Zero-one laws for functional calculus on operator semigroups

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Joint work with Isabelle Chalendar (Lyon) and Jean Esterle (Bordeaux)
1. Some classical results.
2. Newer dichotomy laws.
3. Functional calculus.
We work with one-parameter families \((T(t))_{0 < t < \infty}\) in a Banach algebra \(\mathcal{A}\).

Often \(\mathcal{A}\) is the algebra of bounded linear operators on a Banach space \(\mathcal{X}\), and indeed given \(\mathcal{A}\) we can take \(\mathcal{X} = \mathcal{A}\).

As usual, a semigroup satisfies

\[
T(s + t) = T(s)T(t) \quad \text{for all } t, s > 0.
\]

We assume strong continuity for \(t > 0\) but not necessarily at \(t = 0\), that is,

the mapping \(t \mapsto T(t)x\) is continuous in norm for all \(x \in \mathcal{X}\).
An example

Consider the semigroup

$$T(t) : x \mapsto x^t$$

in $\mathcal{A} = \mathcal{X} = C[0,1]$.  

Even with $T(0) \equiv 1$, we don’t have strong continuity at $0$.

Note that $\| T(t) - I \| = 1$ for all $t > 0$. 

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The classical zero-one law, elementary to prove:

If \( L := \limsup_{t \to 0^+} \| T(t) - I \| < 1 \), then

\[ \| T(t) - I \| \to 0 \]

and hence the semigroup is uniformly continuous and so has the form \( e^{At} \) where \( A \), the generator, is bounded.

**PROOF** (Coulhon): The identity \( 2(x - 1) = x^2 - 1 - (x - 1)^2 \) implies that

\[
2(T(t) - I) = T(2t) - I - (T(t) - I)^2,
\]

so we have \( 2L \leq L + L^2 \) and thus \( L = 0 \) or \( L \geq 1 \).
Hille’s theorem

Hille (1950) had an analogous result for differentiable semigroups. Suppose that \((T(t))_{t>0}\) is an \(n\)-times continuously differentiable semigroup.

If 

\[
\limsup_{t \to 0^+} \| t^n T^{(n)}(t) \| < \left( \frac{n}{e} \right)^n,
\]

then the semigroup has a bounded generator.

The usual case is \(n = 1\), when we get a \textbf{zero-1/e law}, but the argument works more generally, and these constants are sharp.
Analytic semigroups

We will also look at semigroups \((T(t))_{t \in S_\alpha}\), where \(t\) lies in a sector

\[ S_\alpha := \{ z \in \mathbb{C} : |\arg z| < \alpha \} \]

for some \(0 < \alpha < \pi/2\), the semigroup being supposed to be analytic (holomorphic).

Typically we examine \(T(t)\) in \(|t| < \delta\), with \(t \in S_\alpha\).
Beurling (1970) proved that a semigroup defined on $\mathbb{R}_+$ has an analytic extension to some sector $S_\alpha$ if and only if

$$\limsup_{t \to 0^+} \| p(T(t)) \| < \sup \{ |p(z)| : |z| \leq 1 \}$$

for some polynomial $p$.

Kato and Neuberger (both 1970) proved that $p(z) = z - 1$ is sufficient, giving a zero-two law for analyticity, i.e., that

$$\limsup_{t \to 0^+} \| T(t) - I \| < 2$$

implies analyticity.
Mokhtari’s zero-quarter law

Suppose that the semigroup \((T(t))_{t>0}\) is bounded at the origin; then \((T(t_n))_n\) forms a bounded approximate identity in the algebra \(A\) generated by the semigroup, whenever \(t_n \to 0\).

Moreover, if

\[
\limsup_{t \to 0^+} \|T(t) - T(2t)\| < \frac{1}{4},
\]

then either \(T(t) = 0\) for \(t > 0\), or else the semigroup has a bounded generator \(A\).

Esterle and Mokhtari (2002): similar results for \(n \geq 1\), with

\[
\limsup_{t \to 0^+} \|T(t) - T((n + 1)t)\| < \frac{n}{(n + 1)^{1 + 1/n}} = \sup_{[0,1]} |x - x^{n+1}|.
\]
Quasinilpotent semigroups

Recall that a semigroup is quasinilpotent if the spectral radius satisfies $\rho(T(t)) = 0$ for all $t$.

Standard examples can be found in the convolution algebra $L^1(0, 1)$.

Excluding the trivial case, it then turns out that for each $\gamma > 0$ there is a $\delta > 0$ such that

$$\| T(t) - T((\gamma + 1)t)\| > \frac{\gamma}{(\gamma + 1)^{1+1/\gamma}}$$

for $0 < t < \delta$ (Esterle, 2005), an improvement on the Esterle–Mokhtari result.
Suppose that the algebra $\mathcal{A}$ is semi-simple, so no quasinilpotent elements except 0.

**Theorem** (Bendaoud-Chalendar–Esterle–P., 2010). If for some $\gamma > 0$ we have

$$\rho(T(t) - T((\gamma + 1)t)) < \frac{\gamma}{(\gamma + 1)^{1+1/\gamma}}$$

for $0 < t < \delta$ (some $\delta > 0$), then $\mathcal{A}$ is unital and we have $T(t) = e^{tA}$ for some bounded $A \in \mathcal{A}$.

