Haagerup property for von Neumann algebras – old and new

based on joint work with M. Caspers,
building on earlier work with M. Daws, P. Fima and S. White
related to the work of R. Okayasu and R. Tomatsu

Adam Skalski

IMPAN and University of Warsaw

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Equivalent definitions of the Haagerup property

A discrete group G has the Haagerup property (HAP) if the following equivalent properties hold:

- there exists a normalised sequence of positive definite functions on G vanishing at infinity pointwise convergent to 1;
- G admits a mixing unitary representation which weakly contains the trivial representation;
- there exists a real, proper, conditionally negative definite function on G;
- G admits a proper affine action on a real Hilbert space.

A function $\varphi : G \to \mathbb{C}$ is called positive definite if $\phi(e) = 1$ and for all $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$

$$\sum_{i,j=1}^{n} \varphi(g_i^{-1} g_j)\overline{\lambda_i}\lambda_j \geq 0.$$

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Basic properties and examples

- amenable groups have HAP;
- G has both HAP and property (T) if and only if G is compact;
- free groups, finitely generated Coxeter groups have HAP;
- $SL(2, \mathbb{Z})$ has HAP.

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Haagerup approximation property for finite von Neumann algebras

A vNa M with a faithful normal tracial state τ has the **von Neumann algebraic Haagerup approximation property** if there exists a net of completely positive, τ-reducing, normal maps $(\Phi_i)_{i\in\mathcal{I}}$ on M such that the GNS-induced maps T_i on $L^2(M,\tau)$ are compact and the net $(T_i)_{i\in\mathcal{I}}$ converges to $I_{L^2(M,\tau)}$ strongly.

$L^2(M,\tau)$ – the GNS Hilbert space of the pair (M,τ)

$$T_i(x\Omega_\tau) = \Phi_i(x)\Omega_\tau, \quad x \in \mathrm{M}.$$

P.Jolissaint showed that this property does not depend on the choice of τ – so the vNa HAP is a property of a (finite) von Neumann algebra – and that the maps Φ_i can be chosen Markov – i.e. unital and trace preserving.

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Classical HAP via the approximation property for the von Neumann algebra

Theorem (M. Choda)

A discrete group G has HAP if and only if $\mathrm{VN}(G)$ has the von Neumann algebraic Haagerup approximation property.

Proof.

If G has HAP, we use 'good' positive definite functions φ_i, to construct ucp Herz-Schur multipliers on $\mathrm{VN}(G)$, which are L^2-compact and converge to identity pointwise σ-weakly:

$$\lambda_g \mapsto \varphi_i(g)\lambda_g$$

Converse: we 'average' approximating maps into multipliers $\mathrm{VN}(G)$: defining

$$\varphi_i(g) = \tau(\Phi_i(\lambda_{g^{-1}})\lambda_g)$$

yields 'good' positive definite functions. □

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Discrete quantum groups – general notations

$\mathbb{G}$ – a discrete quantum group à la Woronowicz, i.e.

$\ell^\infty(\mathbb{G})$ – a von Neumann algebra, which is of the form $\prod_{i\in\mathcal{I}} M_{n_i}$, equipped with the coproduct

$$\Delta : \ell^\infty(\mathbb{G}) \to \ell^\infty(\mathbb{G})\overline{\otimes}\ell^\infty(\mathbb{G})$$

carrying all the information about $\mathbb{G}$

$c_0(\mathbb{G}) = \bigoplus_{i\in\mathcal{I}} M_{n_i}$ – the corresponding C^*-object

$\mathbb{G}$ has *right and left invariant Haar weights* – if they coincide we say $\mathbb{G}$ is **unimodular**

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Dual (compact) quantum groups

Each discrete quantum group $\mathbb{G}$ admits the dual compact quantum group $\widehat{\mathbb{G}}$.

$L^\infty(\widehat{\mathbb{G}})$, $\mathrm{C}(\widehat{\mathbb{G}})$ – subalgebras of $B(\ell^2(\mathbb{G}))$
$\mathrm{C}^u(\widehat{\mathbb{G}})$ – a 'universal' version of $\mathrm{C}(\widehat{\mathbb{G}})$
$\widehat{\mathbb{G}}$ admits a Haar state h – tracial if and only if $\mathbb{G}$ unimodular

$\ell^2(\mathbb{G}) \approx L^2(\widehat{\mathbb{G}})$ – the GNS space of h

$\mathrm{C}^u(\widehat{\mathbb{G}})$ contains a natural dense Hopf *-algebra, $\mathrm{Pol}(\widehat{\mathbb{G}})$

In particular for G – discrete group

$$L^\infty(\widehat{G}) = \mathrm{VN}(G)$$

$$\mathrm{C}(\widehat{G}) = C^*_r(G), \quad \mathrm{C}^u(\widehat{\mathbb{G}}) = \mathrm{C}^*(G),, \quad \mathrm{Pol}(\widehat{\mathbb{G}}) = \mathbb{C}[G]$$

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Positive definite functions in the quantum world

What should positive definite functions on $\mathbb{G}$ be? There are at least two possible points of view:

- via Bochner's theorem, states on $C^u(\widehat{\mathbb{G}})$ (i.e. states on '$C^*(\mathbb{G})$');
- elements in $\ell^\infty(\mathbb{G})$ yielding 'ucp Herz-Schur multipliers' on $L^\infty(\widehat{\mathbb{G}})$ (i.e. multipliers on '$VN(\mathbb{G})$').

