Zero-one laws for functional calculus on operator semigroups

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One-parameter semigroups

We work with one-parameter families $(T(t))_{0<t<\infty}$ in a Banach algebra $\mathcal{A}$.

Often $\mathcal{A}$ is the algebra of bounded linear operators on a Banach space $\mathcal{X}$, and indeed given $\mathcal{A}$ we can take $\mathcal{X} = \mathcal{A}$.

As usual, a semigroup satisfies

$$T(s+t) = T(s)T(t) \qquad \text{for all } t, s > 0.$$

We assume strong continuity for $t > 0$ but not necessarily at $t = 0$, that is,

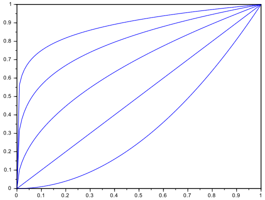
the mapping $t \mapsto T(t)x$ is continuous in norm for all $x \in \mathcal{X}$.

An example

Consider the semigroup

$$T(t) : x \mapsto x^t$$

in $\mathcal{A} = \mathcal{X} = C[0, 1]$.



Even with $T(0) \equiv 1$, we don't have strong continuity at 0.

Note that $\|T(t) - I\| = 1$ for all $t > 0$.

The classical zero-one law

The **classical zero-one law**, elementary to prove:

If $L := \limsup_{t \to 0+} \|T(t) - I\| < 1$, then

$\|T(t) - I\| \to 0$ and hence the semigroup is uniformly continuous and so has the form e^{At} where A, the **generator**, is bounded.

PROOF (Coulhon): The identity $2(x-1) = x^2 - 1 - (x-1)^2$ implies that

$$2(T(t) - I) = T(2t) - I - (T(t) - I)^2,$$

so we have $2L \leq L + L^2$ and thus $L = 0$ or $L \geq 1$.

Hille's theorem

Hille (1950) had an analogous result for differentiable semigroups.

Suppose that $(T(t))_{t>0}$ is an n-times continuously differentiable semigroup.

If

$$\limsup_{t\to 0+} \|t^n T^{(n)}(t)\| < \left(\frac{n}{e}\right)^n,$$

then the semigroup has a bounded generator.

The usual case is $n = 1$, when we get a **zero-**$1/e$ **law**, but the argument works more generally, and these constants are sharp.

Analytic semigroups

We will also look at semigroups $(T(t))_{t\in S_\alpha}$, where t lies in a sector

$$S_\alpha := \{z \in \mathbb{C} : |\arg z| < \alpha\}$$

for some $0 < \alpha < \pi/2$, the semigroup being supposed to be analytic (holomorphic).

Typically we examine $T(t)$ in $|t| < \delta$, with $t \in S_\alpha$.

Beurling's extension theorem

Beurling (1970) proved that a semigroup defined on $\mathbb{R}_+$ has an analytic extension to some sector S_α if and only if

$$\limsup_{t \to 0+} \|p(T(t))\| < \sup\{|p(z)| : |z| \le 1\}$$

for some polynomial p.

Kato and Neuberger (both 1970) proved that $p(z) = z - 1$ is sufficient, giving a zero-two law for analyticity, i.e., that

$$\limsup_{t \to 0+} \|T(t) - I\| < 2$$

implies analyticity.

Mokhtari's zero-quarter law

Suppose that the semigroup $(T(t))_{t>0}$ is bounded at the origin; then $(T(t_n))_n$ forms a bounded approximate identity in the algebra $\mathcal{A}$ generated by the semigroup, whenever $t_n \to 0$.

Moreover, if

$$\limsup_{t\to 0+} \|T(t) - T(2t)\| < \frac{1}{4},$$

then either $T(t) = 0$ for $t > 0$, or else the semigroup has a bounded generator A.

Esterle and Mokhtari (2002): similar results for $n \geq 1$, with

$$\limsup_{t\to 0+} \|T(t) - T((n+1)t)\| < \frac{n}{(n+1)^{1+1/n}} = \sup_{[0,1]} |x - x^{n+1}|.$$

Quasinilpotent semigroups

Recall that a semigroup is **quasinilpotent** if the spectral radius satisfies $\rho(T(t)) = 0$ for all t.

Standard examples can be found in the convolution algebra $L^1(0,1)$.

Excluding the trivial case, it then turns out that for each $\gamma > 0$ there is a $\delta > 0$ such that

$$\|T(t) - T((\gamma+1)t)\| > \frac{\gamma}{(\gamma+1)^{1+1/\gamma}}$$

for $0 < t < \delta$ (Esterle, 2005), an improvement on the Esterle–Mokhtari result.

The other extreme

Suppose that the algebra $\mathcal{A}$ is semi-simple, so no quasinilpotent elements except 0.

Theorem (Bendaoud-Chalendar–Esterle–P., 2010). If for some $\gamma > 0$ we have

$$\rho(T(t) - T((\gamma+1)t)) < \frac{\gamma}{(\gamma+1)^{1+1/\gamma}}$$

for $0 < t < \delta$ (some $\delta > 0$), then $\mathcal{A}$ is unital and we have $T(t) = e^{tA}$ for some bounded $A \in \mathcal{A}$.

