Unitarizable representations and amenable operator algebras

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Today's menu

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1. Unitarizable representations
2. Amenable operator algebras?
3. Non-unitarizable representations
4. Unitarizable representations (reprise)
5. Open questions

What do I mean by "a representation (θ, Γ, A)"?

In this talk: a discrete group Γ, a unital C^*-algebra A, and a HM $\theta : \Gamma \to A_{\mathrm{inv}}$. We say the representation is bounded if $\sup_{x \in \Gamma} \|\theta(x)\| < \infty$.

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A (bounded) representation (θ, Γ, A) is unitarizable, or similar to a $*$-representation, if there exists $s \in A_{\mathrm{inv}}$ such that

$$s\theta(x)s^{-1} \in \mathcal{U}(A) \quad \text{for all } x \in \Gamma.$$

We say that s is a similarity element for θ.

Let $A_{\rm inv}^+ = A_{\rm inv} \cap A^+$. Then Γ acts on $A_{\rm inv}^+$, as follows;

$$\theta^+(x) : h \mapsto \theta(x) h \theta(x)^* \,.$$

Exercise

(θ, Γ, A) is unitarizable if and only if $\theta^+ : \Gamma \curvearrowright A_{\rm inv}^+$ has a fixed point.

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Exercise

Suppose Γ is finite and let $\theta : \Gamma \to A_{\text{inv}}$ be a HM. Show that the action $\theta^{+} : \Gamma \curvearrowright A_{\text{inv}}^{+}$ has a fixed point. (**Hint:** average over orbits.)

Thus every finite subgroup of A_{inv} is similar to a subgroup of $\mathcal{U}(A)$.

Example 1. Let $\varepsilon > 0$ and consider

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad y = \begin{pmatrix} 1 & \varepsilon \\ 0 & -1 \end{pmatrix}.$$

These give a pair of representations $\theta_x, \theta_y : \mathbb{Z}/2\mathbb{Z} \to (\mathbb{M}_2)_{\mathrm{inv}}$.

θ_x^+ and θ_y^+ act on $(\mathbb{M}_2)_{\mathrm{inv}}^+$, but have no common fixed point. Hence there is no $s \in (\mathbb{M}_2)_{\mathrm{inv}}$ which **simultaneously** unitarizes θ_x and θ_y.

Theorem (Day, 1950; Dixmier, 1950)

Let Γ be an amenable discrete group and $\mathcal{M}$ a von Neumann algebra. Then every bounded representation $(\theta, \Gamma, \mathcal{M})$ is unitarizable.

The case $\Gamma = \mathbb{Z}$, $\mathcal{M} = \mathcal{B}(\mathsf{H})$ was proved by Sz.-Nagy (1947) and contains the essential ideas for the general case.

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Theorem (Pisier, 2007)

If Γ is a discrete non-amenable group, then there is **some** *von Neumann algebra $\mathcal{M}$ and some bounded, non-unitarizable rep $(\theta, \Gamma, \mathcal{M})$.*

Unknown if we can always take $\mathcal{M} = \mathcal{B}(\mathcal{H})$!

Quick definition: a Banach algebra $\mathfrak{A}$ is amenable if it has a bounded approximate diagonal, i.e. a bounded net $(m_\alpha) \in \mathfrak{A} \widehat{\otimes} \mathfrak{A}$ satisfying $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $a\pi(m_\alpha) \to a$ for each $a \in \mathfrak{A}$.

Example 2. [Johnson, 1972] If Γ is a discrete amenable group, then $\ell^1(\Gamma)$ is amenable.

In particular, $\ell^1(\Gamma)$ is amenable whenever Γ is abelian.

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Some hereditary properties

- if $\mathfrak{A}$ is amenable and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a HM with dense range, then $\mathfrak{B}$ is amenable;
- if $\mathfrak{A}$ is a Banach algebra, $\mathfrak{J}$ is a closed ideal in $\mathfrak{A}$, and $\mathfrak{J}$ and $\mathfrak{A}/\mathfrak{J}$ are both amenable, then so is $\mathfrak{A}$.

Example 3. [Johnson, 1972] $C(X)$ and $\mathcal{K}(\mathsf{H})$ are amenable.

Both examples are closures of HM'ic images of $\ell^1(\Gamma)$, for some choice of amenable Γ.