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Haagerup property for discrete quantum groups

Theorem (M.Daws, P.Fima, S.White, AS)

Let $\mathbb{G}$ be a discrete quantum group. The following conditions are equivalent (and can be used as the definition of HAP):

- $c_0(\mathbb{G})$ admits an approximate unit built of 'positive definite functions';
- $\mathbb{G}$ admits a mixing representation weakly containing the trivial representation;
- $\hat{\mathbb{G}}$ admits a symmetric proper generating functional ('real proper cond.neg.def. function');
- $\mathbb{G}$ admits a real proper cocycle ('affine part of an action on a real Hilbert space').

Quantum group HAP via the approximation property for the vNa

Recall that if $\mathbb{G}$ is unimodular, then the Haar state of $\widehat{\mathbb{G}}$ is a trace (in particular, $L^\infty(\widehat{\mathbb{G}})$ is a finite von Neumann algebra).

Theorem

Let $\mathbb{G}$ be a discrete unimodular quantum group. Then $\mathbb{G}$ has HAP if and only if $L^\infty(\widehat{\mathbb{G}})$ has the von Neumann algebraic Haagerup approximation property.

Proof.

Follows the classical idea of Choda: if $\mathbb{G}$ has HAP, we have good positive definite functions, so constructing multipliers out of them (see M.Junge + M.Neufang + Z.J.Ruan, later also M.Daws) yields the approximation property for $L^\infty(\hat{\mathbb{G}})$ (this does not use the unimodularity).
The other direction is based on 'averaging' approximating maps on $L^\infty(\hat{\mathbb{G}})$ into Schur multipliers. Here unimodularity seems crucial. □

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The other direction is based on 'averaging' approximating maps on $L^\infty(\hat{\mathbb{G}})$ into Schur multipliers. Here unimodularity seems crucial.

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Quantum group HAP via the approximation property for the vNa revisited

What if a discrete quantum group $\mathbb{G}$ is not unimodular?
The Haar state h on $L^\infty(\widehat{\mathbb{G}})$ is no longer tracial. But...

Theorem

Let $\mathbb{G}$ be a discrete quantum group with the Haagerup property. Let $M = L^\infty(\widehat{\mathbb{G}})$. There exists a net of normal completely positive, unital, h-preserving maps $(\Phi_i)_{i\in\mathcal{I}}$ on M such that each of the respective GNS-induced maps T_i on $L^2(\widehat{\mathbb{G}}) \approx L^2(M,h)$ is compact and the net $(T_i)_{i\in\mathcal{I}}$ converges to $I_{L^2(M,h)}$ strongly. Moreover one can choose Φ_i commuting with the action of the modular group.

von Neumann algebraic HAP for arbitrary vNa

Definition (DFWS)

Let (M, φ) be a von Neumann algebra with a faithful normal state. We say that (M, φ) has the Haagerup property if there exists a net of normal completely positive, unital, φ-preserving maps $(\Phi_i)_{i\in\mathcal{I}}$ on M such that the GNS-induced maps T_i on $L^2(M, \varphi)$ are compact and the net $(T_i)_{i\in\mathcal{I}}$ converges to $I_{L^2(M,\varphi)}$ strongly.

von Neumann algebraic HAP for arbitrary vNa – take II

Definition (M.Caspers + AS)

Let (M, φ) be a von Neumann algebra with a faithful normal state. We say that (M, φ) has the Haagerup property if there exists a net of normal completely positive, φ-reducing maps $(\Phi_i)_{i\in\mathcal{I}}$ on M such that the GNS-induced maps T_i on $L^2(M, \varphi)$ are compact and the net $(T_i)_{i\in\mathcal{I}}$ converges to $I_{L^2(M,\varphi)}$ strongly.

von Neumann algebraic HAP for arbitrary vNa – take III

Definition (M.Caspers + AS)

Let (M, φ) be a von Neumann algebra with a faithful normal semifinite weight. We say that (M, φ) has the Haagerup property if there exists a net of normal completely positive, φ-reducing maps $(\Phi_i)_{i\in\mathcal{I}}$ on M such that the GNS-induced maps T_i on $L^2(M, \varphi)$ are compact and the net $(T_i)_{i\in\mathcal{I}}$ converges to $I_{L^2(M,\varphi)}$ strongly.

