Ideals of operators on the Banach space of continuous functions on the first uncountable ordinal

Niels Laustsen

Lancaster University, UK

Quantum Groups, **O**perators and Non-commutative **P**robability

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Joint work with
Tomasz Kania (Lancaster) and Piotr Koszmider (IMPAN, Warsaw)

Motivation: the ideal structure of $\mathscr{B}(X)$

For a Banach space X, consider the Banach algebra

$$\mathscr{B}(X) = \{T\colon X \to X : T \text{ is bounded and linear}\}.$$

Overall aim: to understand the lattice of (closed, two-sided) ideals of $\mathscr{B}(X)$.

This is a very difficult problem; the only known complete classifications are:

- $\dim X < \infty$;
- $X = \ell_p(\mathbb{I})$ for $1 \leqslant p < \infty$ and $X = c_0(\mathbb{I})$, where $\mathbb{I}$ is an arbitrary infinite index set (Calkin 1941; Gohberg–Markus–Feldman 1960; Gramsch 1967/ Luft 1968; Daws 2006);
- $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{c_0}$ (L–Loy–Read 2004) and its dual $X = \left(\bigoplus_{n\in\mathbb{N}} \ell_2^n\right)_{\ell_1}$ (L–Schlumprecht–Zsák 2006);
- Argyros and Haydon's Banach space with very few operators (2011), and some variants of it (Tarbard 2012; Motakis–Puglisi–Zisimopoulou 2014; Kania–L 2014);
- $X = C(K)$, where K is the 'Mrówka space' constructed by Koszmider (2005), assuming CH (Brooker (unpublished)/Kania–Kochanek 2014).

Maximal ideals of $\mathscr{B}(X)$

Easier goal: to understand the maximal ideals of $\mathscr{B}(X)$ for a Banach space X.

Note:

- $\mathscr{B}(X)$ always contains a unique minimal non-zero ideal: $\overline{\mathscr{F}(X)}$.
- $\mathscr{B}(X)$ is unital, hence:
  – every proper ideal of $\mathscr{B}(X)$ is contained in a maximal ideal;
  – the maximal ideals of $\mathscr{B}(X)$ are automatically closed.

Observation (Dosev–Johnson 2010). The set

$$\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}$$

is the unique maximal ideal of $\mathscr{B}(X)$ if (and only if) it is closed under addition.

Banach spaces X such that $\mathscr{M}_X$ is the unique maximal ideal of $\mathscr{B}(X)$

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}.$

- $c_0(\mathbb{I})$ and $\ell_p(\mathbb{I})$ for $1 \leqslant p < \infty$ and an arbitrary infinite index set $\mathbb{I}$, $\big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_{c_0}$ and $\big(\bigoplus_{n\in\mathbb{N}} \ell_2^n\big)_{\ell_1}$;
- $\big(\bigoplus_{n\in\mathbb{N}} \ell_q^n\big)_E$ for $1 \leqslant q \leqslant \infty$ and $E = c_0$ or $E = \ell_p$ for $1 \leqslant p < \infty$ (L–Odell–Schlumprecht–Zsák 2012; Leung ($\times 2$) 2014; Kania–L 2014);
- $\big(\bigoplus_{\mathbb{N}} \ell_q\big)_{\ell_p}$ for $1 \leqslant q < p < \infty$ (Chen–Johnson–Zheng 2011);
- Lorentz sequence spaces (Kamińska–Popov–Spinu–Tcaciuc–Troitsky 2011);
- certain Orlicz sequence spaces (Lin–Sari–Zheng 2014);
- the quasi-reflexive James spaces J_p for $1 < p < \infty$ (L 2002);
- Edgar's long James spaces $J_p(\omega_1)$ for $1 < p < \infty$ (Kania–Kochanek 2014);
- the James tree space and the James function space (Apatsidis–Argyros–Kanellopoulos 2008).

Banach spaces X such that $\mathscr{M}_X$ is the unique max. ideal of $\mathscr{B}(X)$ (cont.)

