Convergence of the sequence of powers of an element of a Banach algebra

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September 2014

Abstract

- We first discuss some of my joint work with Kamowitz on the powers of quasicompact endomorphisms of Banach algebras.
- We then discuss some work of David Moore (one of my current EPSRC-funded research students) concerning the essential spectral radius.

  In particular, we look at some extensions of Kamowitz's results.
- We discuss some necessary and sufficient conditions for the convergence of the sequence of powers of an element of a Banach algebra due to Koliha (1974).

Definitions and background

Unless otherwise specified, we will work with complex Banach spaces and Banach algebras throughout.

Let A be a commutative (complex) Banach algebra.
We denote by Φ_A the character space of A, and, for $a \in A$, we denote by $\hat{a}$ the Gelfand transform of a.

We recall that an algebra B is **semiprime** if $J = \{0\}$ is the only ideal in B with $J^2 = \{0\}$. Certainly semisimple algebras are semiprime.

It is standard that a **commutative** algebra B is semiprime if and only if B has no non-zero nilpotent elements.

A linear map T from a Banach algebra B to itself is an **algebra endomorphism** if T preserves multiplication.

Further if the algebra B is unital, then the endomorphism T is said to be **unital** if T maps the identity to itself.

In this case, $\phi := T^*|_{\Phi_A}$ is a selfmap of Φ_A; we shall call ϕ the selfmap of Φ_A associated with T. Note that then, for all $a \in A$, we have

$$\widehat{Ta} = \hat{a} \circ \phi\,.$$

In particular, if A is semisimple, then we may recover the endomorphism from the associated selfmap ϕ.

If A is not semisimple, then ϕ may give little information about the endomorphism T.

Even in the latter case, however, the existence or otherwise of fixed points of ϕ is relevant to our study of endomorphisms.

Let E be a Banach space.

Following Conway, we write $\mathrm{B}(E)$ and $\mathrm{B}_0(E)$ (respectively) for the algebras of bounded and of compact linear operators from E to itself.

Let A be a unital Banach algebra. Then $\mathrm{End}(A)$ denotes the set of all bounded unital algebra endomorphisms of A.
With T as above and $T + \mathrm{B}_0(E)$ the coset of T in the **Calkin algebra** $\mathrm{B}(E)/\mathrm{B}_0(E)$, we define the **essential spectrum** of T, $\sigma_e(T)$ by setting

$$\sigma_e(T) = \begin{cases} \emptyset & \text{if } E \text{ is finite dimensional,} \\ \sigma(T + \mathrm{B}_0(E)) & \text{otherwise,} \end{cases}$$

The **essential spectral radius**, $r_e(T)$, of T is then given by

$$r_e(T) = \begin{cases} 0 & \text{if } E \text{ is finite dimensional,} \\ \sup\{|\lambda| : \lambda \in \sigma_e(T)\} & \text{otherwise,} \end{cases}$$

This convention allows us to consider operators on finite and infinite dimensional spaces simultaneously, even though the Calkin algebra will be trivial in the former case.

By Atkinson's theorem, $\sigma_e(T)$ is also equal to the set of all $\lambda \in \mathbb{C}$ for which $\lambda - T$ is not a Fredholm operator.

We shall discuss operators T such that $r_e(T) < 1$.

Following Heuser such an operator T is called **quasicompact**.

If $r_e(T) = 0$ the operator T is a **Riesz operator**.

We have the following obvious implications for an operator T:

compact $\Rightarrow$ power compact $\Rightarrow$ Riesz $\Rightarrow$ quasicompact

Starting in the 1970's, Kamowitz studied compact endomorphisms of commutative Banach algebras. In 1980, he proved the following theorem.

Theorem (Kamowitz, 1980)

Let A be a semisimple, commutative, unital Banach algebra with character space Φ_A.

Let $T \in \mathrm{End}(A)$, and suppose that T is compact. Then $\bigcap_{k=1}^{\infty} T^{*k}(\Phi_A)$ is a finite set.

If Φ_A is connected, then $\bigcap_{k=1}^{\infty} T^{*k}(\Phi_A)$ is a singleton.

Since Kamowitz's initial work, a number of authors have studied various classes of endomorphisms of Banach algebras (often specialising to the case of uniform algebras).

In particular I obtained with Kamowitz the following results concerning quasicompact endomorphisms of commutative, unital semiprime Banach algebras.

Theorem (F.–Kamowitz 2010)

Let B be a unital commutative semiprime Banach algebra with connected character space, let T be a bounded, unital endomorphism of B, and let ϕ be the associated selfmap of Φ_B.

Then T is quasicompact if and only if the operators T^n converge in operator norm to a rank-one unital endomorphism of B.

In this case ϕ has a unique fixed point $x_0 \in \Phi_A$, and the rank-one endomorphism above must be the endomorphism $b \mapsto \hat{b}(x_0)1$.

In the case where the character space is disconnected, a standard method using orthogonal idempotents can be used, and gives us.

