Ideals of operators on the Banach space of continuous functions on the first uncountable ordinal

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Motivation: the ideal structure of $\mathcal{B}(X)$

For a Banach space $X$, consider the Banach algebra

$$\mathcal{B}(X) = \{ T : X \to X : T \text{ is bounded and linear} \}.$$ 

**Overall aim:** to understand the lattice of (closed, two-sided) ideals of $\mathcal{B}(X)$.

This is a very difficult problem; the only known complete classifications are:

- $\dim X < \infty$;
- $X = \ell_p(\mathbb{I})$ for $1 \leq p < \infty$ and $X = c_0(\mathbb{I})$, where $\mathbb{I}$ is an arbitrary infinite index set (Calkin 1941; Gohberg–Markus–Feldman 1960; Gramsch 1967/Luft 1968; Daws 2006);
- $X = (\bigoplus_{n \in \mathbb{N}} \ell_2^n)_{c_0}$ (L–Loy–Read 2004) and its dual $X = (\bigoplus_{n \in \mathbb{N}} \ell_2^n)_{\ell_1}$ (L–Schlumprecht–Zsák 2006);
- Argyros and Haydon’s Banach space with very few operators (2011), and some variants of it (Tarbard 2012; Motakis–Puglisi–Zisimopoulou 2014; Kania–L 2014);
- $X = C(K)$, where $K$ is the ‘Mrówka space’ constructed by Koszmider (2005), assuming CH (Brooker (unpublished)/Kania–Kochanek 2014).
Maximal ideals of $\mathcal{B}(X)$

**Easier goal:** to understand the maximal ideals of $\mathcal{B}(X)$ for a Banach space $X$.

**Note:**
- $\mathcal{B}(X)$ always contains a unique minimal non-zero ideal: $\mathcal{F}(X)$.
- $\mathcal{B}(X)$ is unital, hence:
  - every proper ideal of $\mathcal{B}(X)$ is contained in a maximal ideal;
  - the maximal ideals of $\mathcal{B}(X)$ are automatically closed.

**Observation** (Dosev–Johnson 2010). The set

$$\mathcal{M}_X = \{ T \in \mathcal{B}(X) : \forall R, S \in \mathcal{B}(X) : I \neq STR \}$$

is the unique maximal ideal of $\mathcal{B}(X)$ if (and only if) it is closed under addition.
Banach spaces $X$ such that $\mathcal{M}_X$ is the unique maximal ideal of $\mathcal{B}(X)$

Recall: $\mathcal{M}_X = \{ T \in \mathcal{B}(X) : \forall R, S \in \mathcal{B}(X), I \neq STR \}$.

- $c_0(\mathbb{I})$ and $\ell_p(\mathbb{I})$ for $1 \leq p < \infty$ and an arbitrary infinite index set $\mathbb{I}$, $(\bigoplus_{n \in \mathbb{N}} \ell_2^n)_{c_0}$ and $(\bigoplus_{n \in \mathbb{N}} \ell_2^n)_{\ell_1}$;
- $(\bigoplus_{n \in \mathbb{N}} \ell_q^n)_E$ for $1 \leq q \leq \infty$ and $E = c_0$ or $E = \ell_p$ for $1 \leq p < \infty$ (L–Odell–Schlumprecht–Zsák 2012; Leung (×2) 2014; Kania–L 2014);
- $(\bigoplus_{n \in \mathbb{N}} \ell_q^n)_{\ell_p}$ for $1 \leq q < p < \infty$ (Chen–Johnson–Zheng 2011);
- Lorentz sequence spaces (Kamińska–Popov–Spinu–Tcaciuc–Troitsky 2011);
- certain Orlicz sequence spaces (Lin–Sari–Zheng 2014);
- the quasi-reflexive James spaces $J_p$ for $1 < p < \infty$ (L 2002);
- Edgar’s long James spaces $J_p(\omega_1)$ for $1 < p < \infty$ (Kania–Kochanek 2014);
- the James tree space and the James function space (Apatidis–Argyros–Kanellopoulos 2008).
Recall: \( \mathcal{M}_X = \{ T \in \mathcal{B}(X) : \forall R, S \in \mathcal{B}(X) : I \neq STR \} \).

