the unital algebra generated by $\{b_i, b_i^*: i \in I\}$.

**Lemma 18**

*For each $x \in \mathcal{A}_0$ there exists a finite subset $J \subseteq I$ such that $x$ lies in the finite-dimensional $*$-subalgebra*

$$\mathcal{A}_J := \text{lin}\{b_{j_1}^* \cdots b_{j_q}^* b_{i_1} \cdots b_{i_p} : \text{distinct } i_1, \ldots, i_p \in J, \ j_1, \ldots, j_q \in J\}.$$ 

*Consequently, $\mathcal{A}$ is an AF algebra and $\mathcal{A}_0$ contains its square roots.*
Let \{\alpha_{i,j} : i, j \in I\} \subseteq \mathbb{C} be a fixed collection of amplitudes, so that (I, \{\alpha_{i,j}\}) is a complex digraph. For all \(i \in I\), let
\[
supp(i) := \{j \in I : \alpha_{i,j} \neq 0\} \quad \text{and} \quad supp^+(i) := supp(i) \cup \{i\}.
\]
Thus \(supp(i)\) is the set of sites with which site \(i\) interacts and \(|supp(i)|\) is the valency of the vertex \(i\). Suppose
\[
|supp(i)| < \infty \quad (i \in I).
\]
The transport of a particle from site \(i\) to site \(j\) with amplitude \(\alpha_{i,j}\) is described by the operator
\[
t_{i,j} := \alpha_{i,j} b_j^* b_i.
\]
The symmetric quantum exclusion process

Energies

Let \( \{ \eta_i : i \in I \} \subseteq \mathbb{R} \) be fixed. The total energy in the system is given by

\[
h := \sum_{i \in I} \eta_i \ b_i^* b_i,
\]

where \( \eta_i \) gives the energy of a particle at site \( i \).

Lemma 19

Setting

\[
\tau(x) := i \sum_{i \in I} \eta_i [b_i^* b_i, x] - \frac{1}{2} \sum_{i,j \in I} \tau_{i,j}(x)
\]

defines a \(*\)-linear map \( \tau : A_0 \to A_0 \), where

\[
\tau_{i,j}(x) := t_{i,j}^* [t_{i,j}, x] + [x, t_{i,j}^*] t_{i,j}.
\]
Lemma 20

Let $k$ be a Hilbert space with orthonormal basis $\{f_{i,j} : i, j \in I\}$. Setting

$$\delta(x) := \sum_{i,j \in I} [t_{i,j}, x] \otimes |f_{i,j}\rangle \quad (x \in A_0),$$

where

$$|f_{i,j}\rangle : \mathbb{C} \mapsto k; \quad \lambda \mapsto \lambda f_{i,j},$$

defines a linear map $\delta : A_0 \to A_0 \otimes B(\mathbb{C}; k)$ such that

$$\delta(xy) = \delta(x)y + (x \otimes 1_k)\delta(y) \quad \text{and} \quad \delta^\dagger(x)\delta(y) = \tau(xy) - \tau(x)y - x\tau(y) \quad (x, y \in A_0),$$

with $\tau$ defined as in Lemma 19. Hence

$$\phi : A_0 \to A_0 \otimes B; \quad x \mapsto \begin{bmatrix} \tau(x) & \delta^\dagger(x) \\ \delta(x) & 0 \end{bmatrix}$$

is a flow generator.
Lemma 21

If the amplitudes satisfy the symmetry condition

\[ |\alpha_{i,j}| = |\alpha_{j,i}| \quad \text{for all } i, j \in I \]  

(4)

then

\[ \phi_n(b_{i_0}) = \sum_{i_1 \in \text{supp}^+(i_0)} \cdots \sum_{i_n \in \text{supp}^+(i_{n-1})} b_{i_n} \otimes B_{i_{n-1},i_n} \otimes \cdots \otimes B_{i_0,i_1} \]

for all \( n \in \mathbb{N} \) and \( i_0 \in I \), where

\[ B_{i,j} := |\omega\rangle \langle \omega| + |\omega\rangle \langle \alpha_{i,j} f_{i,j}| - |\alpha_{j,i} f_{j,i}\rangle \langle \omega| \quad (i, j \in I) \]

and

\[ \lambda_i := -i\eta_i - \frac{1}{2} \sum_{j \in \text{supp}(i)} |\alpha_{j,i}|^2 \quad (i \in I). \]
Example