In general, one can deduce similar properties of $\mathcal{A}/\operatorname{Rad}\mathcal{A}$ (quotienting out the radical).

Sectorial semigroups

An easy-stated result for the half-plane $\mathbb{C}_+$:

Theorem (Bendaoud-Chalendar–Esterle–P., 2010). If

$$\sup_{t\in\mathbb{C}_+,|t|<\delta} \rho(T(t) - T((\gamma+1)t)) < 2$$

then $\mathcal{A}/\operatorname{Rad}\mathcal{A}$ is a unital algebra, and the projection of the semigroup onto it has a bounded generator.

Our aim now: look at more general expressions, and "explain" the constants.

More general expressions

Theorem (BCEP, 2010). Let f be a real linear combination of functions $z^m \exp(-zw)$ with $m = 0, 1, 2, \ldots$ and $w > 0$, such that $f(0) = 0$ and $f(z) \to 0$ as $\mathrm{Re}\, z \to \infty$.

Let $(T(t))_{t \in S_\alpha}$ be analytic and non-quasinilpotent.

Define $k_\alpha = \sup_{z \in S_\alpha} |f(z)|$. If

$$\sup_{t \in S_\alpha, |t| < \delta} \rho(f(-tA)) < k_\alpha$$

then $\mathcal{A} / \mathrm{Rad}\, \mathcal{A}$ is unital and the projection of the semigoup has a bounded generator.

Examples

For $f(z)$ we may take $p(z)\exp(-z)$, p a suitable polynomial.

Or take combinations $\exp(-z) - \exp(-(\gamma+1)z)$, as we did earlier.

Thus we may estimate expressions such as $t^n A^n T(t) = t^n T^{(n)}(t)$ and $T(t) - T((\gamma+1)t)$.

In the first case

$$k_\alpha = \left(\frac{n}{e\cos\alpha}\right)^n,$$

recovering and extending the Hille result.

In the second, $k_\alpha \nearrow 2$ as $\alpha \nearrow \pi/2$.

All constants are sharp, as examples in $C[0,1]$ show.

A note on techniques

The methods here are largely based on complex analysis ideas.

In the quasinilpotent case we make estimates of the resolvent of A (which is an entire function).

In the non-quasinilpotent case we have Banach algebra ideas available.

In particular there are nontrivial characters $\chi : \mathcal{A} \to \mathbb{C}$.

We may check that $\chi(T(t)) = \exp(\lambda t)$ for some $\lambda \in \mathbb{C}$ and proceed from there to show that the Gelfand space $\widehat{\mathcal{A}}$ is compact.

A curiosity about quasinilpotent semigroups

First, an analytic semigroup $(T(t))_{t \in S_\alpha}$ bounded near the origin has an extension to $\overline{S_\alpha}$ making it strongly continuous at boundary points.

Second, if the semigroup is quasinilpotent and bounded on the half-plane $\mathbb{C}_+$, then it is trivial.

Indeed, if its boundary values satisfy

$$\int_{-\infty}^{\infty} \frac{\log^+ \|T(iy)\|}{1+y^2} < \infty,$$

then $T(t) = 0$ for $t \in \mathbb{C}_+$ (Chalendar–Esterle-P., 2010).

Basic Functional Calculus

We begin with semigroups on $\mathbb{R}_+$.

If $(T(t))_{t>0}$ is uniformly bounded and strongly continuous, then we may write

$$(A + \lambda I)^{-1} = -\int_0^\infty e^{\lambda t} T(t)\, dt,$$

for $\operatorname{Re} \lambda < 0$ (Bochner integral with respect to strong operator topology).

If in addition $(T(t))_{t>0}$ is quasinilpotent, then we have the above for all $\lambda \in \mathbb{C}$.

Functions defined by measures

Take $\mu \in M_c(0,\infty)$, i.e., complex finite Borel measure of compact support.
Then its Laplace transform is, as usual,

$$F(s) := \mathcal{L}\mu(s) = \int_0^\infty e^{-s\xi}\, d\mu(\xi).$$

Now we can define a functional calculus for the generator of a semigroup on $\mathcal{X}$ by

$$F(-A)x = \int_0^\infty T(\xi)x\, d\mu(\xi) \qquad (x \in \mathcal{X}).$$

Examples

The results will apply to examples with $\int_0^\infty d\mu(t) = 0$.

For instance, take $\mu = \delta_1 - \delta_2$; then

$$F(s) = e^{-s} - e^{-2s}$$

and

$$F(-tA) = T(t) - T(2t).$$

More exotic examples:

$$d\mu(t) = (\chi_{(1,2)} - \chi_{(2,3)})(t)\, dt$$

or

$$\mu = \delta_1 - 3\delta_2 + \delta_3 + \delta_4.$$

Theorem for quasinilpotent semigroups

Theorem (Chalendar–Esterle–P., 2013) Let $\mu \in M_c(0,\infty)$ be real with $\int_0^\infty d\mu(t) = 0$.