Immediate questions

- does the property depend on the choice of φ?
- can one always get the approximating maps unital and φ-preserving (i.e. Markov)?
- are there any other possible choices?
- does it characterize the Haagerup property for discrete quantum groups?

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Non-immediate answers

- does the property depend on the choice of φ? — NO
- can one always get the approximating maps unital and φ-preserving (i.e. Markov)? — SOMETIMES
- are there any other possible choices? — OH, YES
- does it characterize the Haagerup property for discrete quantum groups? — WHO KNOWS?

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The magic of crossed product duality

Theorem (CS)

The Haagerup property does not depend on the choice of a faithful normal semifinite weight.

Idea of the proof:

- show that (M, φ) has HAP iff all its 'nice' corners have HAP;
- prove that one can change weights if the algebra is semifinite;
- show that in the semifinite case the approximating maps can be chosen contractive;
- prove that HAP is stable under passing to crossed products by (φ-preserving) actions of amenable groups;
- use the Takesaki-Takai duality for the crossed products by the modular action.

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Standard form approach

At the same time R.Okayasu and R.Tomatsu developed another approach to the Haagerup property based on the standard form of the algebra M (the approximating maps in their approach act directly on the Hilbert space).

Theorem (COST)

A vNa M has the Haagerup property in the sense of CS if and only if it has the Haagerup property in the sense of OT.

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Modularity

The approach of OT is related to considering the KMS–induced maps.

Definition

Let (M, φ) be a vNa with a faithful normal semifinite weight, $\Phi : M \to M$ a normal completely positive, φ-reducing map. Its *KMS*-implementation on $L^2(M, \varphi)$ is (informally!) given by the formula

$$T^{KMS}(\Omega_\varphi^{\frac{1}{2}} x \Omega_\varphi^{\frac{1}{2}}) = \Omega_\varphi^{\frac{1}{2}} \Phi(x) \Omega_\varphi^{\frac{1}{2}}$$

Once again the crossed product technique yields the fact that the Haagerup property is equivalent to the KMS Haagerup property:

Definition

(M, φ) has the KMS Haagerup property if there exists a net of normal completely positive, φ-reducing maps $(\Phi_i)_{i \in \mathcal{I}}$ on M such that the KMS-induced maps T_i^{KMS} on $L^2(M, \varphi)$ are compact and the net $(T_i^{KMS})_{i \in \mathcal{I}}$ converges to $I_{L^2(M,\varphi)}$ strongly.

The KMS– and GNS–induced maps coincide if the map in question commutes with the modular group. We do not know if one can always achieve it!

We say that Φ is KMS–symmetric if the induced map T^{KMS} is self-adjoint.

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$$T^{KMS}(\Omega_\varphi^{\frac{1}{2}} x \Omega_\varphi^{\frac{1}{2}}) = \Omega_\varphi^{\frac{1}{2}} \Phi(x) \Omega_\varphi^{\frac{1}{2}}$$

Once again the crossed product technique yields the fact that the Haagerup property is equivalent to the KMS Haagerup property:

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(M, φ) has the KMS Haagerup property if there exists a net of normal completely positive, φ-reducing maps $(\Phi_i)_{i \in \mathcal{I}}$ on M such that the KMS-induced maps T_i^{KMS} on $L^2(M, \varphi)$ are compact and the net $(T_i^{KMS})_{i \in \mathcal{I}}$ converges to $I_{L^2(M,\varphi)}$ strongly.

The KMS– and GNS–induced maps coincide if the map in question commutes with the modular group. We do not know if one can always achieve it!
We say that Φ is KMS–symmetric if the induced map T^{KMS} is self-adjoint.

Modularity

The approach of OT is related to considering the KMS–induced maps.

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Markov property

The next result is surprisingly rather technical.

Theorem (CS)

Suppose that M has the Haagerup property and φ is a faithful normal state on M. Then one can choose the approximating (in the KMS-sense) maps to be unital, φ-preserving and KMS-symmetric (i.e. their KMS-implementations are selfadjoint operators on $L^2(M, \varphi)$).

Corollary

Free product of vNas with faithful normal states which have the Haagerup property has the Haagerup property (allowing amalgamation over a finite-dimensional subalgebra).

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Cnd functions...

A function $\psi : G \to \mathbb{C}$ is called conditionally negative definite if $\psi(e) = 0$ and for all $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$

$$\sum_{i=1}^{n} \lambda_i = 0 \Longrightarrow \sum_{i,j=1}^{n} \psi(g_i^{-1} g_j)\overline{\lambda_i}\lambda_j \leq 0.$$

Schönberg correspondence says that the conditionally negative definite functions are 'generators of families of positive definite functions':

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… and convolution semigroups of states

States on $\mathrm{Pol}(\widehat{\mathbb{G}})$ are in one-to-one correspondence with the states on $\mathsf{C}^u(\widehat{\mathbb{G}})$.