Recall: $\mathscr{M}_X = \{T \in \mathscr{B}(X) : \forall\, R, S \in \mathscr{B}(X) : I \neq STR\}.$

- $L_p[0,1]$ for $1 \leqslant p < \infty$ (Dosev–Johnson–Schechtman 2011);
- $L_\infty[0,1] \cong \ell_\infty$ (L–Loy 2005, using Pełczyński and Rosenthal);
- $\ell_\infty(\mathbb{I})$ and $\ell_\infty^c(\mathbb{I})$ for an arbitrary infinite index set $\mathbb{I}$ (Johnson–Kania–Schechtman 2014);
- ℓ_∞/c_0 (using Drewnowski–Roberts 1991);
- $C[0,1]$ (Brooker 2010, using Pełczyński and Rosenthal);
- $C(A)$, where A is the 'double arrow space' (Michalak 2003);
- $C[0,\omega^\omega]$ and $C[0,\alpha]$, where α is a countable ordinal satisfying $\alpha = \omega^\alpha$ (Brooker, using Bourgain and Pełczyński);
- $C[0,\omega_1]$ (Kania–L 2012);
- $\left(\bigoplus_{\alpha<\omega_1} C[0,\alpha]\right)_{c_0}$ (Kania–L 2014).

$C(K)$-spaces

For a compact Hausdorff space K, consider the Banach space

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}.$$

Fact. *$C(K)$ separable $\iff$ K metrizable.*

Classification. *Let K be a compact metric space. Then:*

- *K has $n \in \mathbb{N}$ elements $\iff$ $C(K) \cong \ell_\infty^n$;*
- (Milutin) *K is uncountable $\iff$ $C(K) \cong C[0,1]$;*
- (Bessaga and Pełczyński) *K is countably infinite $\iff$ $C(K) \cong C[0, \omega^{\omega^\alpha}]$ for a unique countable ordinal α.*

Here, for an ordinal σ, the interval $[0, \sigma] = \{\alpha \text{ ordinal} : \alpha \leqslant \sigma\}$ is equipped with the *order topology*, which is determined by the basis

$$[0, \beta), \qquad (\alpha, \beta), \qquad (\alpha, \sigma] \qquad (0 \leqslant \alpha < \beta \leqslant \sigma).$$

Note: $C[0, \omega_1]$, where ω_1 is the first uncountable ordinal, is the "next" $C(K)$-space after the separable ones $C[0, \omega^{\omega^\alpha}]$ for countable α.

Theorem (Semadeni 1960). $C[0, \omega_1] \not\cong C[0, \omega_1] \oplus C[0, \omega_1]$.

The topological dichotomy

For convenience, consider the hyperplane

$$C_0[0,\omega_1) = \{f \in C[0,\omega_1] : f(\omega_1) = 0\}$$

instead of $C[0,\omega_1]$.

Theorem (Kania–Koszmider–L). *Let K be a weak*-compact subset of $C_0[0,\omega_1)^*$. Then exactly one of the following two alternatives holds:*

- *K is* uniformly Eberlein compact, *in the sense that K is homeomorphic to a weakly compact subset of a Hilbert space;*
- *K contains a homeomorphic copy of $[0,\omega_1]$ of the form*

$$\{\rho + \lambda\delta_\alpha : \alpha \in D\} \cup \{\rho\},$$

where $\rho \in C_0[0,\omega_1)^*$, $\lambda \in \mathbb{C} \setminus \{0\}$, δ_α is the Dirac measure at α, and D is a closed and unbounded subset of $[0,\omega_1)$.

Note:

(i) $[0,\omega_1]$ is not contained in any uniformly Eberlein compact space;

(ii) the unit ball of $C_0[0,\omega_1)^*$ in the weak* top. contains a homeomorphic copy of every uniformly Eberlein compact space of density at most $\aleph_1$.

Operator-theoretic application: consider $K = T^*(\text{the unit ball of } C_0[0,\omega_1)^*)$ for $T \in \mathscr{B}(C_0[0,\omega_1))$.