In general, one can deduce similar properties of $\mathcal{A}/\text{Rad} \mathcal{A}$ (quotienting out the radical).
An easy-stated result for the half-plane $\mathbb{C}_+$:

**Theorem** (Bendaoud-Chalendar–Esterle–P., 2010). If

$$\sup_{t \in \mathbb{C}_+, |t| < \delta} \rho(T(t) - T((\gamma + 1)t)) < 2$$

then $\mathcal{A}/\text{Rad} \mathcal{A}$ is a unital algebra, and the projection of the semigroup onto it has a bounded generator.

Our aim now: look at more general expressions, and “explain” the constants.
More general expressions

**Theorem** (BCEP, 2010). Let \( f \) be a real linear combination of functions \( z^m \exp(-zw) \) with \( m = 0, 1, 2, \ldots \) and \( w > 0 \), such that \( f(0) = 0 \) and \( f(z) \to 0 \) as \( \Re z \to \infty \).

Let \( (T(t))_{t \in S_\alpha} \) be analytic and non-quasinilpotent.

Define \( k_\alpha = \sup_{z \in S_\alpha} |f(z)|. \) If

\[
\sup_{t \in S_\alpha, |t| < \delta} \rho(f(-tA)) < k_\alpha
\]

then \( \mathcal{A}/\text{Rad} \mathcal{A} \) is unital and the projection of the semigoup has a bounded generator.
For $f(z)$ we may take $p(z)\exp(-z)$, $p$ a suitable polynomial.

Or take combinations $\exp(-z) - \exp(-(\gamma + 1)z)$, as we did earlier.

Thus we may estimate expressions such as $t^n A^n T(t) = t^n T^{(n)}(t)$ and $T(t) - T((\gamma + 1)t)$.

In the first case

$$k_\alpha = \left(\frac{n}{e \cos \alpha}\right)^n,$$

recovering and extending the Hille result.

In the second, $k_\alpha \nearrow 2$ as $\alpha \nearrow \pi/2$.

All constants are sharp, as examples in $C[0, 1]$ show.
The methods here are largely based on complex analysis ideas.

In the quasinilpotent case we make estimates of the resolvent of $A$ (which is an entire function).

In the non-quasinilpotent case we have Banach algebra ideas available.

In particular there are nontrivial characters $\chi : \mathcal{A} \to \mathbb{C}$.

We may check that $\chi(T(t)) = \exp(\lambda t)$ for some $\lambda \in \mathbb{C}$ and proceed from there to show that the Gelfand space $\hat{\mathcal{A}}$ is compact.
First, an analytic semigroup \( (T(t))_{t \in S_\alpha} \) bounded near the origin has an extension to \( \overline{S_\alpha} \) making it strongly continuous at boundary points.

Second, if the semigroup is quasinilpotent and bounded on the half-plane \( \mathbb{C}_+ \), then it is trivial.

Indeed, if its boundary values satisfy

\[
\int_{-\infty}^{\infty} \frac{\log^+ \| T(iy) \|}{1 + y^2} < \infty,
\]

then \( T(t) = 0 \) for \( t \in \mathbb{C}_+ \) (Chalendar–Esterle-P., 2010).
We begin with semigroups on $\mathbb{R}_+$. If $(T(t))_{t>0}$ is uniformly bounded and strongly continuous, then we may write

$$(A + \lambda I)^{-1} = -\int_0^\infty e^{\lambda t} T(t) \, dt,$$

for $\text{Re} \, \lambda < 0$ (Bochner integral with respect to strong operator topology).

If in addition $(T(t))_{t>0}$ is quasinilpotent, then we have the above for all $\lambda \in \mathbb{C}$. 
Take $\mu \in M_c(0, \infty)$, i.e., complex finite Borel measure of compact support.

Then its Laplace transform is, as usual,

$$F(s) := \mathcal{L}_\mu(s) = \int_0^\infty e^{-s\xi} \, d\mu(\xi).$$

Now we can define a functional calculus for the generator of a semigroup on $\mathcal{X}$ by

$$F(-A)x = \int_0^\infty T(\xi)x \, d\mu(\xi) \quad (x \in \mathcal{X}).$$
Examples

The results will apply to examples with $\int_0^\infty d\mu(t) = 0$.

For instance, take $\mu = \delta_1 - \delta_2$; then

$$F(s) = e^{-s} - e^{-2s}$$

and

$$F(-tA) = T(t) - T(2t).$$

More exotic examples:

$$d\mu(t) = (\chi_{(1,2)} - \chi_{(2,3)}) (t) \, dt$$

or

$$\mu = \delta_1 - 3\delta_2 + \delta_3 + \delta_4.$$
Theorem (Chalendar–Esterle–P., 2013) Let $\mu \in M_c(0, \infty)$ be real with $\int_0^\infty d\mu(t) = 0$.