Definition

A convolution semigroup of states on $\mathrm{Pol}(\widehat{\mathbb{G}})$ is a family $(\mu_t)_{t\geq 0}$ of states on $\mathrm{Pol}(\widehat{\mathbb{G}})$ such that

1. $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta_{\widehat{\mathbb{G}}}, \quad t, s \geq 0;$
2. $\mu_t(a) \stackrel{t\to 0^+}{\longrightarrow} \mu_0(a) := \epsilon(a), \quad a \in \mathrm{Pol}(\widehat{\mathbb{G}}).$

Conditionally negative definite functions versus generating functionals

The following theorem is essentially due to M. Schürmann.

Theorem (Quantum Schönberg correspondence)

Each convolution semigroup of states on $\mathrm{Pol}(\widehat{\mathbb{G}})$ possesses a generating functional $L : \mathrm{Pol}(\widehat{\mathbb{G}}) \to \mathbb{C}$:

$$L(a) = \lim_{t \to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in \mathrm{Pol}(\hat{\mathbb{G}}).$$

The functional L is selfadjoint, vanishes at 1 and is positive on the kernel of the counit; in turn each functional enjoying these properties generates a convolution semigroup of states.

Thus – conditionally negative functions on $\mathbb{G}$ correspond to **generating functionals** on $\mathrm{Pol}(\widehat{\mathbb{G}})$.

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HAP and convolution semigroups of states

Theorem (DFWS)

Let $\mathbb{G}$ be a discrete quantum group. The following are equivalent:

1. $\mathbb{G}$ has HAP;
2. there exists a convolution semigroup of states $(\mu_t)_{t\geq 0}$ on $\mathrm{C}^u(\widehat{\mathbb{G}})$ such that the associated 'functions' form an approximate identity in $c_0(\mathbb{G})$;
3. $\widehat{\mathbb{G}}$ admits a symmetric proper generating functional.

In addition the states in (ii) can be chosen invariant under both the scaling automorphism group and the unitary antipode R.

Corollary

If $\mathbb{G}$ is a discrete quantum group which has HAP, then $L^\infty(\widehat{\mathbb{G}})$ admits a semigroup of Markov maps, which are L^2-compact and converge pointwise σ-weakly to identity. They can also be chosen KMS-symmetric and commuting with the modular group (of h).

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vNa HAP via semigroups

(M, φ) – vNa with a faithful normal state

Definition

A Markov semigroup $\{\Phi_t : t \geq 0\}$ on (M, φ) is a semigroup of Markov maps on M such that for all $x \in M$ we have $\Phi_t(x) \xrightarrow{t \to 0^+} \Phi_0(x) = x$ σ-weakly. It is *KMS-symmetric* if each Φ_t is KMS symmetric, and *immediately L^2-compact* if each of the maps Φ_t^{KMS} with $t > 0$ is compact.

The next result was inspired by the theorem for finite von Neumann algebras due to P.Jolissaint and F.Martin.

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The next result characterises the Haagerup property for a von Neumann algebra in terms of objects playing the role of the generators of the approximating semigroup – **quantum Dirichlet forms**. In the discrete quantum group world they correspond to L, in the classical world to conditionally negative definite functions.

Theorem

The following are equivalent:

1. (M, φ) has the Haagerup property;
2. $L^2(M, \varphi)$ admits an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and a non-decreasing sequence of non-negative numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \lambda_n = +\infty$ and the prescription

$$Q(\xi) = \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi\rangle|^2, \quad \xi \in \mathrm{Dom}\, Q,$$

where $\mathrm{Dom}\, Q = \{\xi \in H_\varphi : \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi\rangle|^2 < \infty\}$, defines a conservative completely Dirichlet form.

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Open questions

- how to characterise the HAP for discrete non-unimodular $\mathbb{G}$ via the von Neumann algebra $L^\infty(\widehat{\mathbb{G}})$?
- is the modular vNa HAP equivalent to the usual HAP?
- can the approximating maps in HAP be chosen so that they are Markov even in the weight case?

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Invitation

Graduate School Topological quantum groups

Będlewo (Poland), 28th June – 11th July 2015

http://bcc.impan.pl/15TQG/

Speakers: Teodor Banica, Michael Brannan, Kenny De Commer, Matthew Daws, Sergey Neshveyev, Zhong–Jin Ruan, Roland Speicher, Reiji Tomatsu

Topics: Quantum groups and... Hadamard matrices, approximation properties, harmonic analysis, (ergodic) actions, categories, free combinatorics, random walks, Poisson boundaries