Characterizations of the unique maximal ideal of $\mathscr{B}(C_0[0,\omega_1))$

Theorem (Kania–Koszmider–L). *Let $T \in \mathscr{B}(C_0[0,\omega_1))$. Then TFAE:*

(a) $T \in \mathscr{M}_{C_0[0,\omega_1)}$ *(that is, $I \neq STR$ for all $R, S \in \mathscr{B}(C_0[0,\omega_1))$);*

(b) *T does not fix a copy of $C_0[0,\omega_1)$;*

(c) *T is a* Semadeni operator, *in the sense that T^{**} maps the subspace*

$$\{\Lambda \in C_0[0,\omega_1)^{**} : \langle \lambda_n, \Lambda\rangle \to 0 \text{ as } n \to \infty \\ \text{for every weak}^*\text{-null sequence } (\lambda_n) \text{ in } C_0[0,\omega_1)^*\}$$

into the canonical copy of $C_0[0,\omega_1)$ in its bidual;

(d) *there is a closed, unbounded subset D of $[0,\omega_1)$ such that*

$$(Tf)(\alpha) = 0 \qquad (f \in C_0[0,\omega_1),\ \alpha \in D);$$

(e) *T factors through the Banach space $\left(\bigoplus_{\alpha<\omega_1} C[0,\alpha]\right)_{c_0}$;*

(f) *the range of T is contained in a* Hilbert-generated subspace *of $C_0[0,\omega_1)$; that is, there exist a Hilbert space H and an operator $U\colon H \to C_0[0,\omega_1)$ such that $T(C_0[0,\omega_1)) \subseteq \overline{U(H)}$;*

(g) *the range of T is contained in a* weakly compactly generated subspace *of $C_0[0,\omega_1)$; that is, there exist a reflexive Banach space X and an operator $V\colon X \to C_0[0,\omega_1)$ such that $T(C_0[0,\omega_1)) \subseteq \overline{V(X)}$.*

Let X be an Asplund space (that is, every separable subspace of X has separable dual), and let $K \subset X^*$ be weak*-compact.
Szlenk associated an ordinal $\operatorname{Sz} K$ with K, its *Szlenk index*.

Set

$$\operatorname{Sz} X = \operatorname{Sz}(\text{the unit ball of } X^*).$$

(We extend this to all Banach spaces by $\operatorname{Sz} X := \infty$ when X is not Asplund.)

Theorem (Samuel 1983). $\operatorname{Sz} C[0, \omega^{\omega^\alpha}] = \omega^{\alpha+1}$ *for each countable ordinal* α.

More generally, for an operator $T\colon X \to Y$, define

$$\operatorname{Sz} T = \operatorname{Sz}(T^*(\text{the unit ball of } Y^*)).$$

(This may also be ∞.) For an ordinal α, set

$$\mathscr{SZ}_\alpha(X, Y) = \{T \in \mathscr{B}(X, Y) : \operatorname{Sz} T \leqslant \omega^\alpha\}.$$

Theorem (Brooker 2012). *The class* $\mathscr{SZ}_\alpha$ *is a closed, injective and surjective operator ideal in the sense of Pietsch for every ordinal* α.

Set $E_{\omega_1} = \left(\bigoplus_{\alpha<\omega_1} C[0,\alpha]\right)_{c_0}$, and recall that

$$T \in \mathscr{M}_{C_0[0,\omega_1)} \iff T \text{ factors through } E_{\omega_1}.$$