Theorem (F.–Kamowitz 2010)

Let B be a unital commutative semiprime Banach algebra, and let T be a bounded unital endomorphism of B.
Then T is quasicompact if and only if there is a natural number n such that the operators $(T^{kn})_{k=1}^{\infty}$ converge in operator norm to a finite-rank unital endomorphism of B.

This looked to be the end of the story. However:

- There was scope for considering suitable non-commutative Banach algebras.
- It looked as though there must be some more general results about convergence of powers of operators underlying these results.

One key part of the proofs of these results is that the set of eigenvalues of the endomorphism should be closed under powers.

This is a consequence of the lack of non-zero nilpotent elements in the algebra.

This is the condition we use in the non-commutative setting.

Theorem (F.-Moore 2013)

Let A be a unital Banach algebra with no non-zero nilpotent elements and no non-zero idempotents other than 1.

Let $T \in \mathrm{End}(A)$. Then T is quasi-compact if and only if there is a closed 2-sided ideal M of codimension 1 in A with $T(M) \subseteq M$ and such that $r(T_{|M}) < 1$.

In this case, the operators T^n converge in operator norm to a rank-one endomorphism of A with kernel M.

While looking for an efficient approach to proving the results of Feinstein and Kamowitz, Moore obtained a number of interesting results concerning the essential spectral radius.

In 1974, Brunel and Revuz gave the following characterisation of quasi-compactness for bounded operators on a Banach space. This result does not appear to be very well known.

Theorem (Brunel & Revuz 1974)

Let E be a Banach space and let $T \in \mathrm{B}(E)$. Then T is quasi-compact if and only if it is equal to the sum of a finite rank operator and an operator of spectral radius less than 1.

This result was re-discovered by Hong Ke Du (1993) who used it to obtain several other characterisations of quasi-compactness.

Moore's version of this theorem (proved independently) is a little stronger.
His methods involve careful use of the holomorphic functional calculus and spectral projections.

Theorem (Moore, 2013)

Let E be a complex Banach space, let $T \in \mathrm{B}(E)$ and let $\beta > 0$. Then $r_e(T) < \beta$ if and only if there is a finite rank idempotent $P \in \mathrm{B}(E)$ which commutes with T, which preserves all T-invariant closed subspaces of E and which satisfies $r(T(I - P)) < \beta$.

Using his stronger version of the theorem, Moore obtained the following inequalities concerning the essential spectral radius.

We have not been able to find these in the literature in this general setting.

Theorem (Moore 2013)

Let E be a complex Banach space, let $T \in \mathrm{B}(E)$ and let F be any closed T-invariant subspace of E.
Then $r_e(T_{|F}) \leq r_e(T)$ and $r_e(T/F) \leq r_e(T)$.

It is not always true that $\sigma_e(T_{|F}) \subseteq \sigma_e(T)$ for an arbitrary T-invariant closed subspace F: we are grateful to Jonathan Partington for pointing out some counterexamples to us.

One application of Moore's results is the following reinforcement of Kamowitz's result from 1980.

Theorem (Moore, 2013)

Let A be a commutative unital Banach algebra with connected character space Φ_A.
Let $T \in \mathrm{End}(A)$, and suppose that T is quasi-compact.
Then there is a character $\varphi \in \Phi_A$ such that $\bigcap\{T^{*k}(\Phi_A) : k \in \mathbb{N}\} = \{\varphi\}$.

A recurrent theme in these results is the convergence of the sequence T^n of powers of an operator T.

This raises the following slightly more general question: let A be a Banach algebra, and let $a \in A$. Can we find useful necessary and/or sufficient conditions for the convergence of the sequence a^n in A?

As part of our investigations into the applications of the holomorphic functional calculus and spectral projections, we obtained some sufficient conditions for the convergence of the sequence a^n.

It turns out, however, that these conditions are actually necessary and sufficient, and that this result was obtained in 1974 by J. J. Koliha.

In fact Koliha's results cover not only powers of elements, but also other 'power like' sequences of functions such as the Cesàro means of the sequence of powers.

We restrict ourselves to discussing the case of powers.

Theorem (Koliha 1974)

Let A be a Banach algebra, and let $a \in A$. Then the sequence of powers a^n converges in A if and only if the following two conditions hold:

(i) $\sigma(a) \setminus \{1\}$ *is a compact subset of the open unit disc;*

(ii) *if $1 \in \sigma(a)$, then 1 is a* ***simple*** *pole of the resolvent function $R_a(\lambda) := (\lambda - a)^{-1}$.*

Koliha's results can be used to give efficient proofs of our results about quasicompact endomorphisms, as Fredholm theory provides the criteria needed to prove that 1 is a simple pole in the given settings.

In order to prove the theorem above, Koliha first proved an easier result, which in the case of powers says the following.

Theorem (Koliha 1974)

Let A be a Banach algebra, and let $a \in A$. Then the sequence of powers a^n converges in A if and only if there are p and c in A such that $a = p + c$, $p^2 = p$, $pc = cp = 0$ and $r(c) < 1$.

If time permits, we will look at some of the details of the proofs.