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Remark

Johnson went on to show (1972) that every GCR **(i.e. Type I)** C^*-algebra is amenable. (In fact, strongly amenable.)
Also, the algebras $\mathcal{O}_n$, $2 \leq n \leq \infty$, are amenable (but not strongly amenable). [Rosenberg, 1977]

None of these proofs need the word "nuclear"

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Question.

Let $\mathfrak{A}$ be an amenable operator algebra. Must $\mathfrak{A}$ be isomorphic to (the underlying Banach algebra of) some C^*-algebra?

In the finite-dimensional setting, the answer is **YES**, by Wedderburn's theorem. This was pushed further by Gifford in his PhD thesis.

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Theorem (Gifford, 1997/2006)

Amenable, closed subalgebras of $\mathcal{K}(\mathrm{H})$ are isomorphic to C^*-algebras.

In full generality, this question resisted attempts over many years...

Let $\mathcal{Q}(\mathsf{H}) := \mathcal{B}(\mathsf{H})/\mathcal{K}(\mathsf{H})$ denote the Calkin algebra and $q : \mathcal{B}(\mathsf{H}) \to \mathcal{Q}(\mathsf{H})$ the quotient HM.

Question.

Is there a bounded, non-unitarizable rep $(\theta, \mathbb{Z}, \mathcal{Q}(\mathsf{H}))$?
What if we replace $\mathbb{Z}$ by some other discrete abelian group?

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The point of Ozawa's question: suppose Γ is abelian; then

- each bounded rep $(\theta, \Gamma, \mathcal{Q}(\mathsf{H}))$ gives an amenable $\mathfrak{A} \subset \mathcal{B}(\mathsf{H})$;
- if $\mathfrak{A}$ is isomorphic to a C^*-algebra then θ is unitarizable.

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So, a bounded **non-unitarizable** $(\theta, \Gamma, \mathcal{Q}(\mathsf{H}))$ gives rise to an amenable operator algebra not isomorphic to any C^*-algebra.

Details

Given $\theta : \Gamma \to \mathcal{Q}(\mathsf{H})$ define $\mathfrak{B} = \overline{\text{lin}}\{\theta(x) : x \in \Gamma\}$. $\mathfrak{B}$ is amenable.

Let $\mathfrak{A} = q^{-1}(\mathfrak{B})$. There is a short exact sequence

$$0 \to \mathcal{K}(\mathsf{H}) \to \mathfrak{A} \xrightarrow{q} \mathfrak{B} \to 0$$

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By hereditary properties, $\mathfrak{A}$ is an amenable operator algebra.

Now suppose $\mathfrak{A}$ is also isomorphic to a C^*-algebra. Then there exists $R \in \mathcal{B}(\mathsf{H})_{\text{inv}}$ such that $R\mathfrak{A}R^{-1}$ is a **self-adjoint subalgebra** of $\mathcal{B}(\mathsf{H})$.

Put $s := q(R)$. Then $s\mathfrak{B}s^{-1}$ is a commutative and self-adjoint subalgebra of $\mathcal{Q}(\mathsf{H})$. Observe: if $x \in \Gamma$, then $s\theta(x)s^{-1}$ is normal with spectrum contained in $\mathbb{T}$, hence is unitary. So s unitarizes θ.

Theorem (see `arXiv:1309.2415v1`)

There is a set $\mathfrak{T}$ of bounded HMs $\mathbb{Z}^{\oplus \mathfrak{c}} \to \mathcal{Q}(\ell_2)$, with $|\mathfrak{T}| = 2^{\mathfrak{c}}$, such that

- *$\mathfrak{T}$ is parametrized by certain "1-cocycles"*
- *Let $\theta \in \mathfrak{T}$; then $(\theta, \mathbb{Z}^{\oplus \mathfrak{c}}, \mathcal{Q}(\ell_2))$ is unitarizable iff θ corresponds to an* **"inner" cocycle**.

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But $|\{\text{inner cocycles}\}| \leq \mathcal{Q}(\ell_2) = \mathfrak{c} < 2^{\mathfrak{c}} = |\mathfrak{T}|$. Therefore

there is a bounded, non-unitarizable $(\theta, \mathbb{Z}^{\oplus\mathfrak{c}}, \mathcal{Q}(\ell_2))$

Hence, by Ozawa's ingenious observation, our infamous question has the answer **NO**.

Corollary (Farah, Ozawa, ibid.)