Theorem (Kania–L). *Let $T \in \mathscr{B}(C_0[0,\omega_1))$. Then TFAE:*

(a) *T fixes a copy of E_{ω_1};*

(b) *the identity operator on E_{ω_1} factors through T;*

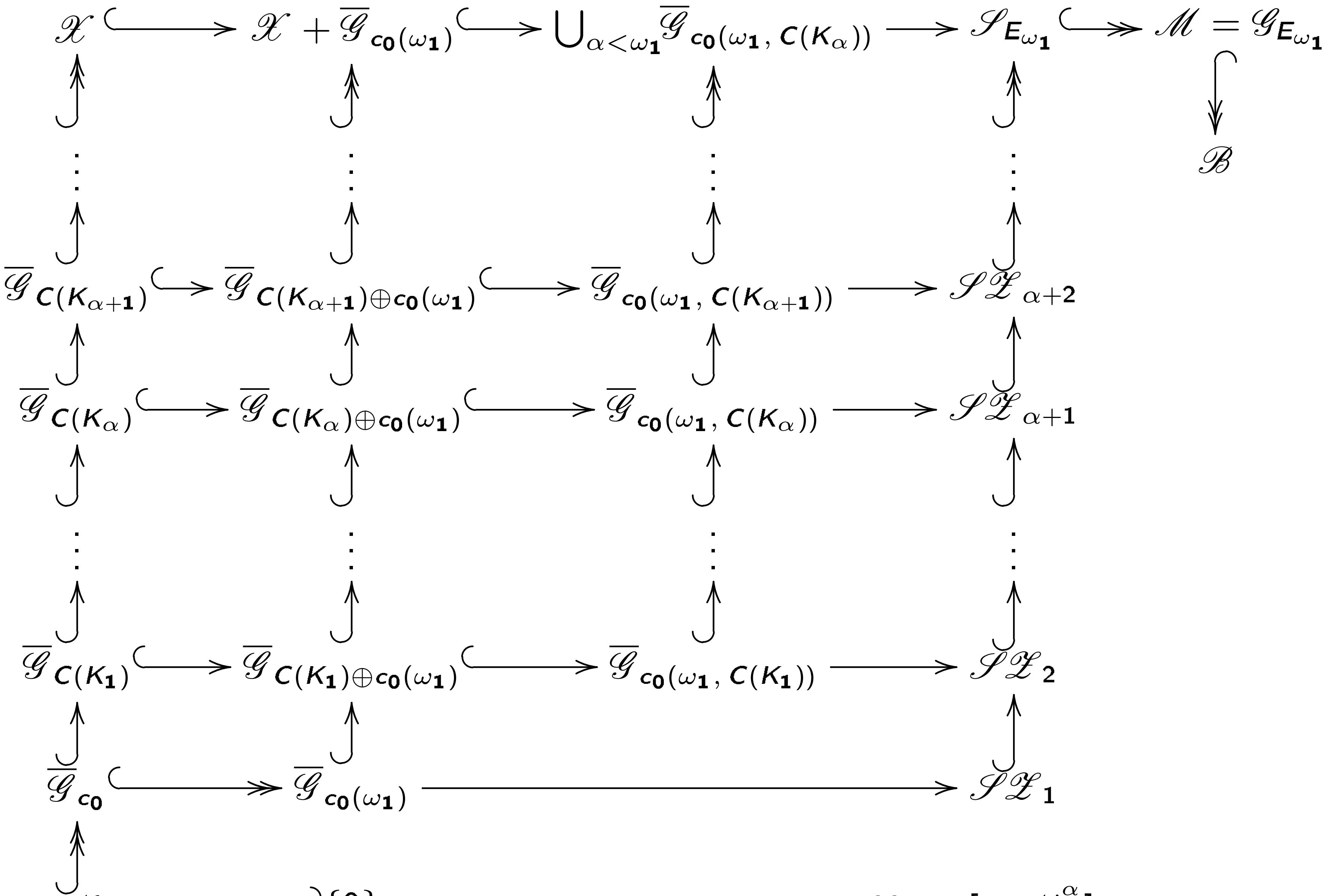
(c) *the Szlenk index of T is uncountable.*

Corollary. *The set*

$$\begin{aligned}\mathscr{S}_{E_{\omega_1}}(C_0[0,\omega_1)) &= \{T \in \mathscr{B}(C_0[0,\omega_1)) : T \text{ does not fix a copy of } E_{\omega_1}\} \\ &= \{T \in \mathscr{B}(C_0[0,\omega_1)) : \text{the identity operator on } E_{\omega_1} \\ &\qquad\qquad \text{does not factor through } T\} \\ &= \{T \in \mathscr{B}(C_0[0,\omega_1)) : \operatorname{Sz} T < \omega_1\} = \bigcup_{\alpha<\omega_1} \mathscr{S}\mathscr{Z}_\alpha(C_0[0,\omega_1))\end{aligned}$$

is the second-largest proper closed ideal of $\mathscr{B}(C_0[0,\omega_1))$: for each proper ideal $\mathscr{I}$ of $\mathscr{B}(C_0[0,\omega_1))$, either $\mathscr{I} = \mathscr{M}_{C_0[0,\omega_1)}$ or $\mathscr{I} \subseteq \mathscr{S}_{E_{\omega_1}}(C_0[0,\omega_1))$.