Let $(T(t))_{t>0}$ be a nontrivial strongly continuous quasinilpotent semigroup. Then there is an $\eta > 0$ such that

$$\|F(-sA)\| > \max_{x \geq 0} |F(x)| \qquad (0 < s \leq \eta).$$

For complex measures we define $\widetilde{F} = \mathcal{L}\overline{\mu}$, so $\widetilde{F}(z) = \overline{F(\overline{z})}$. Then

$$\|F(-sA)\widetilde{F}(-sA)\| > \max_{x \geq 0} |F(x)|^2 \qquad (0 < s \leq \eta).$$

The non-quasinilpotent case

For non-quasinilpotent semigroups there are various similar results, but they are more technical.

For example, in the case of a real measure, if there are $t_k \to 0$ with

$$\|F(-t_k A)\| < \sup_{x>0} |F(x)|,$$

then there are idempotents $P_n \in \mathcal{A}$ (i.e., $P_n^2 = P_n$) such that $\bigcup_{n=1}^{\infty} P_n \mathcal{A}$ is dense in $\mathcal{A}$ and each semigroup $(P_n T(t))$ has a bounded generator.

Analytic semigroups

For analytic semigroups on S_α we can replace measures by distributions.

Take $H(S_\alpha)$ to be the Frechet space of analytic functions on S_α with topology of local uniform convergence.

Now take (K_n) compact increasing, with $\bigcup_{n=1}^\infty K_n = S_\alpha$.

Our distributions are $\varphi : H(S_\alpha) \to \mathbb{C}$, such that

$$|\langle f, \varphi \rangle| \le M \sup\{|f(z)| : z \in K_n\}$$

for some $M > 0$ and $n \ge 1$.

More on distributions

It's easy to see (Hahn–Banach) that such a distribution φ can be represented by a non-unique Borel measure μ supported on K_n, i.e.,

$$\langle f, \varphi \rangle = \int_{K_n} f(\xi)\, d\mu(\xi).$$

For example,

$$\langle f, \varphi \rangle := f'(1) = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z-1)^2},$$

where C is a small circle surrounding 1.

The functional calculus

We need the **Fourier–Borel transform** of φ, given by

$$F(z) := \mathcal{FB}(\varphi)(z) = \langle e_{-z}, \varphi \rangle,$$

where $e_{-z}(\xi) = e^{-z\xi}$. Thus,

$$F(z) = \int_{K_n} e^{-z\xi} \, d\mu(\xi).$$

Then we define

$$F(-A) = \langle T, \varphi \rangle = \int_{K_n} T(\xi) \, d\mu(\xi).$$

as a Bochner integral, and independent of the choice of μ.

Theorem for non-quasinilpotent semigroups

Theorem (CEP 2013). Take S_α for $0 < \alpha < \pi/2$, and φ induced by a symmetric measure, i.e., $\mu(\overline{S}) = \overline{\mu(S)}$, supported on $\overline{S_\beta}$ with $0 \le \beta < \alpha$, such that $\int_{S_\alpha} d\mu(z) = 0$. Let $F = \mathcal{FB}(\varphi)$.

If there exists $\delta > 0$ with

$$\sup_{z \in S_{\alpha-\beta}, |z| \le \delta} \rho(F(-zA)) < \sup_{z \in S_{\alpha-\beta}} |F(z)|,$$

then $\mathcal{A}/\operatorname{Rad}\mathcal{A}$ is unital and the quotient semigroup has bounded generator.

Note that *a priori* $F(-zA)$ only makes sense for $z \in \overline{S_{\alpha-\beta}}$.

The case $\beta = 0$ and $\mathcal{A}$ semisimple

A related result holds for the case $(T(t))_{t \in S_\alpha}$ semisimple (so no nontrivial quasnilpotent elements).

If there exists $\delta > 0$ with

$$\sup_{0<t\leq\delta} \|F(-tA)\| < \sup_{t>0} |F(t)|,$$

then the semigroup has a bounded generator.

For example, $F(t) = e^{-t} - e^{-2t}$ and the sup is $\dfrac{1}{4}$.

The case $C[0,1]$

Consider the "universal" example $T(t): x \to x^t$ in $C[0,1]$.
For $F = \mathcal{FB}(\varphi)$ it is easy to check that

$$F(-tA)(x) = F(-t \log x),$$

and

$$\rho(F(-tA)) = \|F(-tA)\| = \sup_{x>0} F(-t \log x) = \sup_{r>0} |F(tr)|.$$

Thus

$$\sup_{0<t<\delta} \|F(-tA)\| = \sup_{t>0} |F(t)|,$$

and there is no bounded generator.

Final comments

1. The general analytic quasinilpotent case is harder, although the method used to $\mathbb{R}_+$ works, with modifications. Again it gives a lower bound on $F(-sA)$ for s near the origin if the semigroup is non-trivial.

2. Work in progress deals with multivariable functional calculus (several complex variables) and a family of commuting semigroups. One complication here is that functions of several variables can vanish on a line, e.g. $F(z_1, z_2) = z_1 - z_2$.

3. There are many other zero-one laws. Today we have restricted ourselves to estimates near the origin.

Very final comment

The end. Thank you.