Suppose that the amplitudes satisfy the symmetry condition (4), and further that there are uniform bounds on the amplitudes, valencies and energies, so that

\[ M := \sup_{i,j \in I} |\alpha_{i,j}|, \quad V := \sup_{i \in I} |\text{supp}(i)| \quad \text{and} \quad H := \sup_{i \in I} |\eta_i| \]

are all finite. Then

\[ |\lambda_i| \leq |\eta_i| + \frac{1}{2} VM^2 \quad \text{and} \quad \|B_{i,j}\| \leq |\lambda_i| + 2M \leq H + \frac{1}{2} VM^2 + 2M \]

for all \( i, j \in I \). Hence

\[ \|\phi_n(b_i)\| \leq (V + 1)^n (H + \frac{1}{2} VM^2 + 2M)^n \quad (n \in \mathbb{Z}_+, \ i \in I) \]

and \( A_{\phi} = A_0 \).
The universal rotation algebra

**Definition**

Let \( \mathcal{A} \) be the *universal rotation algebra*: this is the universal \( C^* \) algebra with unitary generators \( U, V \) and \( Z \) satisfying the relations

\[
UV = ZVU, \quad UZ = ZU \quad \text{and} \quad VZ = ZV.
\]

It is the group \( C^* \) algebra corresponding to the discrete Heisenberg group

\[
\Gamma := \langle u, v, z \mid uv = zvu, \, uz = zu, \, vz = zv \rangle.
\]

**A pair of derivations**

Letting \( \mathcal{A}_0 \) denote the \(*\)-subalgebra generated by \( U, V \) and \( Z \), there are skew-adjoint derivations

\[
\delta_1 : \mathcal{A}_0 \to \mathcal{A}_0; \quad U^m V^n Z^p \mapsto mU^m V^n Z^p
\]

and

\[
\delta_2 : \mathcal{A}_0 \to \mathcal{A}_0; \quad U^m V^n Z^p \mapsto nU^m V^n Z^p.
\]
Theorem 22

Fix \( c_1, c_2 \in \mathbb{C} \), let \( \delta = c_1 \delta_1 + c_2 \delta_2 \) and define the Bellissard map

\[
\tau : A_0 \to A_0; \\
U^m V^n Z^p \mapsto - \left( \frac{1}{2} |c_1|^2 m^2 + \frac{1}{2} |c_2|^2 n^2 + \overline{c_1} c_2 m n + (\overline{c_1} c_2 - c_1 \overline{c_2}) p \right) U^m V^n Z^p.
\]

Then

\[
\phi : A_0 \to A_0 \otimes B(\mathbb{C}^2); \ x \mapsto \begin{bmatrix} \tau(x) & \delta^\dagger(x) \\ \delta(x) & 0 \end{bmatrix}
\]

is a flow generator. Furthermore, \( U, V, Z \in A_\phi \) and \( A_\phi = A_0 \).
The non-commutative torus

**Definition**

Let $A$ be the non-commutative torus with parameter $\lambda \in \mathbb{T}$, so that $A$ is the universal $C^*$ algebra with unitary generators $U$ and $V$ subject to the relation

$$UV = \lambda VU,$$

and let

$$A_0 := \langle U, V \rangle = \text{lin}\{U^mV^n : m, n \in \mathbb{Z}\}.$$

**An automorphism**

For each $(\mu, \nu) \in \mathbb{T}^2$, let $\pi_{\mu,\nu}$ be the automorphism of $A$ such that

$$\pi_{\mu,\nu}(U^mV^n) = \mu^m\nu^nU^mV^n \quad \text{for all } m, n \in \mathbb{Z}.$$
Fix \((\mu, \nu) \in \mathbb{T}^2\) with \(\mu \neq 1\). There exists a flow generator
\[
\phi : A_0 \to A_0 \otimes \mathcal{B}(\mathbb{C}^2); \quad x \mapsto \begin{bmatrix}
\tau(x) & -\mu \delta(x) \\
\delta(x) & \pi_{\mu, \nu}(x) - x
\end{bmatrix},
\]
where the \(\pi_{\mu, \nu}\)-derivation
\[
\delta : A_0 \to A_0; \quad U^m V^n \mapsto \frac{1 - \mu^m \nu^n}{1 - \mu} U^m V^n
\]
is such that \(\delta^\dagger = -\mu \delta\) and the map
\[
\tau := \frac{\mu}{1 - \mu} \delta : A_0 \to A_0; \quad U^m V^n \mapsto \frac{\mu(1 - \mu^m \nu^n)}{(1 - \mu)^2} U^m V^n.
\]
Furthermore, \(U, V \in A_\phi\) and so \(A_\phi = A_0\).
The full story