- \( L_p[0,1] \) for \( 1 \leq p < \infty \) (Dosev–Johnson–Schechtman 2011);
- \( L_\infty[0,1] \cong \ell_\infty \) (L–Loy 2005, using Pełczyński and Rosenthal);
- \( \ell_\infty(\mathbb{I}) \) and \( \ell^c_\infty(\mathbb{I}) \) for an arbitrary infinite index set \( \mathbb{I} \) (Johnson–Kania–Schechtman 2014);
- \( \ell_\infty/c_0 \) (using Drewnowski–Roberts 1991);
- \( C[0,1] \) (Brooker 2010, using Pełczyński and Rosenthal);
- \( C(A) \), where \( A \) is the ‘double arrow space’ (Michalak 2003);
- \( C[0,\omega^\omega] \) and \( C[0,\alpha] \), where \( \alpha \) is a countable ordinal satisfying \( \alpha = \omega^\alpha \) (Brooker, using Bourgain and Pełczyński);
- \( C[0,\omega_1] \) (Kania–L 2012);
- \( (\bigoplus_{\alpha<\omega_1} C[0,\alpha])_{c_0} \) (Kania–L 2014).
For a compact Hausdorff space $K$, consider the Banach space

$$C(K) = \{ f : K \to \mathbb{C} : f \text{ is continuous} \}.$$ 

**Fact.** $C(K)$ separable $\iff$ $K$ metrizable.

**Classification.** Let $K$ be a compact metric space. Then:
- $K$ has $n \in \mathbb{N}$ elements $\iff$ $C(K) \cong \ell^n$;
- (Milutin) $K$ is uncountable $\iff$ $C(K) \cong C[0,1]$;
- (Bessaga and Pełczyński) $K$ is countably infinite $\iff$ $C(K) \cong C[0,\omega^\alpha]$ for a unique countable ordinal $\alpha$.

Here, for an ordinal $\sigma$, the interval $[0, \sigma] = \{ \alpha \text{ ordinal} : \alpha \leq \sigma \}$ is equipped with the *order topology*, which is determined by the basis

$$[0, \beta), (\alpha, \beta), (\alpha, \sigma] \quad (0 \leq \alpha < \beta \leq \sigma).$$

**Note:** $C[0, \omega_1]$, where $\omega_1$ is the first uncountable ordinal, is the “next” $C(K)$-space after the separable ones $C[0, \omega^\alpha]$ for countable $\alpha$.

**Theorem** (Semadeni 1960). $C[0, \omega_1] \not\cong C[0, \omega_1] \oplus C[0, \omega_1]$. 

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The topological dichotomy

For convenience, consider the hyperplane

\[ C_0[0, \omega_1] = \{ f \in C[0, \omega_1] : f(\omega_1) = 0 \} \]

instead of \( C[0, \omega_1] \).

**Theorem** (Kania–Koszmider–L). Let \( K \) be a weak\(^*\)-compact subset of \( C_0[0, \omega_1]^* \). Then exactly one of the following two alternatives holds:

- \( K \) is uniformly Eberlein compact, *in the sense that \( K \) is homeomorphic to a weakly compact subset of a Hilbert space;*

- \( K \) contains a homeomorphic copy of \( [0, \omega_1] \) of the form

\[ \{ \rho + \lambda \delta_\alpha : \alpha \in D \} \cup \{ \rho \}, \]

where \( \rho \in C_0[0, \omega_1]^* \), \( \lambda \in \mathbb{C} \setminus \{0\} \), \( \delta_\alpha \) is the Dirac measure at \( \alpha \), and \( D \) is a closed and unbounded subset of \( [0, \omega_1] \).

**Note:**

(i) \([0, \omega_1]\) is not contained in any uniformly Eberlein compact space;

(ii) the unit ball of \( C_0[0, \omega_1]^* \) in the weak\(^*\) top. contains a homeomorphic copy of every uniformly Eberlein compact space of density at most \( \aleph_1 \).

**Operator-theoretic application:** consider \( K = T^* \) (the unit ball of \( C_0[0, \omega_1]^* \)) for \( T \in \mathcal{B}(C_0[0, \omega_1]) \).
Characterizations of the unique maximal ideal of $\mathcal{B}(C_0[0, \omega_1])$

**Theorem** (Kania–Koszmider–L). Let $T \in \mathcal{B}(C_0[0, \omega_1])$. Then TFAE:

(a) $T \in \mathcal{M}_{C_0[0, \omega_1]}$ (that is, $I \neq \text{STR}$ for all $R, S \in \mathcal{B}(C_0[0, \omega_1]))$;

(b) $T$ does not fix a copy of $C_0[0, \omega_1]$;

(c) $T$ is a Semadeni operator, in the sense that $T^{**}$ maps the subspace

$$\{ \Lambda \in C_0[0, \omega_1]^{**} : \langle \lambda_n, \Lambda \rangle \to 0 \text{ as } n \to \infty$$

for every weak*–null sequence $(\lambda_n)$ in $C_0[0, \omega_1]^*$

into the canonical copy of $C_0[0, \omega_1]$ in its bidual;