Partial structure of the lattice of closed ideals of $\mathscr{B} = \mathscr{B}(C_0[0, \omega_1))$



$K_\alpha = [0, \omega^{\omega^\alpha}], \ \alpha < \omega_1$

Conventions

- We suppress $C_0[0,\omega_1)$ everywhere, thus writing $\mathscr{K}$ instead of $\mathscr{K}(C_0[0,\omega_1))$ for the ideal of compact operators on $C_0[0,\omega_1)$, *etc.*;
- $\mathscr{I} \hookrightarrow \mathscr{J}$ means that the ideal $\mathscr{I}$ is properly contained in the ideal $\mathscr{J}$;
- $\mathscr{I} \hookrightarrow\!\!\!\!\rightarrow \mathscr{J}$ indicates that there are no closed ideals between $\mathscr{I}$ and $\mathscr{J}$;
- $\mathscr{G}_X$ denotes the set of operators that factor through the Banach space X and $\overline{\mathscr{G}}_X$ its closure;
- $c_0(\omega_1, X)$ denotes the c_0-direct sum of ω_1 copies of the Banach space X, and $c_0(\omega_1) := c_0(\omega_1, \mathbb{C})$;
- $\mathscr{X}$ denotes the ideal of operators with separable range.

Definition. A linear mapping δ from a Banach algebra $\mathscr{A}$ into a Banach $\mathscr{A}$-bimodule is a *derivation* if

$$\delta(ab) = a \cdot \delta(b) + \delta(a) \cdot b \qquad (a, b \in \mathscr{A}).$$

Theorem (Johnson 1967). *Let X be a Banach space such that $X \cong X \oplus X$. Then every derivation from $\mathscr{B}(X)$ into a Banach $\mathscr{B}(X)$-bimodule is automatically continuous.*

Question: what happens when $X \ncong X \oplus X$?

Theorem (Loy–Willis 1989). *Every derivation from $\mathscr{B}(C[0,\omega_1])$ into a Banach $\mathscr{B}(C[0,\omega_1])$-bimodule is automatically continuous.*

Remark. Around the same time, Read constructed a Banach space X such that there is a discontinuous derivation from $\mathscr{B}(X)$ into a Banach $\mathscr{B}(X)$-bimodule.

Bounded approximate identities in the Loy–Willis ideal

Loy and Willis' starting point: $\mathscr{B}(C[0,\omega_1])$ *contains a maximal ideal* $\mathscr{M}$ *of codimension one.*

Note: our work shows that $\mathscr{M} = \mathscr{M}_{C[0,\omega_1]}$. We call $\mathscr{M}$ the *Loy–Willis ideal.*

Loy and Willis' key step: $\mathscr{M}$ *has a bounded right approximate identity, that is, a norm-bounded net* (U_j) *such that* $TU_j \to T$ *for each* $T \in \mathscr{M}$.

Question: does $\mathscr{M}$ also have a bounded left approximate identity, that is, a norm-bounded net (U_j) such that $U_j T \to T$ for each $T \in \mathscr{M}$?

Answer: Yes! — In fact more is true:

Theorem (Kania–Koszmider–L). $\mathscr{M}$ *contains a net* (Q_j) *of projections with* $\|Q_j\| \leqslant 2$ *such that*

$$\forall\, T \in \mathscr{M}\ \exists\, j_0\ \forall\, j \geqslant j_0: \ \ Q_j T = T.$$

Corollary (using Dixon 1973). $\mathscr{M}$ *has a bounded two-sided approximate identity.*

A few references (in chronological order)

- Z. Semadeni, Banach spaces non-isomorphic to their Cartesian squares. II, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.* **8** (1960), 81–84.
- R. J. Loy and G. A. Willis, Continuity of derivations on $\mathscr{B}(E)$ for certain Banach spaces E, *J. London Math. Soc.* **40** (1989), 327–346.
- T. Kania and N. J. Laustsen, Uniqueness of the maximal ideal of the Banach algebra of bounded operators on $C([0,\omega_1])$, *J. Funct. Anal.* **262** (2012), 4831–4850; arXiv:1112.4800.
- T. Kania, P. Koszmider and N. J. Laustsen, A weak*-topological dichotomy with applications in operator theory; *Trans. London Math. Soc.* **1**, 1–28 (2014); arXiv:1303.0020.
- T. Kania and N. J. Laustsen, Operators on two Banach spaces of continuous functions on locally compact spaces of ordinals; *Proc. Amer. Math. Soc.*, to appear; arXiv:1304.4951.