Let $(T(t))_{t>0}$ be a nontrivial strongly continuous quasinilpotent semigroup. Then there is an $\eta > 0$ such that

$$\|F(-sA)\| > \max_{x \geq 0} |F(x)| \quad (0 < s \leq \eta).$$

For complex measures we define $\tilde{F} = \mathcal{L}\mu$, so $\tilde{F}(z) = \overline{F(\overline{z})}$. Then

$$\|F(-sA)\tilde{F}(-sA)\| > \max_{x \geq 0} |F(x)|^2 \quad (0 < s \leq \eta).$$
The non-quasinilpotent case

For non-quasinilpotent semigroups there are various similar results, but they are more technical.

For example, in the case of a real measure, if there are \( t_k \to 0 \) with

\[
\|F(-t_k A)\| < \sup_{x>0} |F(x)|,
\]

then there are idempotents \( P_n \in \mathcal{A} \) (i.e., \( P_n^2 = P_n \)) such that \( \bigcup_{n=1}^{\infty} P_n \mathcal{A} \) is dense in \( \mathcal{A} \) and each semigroup \( (P_n T(t)) \) has a bounded generator.
Analytic semigroups

For analytic semigroups on $S_\alpha$ we can replace measures by distributions.

Take $H(S_\alpha)$ to be the Frechet space of analytic functions on $S_\alpha$ with topology of local uniform convergence.

Now take $(K_n)$ compact increasing, with $\bigcup_{n=1}^\infty K_n = S_\alpha$.

Our distributions are $\varphi : H(S_\alpha) \to \mathbb{C}$, such that

$$|\langle f, \varphi \rangle| \leq M \sup\{|f(z)| : z \in K_n\}$$

for some $M > 0$ and $n \geq 1$. 
It’s easy to see (Hahn–Banach) that such a distribution $\varphi$ can be represented by a non-unique Borel measure $\mu$ supported on $K_n$, i.e.,

$$\langle f, \varphi \rangle = \int_{K_n} f(\xi) \, d\mu(\xi).$$

For example,

$$\langle f, \varphi \rangle := f'(1) = \frac{1}{2\pi i} \int_C \frac{f(z) \, dz}{(z - 1)^2},$$

where $C$ is a small circle surrounding 1.
The functional calculus

We need the **Fourier–Borel transform** of \( \varphi \), given by

\[
F(z) := \mathcal{FB}(\varphi)(z) = \langle e^{-z}, \varphi \rangle,
\]

where \( e_{-z}(\xi) = e^{-z\xi} \). Thus,

\[
F(z) = \int_{K_n} e^{-z\xi} \, d\mu(\xi).
\]

Then we define

\[
F(-A) = \langle T, \varphi \rangle = \int_{K_n} T(\xi) \, d\mu(\xi).
\]

as a Bochner integral, and independent of the choice of \( \mu \).
Theorem (CEP 2013). Take $S_\alpha$ for $0 < \alpha < \pi/2$, and $\varphi$ induced by a symmetric measure, i.e., $\mu(S) = \overline{\mu(S)}$, supported on $S_\beta$ with $0 \leq \beta < \alpha$, such that $\int_{S_\alpha} d\mu(z) = 0$. Let $F = FB(\varphi)$. If there exists $\delta > 0$ with

$$\sup_{z \in S_{\alpha - \beta}, |z| \leq \delta} \rho(F(-zA)) < \sup_{z \in S_{\alpha - \beta}} |F(z)|,$$

then $\mathcal{A}/\text{Rad}\mathcal{A}$ is unital and the quotient semigroup has bounded generator.

Note that a priori $F(-zA)$ only makes sense for $z \in \overline{S_{\alpha - \beta}}$. 

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The case $\beta = 0$ and $A$ semisimple

A related result holds for the case $(T(t))_{t \in S_\alpha}$ semisimple (so no nontrivial quas-nilpotent elements).

If there exists $\delta > 0$ with

$$\sup_{0 < t \leq \delta} \| F(-tA) \| < \sup_{t > 0} |F(t)|,$$

then the semigroup has a bounded generator.

For example, $F(t) = e^{-t} - e^{-2t}$ and the sup is $\frac{1}{4}$. 
Consider the “universal” example $T(t) : x \rightarrow x^t$ in $C[0, 1]$. For $F = FB(\varphi)$ it is easy to check that

$$F(-tA)(x) = F(-t \log x),$$

and

$$\rho(F(-tA)) = \|F(-tA)\| = \sup_{x > 0} F(-t \log x) = \sup_{r > 0} |F(tr)|.$$

Thus

$$\sup_{0 < t < \delta} \|F(-tA)\| = \sup_{t > 0} |F(t)|,$$

and there is no bounded generator.
1. The general analytic quasinilpotent case is harder, although the method used to $\mathbb{R}_+$ works, with modifications. Again it gives a lower bound on $F(-sA)$ for $s$ near the origin if the semigroup is non-trivial.

2. Work in progress deals with multivariable functional calculus (several complex variables) and a family of commuting semigroups. One complication here is that functions of several variables can vanish on a line, e.g. $F(z_1, z_2) = z_1 - z_2$.

3. There are many other zero-one laws. Today we have restricted ourselves to estimates near the origin.
The end. Thank you.