(d) there is a closed, unbounded subset $D$ of $[0, \omega_1)$ such that

$$(Tf)(\alpha) = 0 \quad (f \in C_0[0, \omega_1), \alpha \in D);$$

(e) $T$ factors through the Banach space $(\bigoplus_{\alpha < \omega_1} C[0, \alpha])_{c_0};$

(f) the range of $T$ is contained in a Hilbert-generated subspace of $C_0[0, \omega_1]$; that is, there exist a Hilbert space $H$ and an operator $U : H \to C_0[0, \omega_1)$ such that $T(C_0[0, \omega_1]) \subseteq \overline{U(H)}$;

(g) the range of $T$ is contained in a weakly compactly generated subspace of $C_0[0, \omega_1]$; that is, there exist a reflexive Banach space $X$ and an operator $V : X \to C_0[0, \omega_1)$ such that $T(C_0[0, \omega_1)) \subseteq \overline{V(X)}.$
The Szlenk index

Let $X$ be an Asplund space (that is, every separable subspace of $X$ has separable dual), and let $K \subset X^*$ be weak*-compact. Szlenk associated an ordinal $Sz\ K$ with $K$, its Szlenk index.

Set

$$Sz\ X = Sz(\text{the unit ball of } X^*).$$

(We extend this to all Banach spaces by $Sz\ X := \infty$ when $X$ is not Asplund.)

**Theorem** (Samuel 1983). $Sz\ C[0, \omega^\alpha] = \omega^{\alpha+1}$ for each countable ordinal $\alpha$.

More generally, for an operator $T : X \to Y$, define

$$Sz\ T = Sz(T^*(\text{the unit ball of } Y^*)).$$

(This may also be $\infty$.) For an ordinal $\alpha$, set

$$\mathcal{I}_\alpha^\mathcal{L}(X, Y) = \{ T \in \mathcal{B}(X, Y) : Sz\ T \leq \omega^\alpha \}.$$

**Theorem** (Brooker 2012). The class $\mathcal{I}_\alpha^\mathcal{L}$ is a closed, injective and surjective operator ideal in the sense of Pietsch for every ordinal $\alpha$. 

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The second-largest proper ideal of $\mathcal{B}(C_0[0,\omega_1])$

Set $E_{\omega_1} = (\bigoplus_{\alpha<\omega_1} C[0,\alpha])_{c_0}$, and recall that

$$T \in M_{C_0[0,\omega_1]} \iff T \text{ factors through } E_{\omega_1}.$$ 

**Theorem** (Kania–L). Let $T \in \mathcal{B}(C_0[0,\omega_1])$. Then TFAE:

(a) $T$ fixes a copy of $E_{\omega_1}$;
(b) the identity operator on $E_{\omega_1}$ factors through $T$;
(c) the Szlenk index of $T$ is uncountable.

**Corollary.** The set

$$\mathcal{I}_{E_{\omega_1}}(C_0[0,\omega_1)) = \{ T \in \mathcal{B}(C_0[0,\omega_1)) : T \text{ does not fix a copy of } E_{\omega_1} \}$$

$$= \{ T \in \mathcal{B}(C_0[0,\omega_1)) : \text{the identity operator on } E_{\omega_1}$$

$$\text{ does not factor through } T \}$$

$$= \{ T \in \mathcal{B}(C_0[0,\omega_1)) : \text{Sz } T < \omega_1 \} = \bigcup_{\alpha<\omega_1} \mathcal{I}_{\mathcal{L}_\alpha}(C_0[0,\omega_1))$$

is the second-largest proper closed ideal of $\mathcal{B}(C_0[0,\omega_1))$: for each proper ideal $\mathcal{I}$ of $\mathcal{B}(C_0[0,\omega_1))$, either $\mathcal{I} = M_{C_0[0,\omega_1)}$ or $\mathcal{I} \subseteq \mathcal{I}_{E_{\omega_1}}(C_0[0,\omega_1)).$
Partial structure of the lattice of closed ideals of $\mathcal{B} = \mathcal{B}(C_0[0, \omega_1])$

$$\mathcal{K} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{X} + \mathcal{G}_{c_0}(\omega_1) \hookrightarrow \bigcup_{\alpha < \omega_1} \mathcal{G}_{c_0}(\omega_1, C(K_\alpha)) \hookrightarrow \mathcal{I}_{E_\omega_1} \hookrightarrow \mathcal{M} = \mathcal{G}_{E_\omega_1}$$

$$\mathcal{G}_{c(K_{\alpha+1})} \hookrightarrow \mathcal{G}_{c(K_{\alpha+1})} \oplus c_0(\omega_1) \hookrightarrow \mathcal{G}_{c_0}(\omega_1, C(K_{\alpha+1})) \hookrightarrow \mathcal{I}_{\mathcal{L}_{\alpha+2}}$$

$$\mathcal{G}_{c(K_\alpha)} \hookrightarrow \mathcal{G}_{c(K_\alpha)} \oplus c_0(\omega_1) \hookrightarrow \mathcal{G}_{c_0}(\omega_1, C(K_\alpha)) \hookrightarrow \mathcal{I}_{\mathcal{L}_{\alpha+1}}$$

$$\mathcal{G}_{c(K_1)} \hookrightarrow \mathcal{G}_{c(K_1)} \oplus c_0(\omega_1) \hookrightarrow \mathcal{G}_{c_0}(\omega_1, C(K_1)) \hookrightarrow \mathcal{I}_{\mathcal{L}_2}$$

$$\mathcal{G}_{c_0} \hookrightarrow \mathcal{G}_{c_0}(\omega_1) \hookrightarrow \mathcal{I}_{\mathcal{L}_1}$$

$$\mathcal{K} \hookrightarrow \{0\}$$

$$K_\alpha = [0, \omega^\alpha], \alpha < \omega_1$$
Conventions

- We suppress $C_0[0, \omega_1)$ everywhere, thus writing $\mathcal{K}$ instead of $\mathcal{K}(C_0[0, \omega_1))$ for the ideal of compact operators on $C_0[0, \omega_1)$, etc.;

- $\mathcal{I} \hookrightarrow \mathcal{J}$ means that the ideal $\mathcal{I}$ is properly contained in the ideal $\mathcal{J}$;

- $\mathcal{I} \twoheadrightarrow \mathcal{J}$ indicates that there are no closed ideals between $\mathcal{I}$ and $\mathcal{J}$;

- $G_X$ denotes the set of operators that factor through the Banach space $X$ and $\overline{G}_X$ its closure;

- $c_0(\omega_1, X)$ denotes the $c_0$-direct sum of $\omega_1$ copies of the Banach space $X$, and $c_0(\omega_1) := c_0(\omega_1, \mathbb{C})$;

- $\mathcal{K}$ denotes the ideal of operators with separable range.
**Definition.** A linear mapping $\delta$ from a Banach algebra $\mathbb{A}$ into a Banach $\mathbb{A}$-bimodule is a *derivation* if

$$\delta(ab) = a \cdot \delta(b) + \delta(a) \cdot b \quad (a, b \in \mathbb{A}).$$

**Theorem** (Johnson 1967). *Let $X$ be a Banach space such that $X \cong X \oplus X$. Then every derivation from $\mathcal{B}(X)$ into a Banach $\mathcal{B}(X)$-bimodule is automatically continuous.*

**Question:** what happens when $X \not\cong X \oplus X$?

**Theorem** (Loy–Willis 1989). *Every derivation from $\mathcal{B}(C[0, \omega_1])$ into a Banach $\mathcal{B}(C[0, \omega_1])$-bimodule is automatically continuous.*

**Remark.** Around the same time, Read constructed a Banach space $X$ such that there is a discontinuous derivation from $\mathcal{B}(X)$ into a Banach $\mathcal{B}(X)$-bimodule.
Loy and Willis’ starting point: \( B(C[0,\omega_1]) \) contains a maximal ideal \( \mathcal{M} \) of codimension one.

Note: our work shows that \( \mathcal{M} = \mathcal{M}_{C[0,\omega_1]} \). We call \( \mathcal{M} \) the Loy–Willis ideal.

Loy and Willis’ key step: \( \mathcal{M} \) has a bounded right approximate identity, that is, a norm-bounded net \( (U_j) \) such that \( TU_j \rightarrow T \) for each \( T \in \mathcal{M} \).

Question: does \( \mathcal{M} \) also have a bounded left approximate identity, that is, a norm-bounded net \( (U_j) \) such that \( U_j T \rightarrow T \) for each \( T \in \mathcal{M} \)?

Answer: Yes! — In fact more is true:

**Theorem** (Kania–Koszmider–L). \( \mathcal{M} \) contains a net \( (Q_j) \) of projections with \( \|Q_j\| \leq 2 \) such that

\[
\forall T \in \mathcal{M} \exists j_0 \forall j \geq j_0 : Q_j T = T.
\]

**Corollary** (using Dixon 1973). \( \mathcal{M} \) has a bounded two-sided approximate identity.
A few references (in chronological order)