There is an amenable closed $\mathfrak{A} \subset \mathcal{B}(\ell_2)$ not isomorphic to any C^*-algebra.

A surprising feature is that $\mathfrak{A}$ is "locally very nice":

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A surprising feature is that $\mathfrak{A}$ is "locally very nice":

for every countable subset $X \subset \mathfrak{c}$, the rep $(\theta, \mathbb{Z}^{\oplus X}, \mathcal{Q}(\ell_2))$ **is** unitarizable.

Thus $\mathfrak{A} = \varinjlim_X \mathfrak{A}_X$, where each $\mathfrak{A}_X$ is separable, amenable and similar to a C^*-algebra.

Moreover, similarity elements s_X exist with $\sup_X \|s_X\| \left\|s_X^{-1}\right\| < \infty$.

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Theorem (C.–Farah–Ozawa, 2014)

Let $\mathcal{C} = \ell^\infty / c_0$. *There is a bounded HM* $\theta : (\mathbb{Z}/2\mathbb{Z})^{\oplus \aleph_1} \to \mathcal{C} \otimes \mathbb{M}_2$ *which is not unitarizable (inside* $\mathcal{C} \otimes \mathbb{M}_2$*).*

This gives rise to $\mathfrak{A} \subset \ell^\infty \otimes \mathbb{M}_2$ which has density character $\aleph_1$, and is amenable, but not isomorphic to any C^*-algebra.

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Remark

Any amenable closed subalgebra of ℓ^∞ is automatically self-adjoint (corollary of Sheinberg, 1977), hence isomorphic to some $C_0(X)$.

Let $\Gamma = (\mathbb{Z}/2\mathbb{Z})^{\oplus \aleph_1}$.

We will construct two commuting, bounded representations $\theta_x, \theta_y : \Gamma \to \mathcal{C} \otimes \mathbb{M}_2$ which cannot be simultaneously unitarized.

Then $\theta_x \times \theta_y : \Gamma \times \Gamma \to (\ell^\infty / c_0) \otimes \mathbb{M}_2$ is the desired bounded but non-unitarizable representation.

All the real work takes place inside $\mathcal{C} = \ell_\infty / c_0 = C(\beta\mathbb{N} \setminus \mathbb{N})$.

We can find $\mathcal{F}, \mathcal{G} \subset 2^{\mathbb{N}}$, with $|\mathcal{F}| = |\mathcal{G}| = \aleph_1$, such that

$$(q(1_\mathsf{J}))_{\mathsf{J}\in\mathcal{F}} \cup (q(1_\mathsf{K}))_{\mathsf{K}\in\mathcal{G}}$$

is a family of non-zero, pairwise-orthogonal projections in ℓ_∞/c_0.

We can also arrange for the following condition to hold.

"Gap condition"

For each partition $\mathbb{N} = \mathsf{X} \sqcup \mathsf{Y}$, either there exists $\mathsf{J} \in \mathcal{F}$ such that $\mathsf{X} \cap \mathsf{J}$ is infinite, or there exists $\mathsf{K} \in \mathcal{G}$ such that $\mathsf{Y} \cap \mathsf{K}$ is infinite.

Pick two involutions $x, y \in \mathbb{M}_2$ such that θ_x^+, θ_y^+ have no common fixed point. (We saw easy examples earlier!)

For each $\mathsf{J} \in \mathcal{F}$ and $\mathsf{K} \in \mathcal{G}$, define involutions in $\ell_\infty \otimes \mathbb{M}_2$ by

$$x_{\mathsf{J}} = 1_{\mathsf{J}} \otimes x + 1_{\mathbb{N}\setminus\mathsf{J}} \otimes I_2 \quad \text{and} \quad y_{\mathsf{K}} = 1_{\mathsf{K}} \otimes y + 1_{\mathbb{N}\setminus\mathsf{K}} \otimes I_2.$$

Define $\Theta_x : \Gamma \to (\ell_\infty/c_0) \otimes \mathbb{M}_2$ by $\Theta_x(e_{\mathsf{J}}) = (q \otimes \mathrm{id})(x_{\mathsf{J}})$, and define Θ_y similarly. These representations of Γ are bounded, **and their ranges commute**, as required.

We get actions $\Theta_x^+, \Theta_y^+ : \Gamma \curvearrowright ((\ell_\infty/c_0) \otimes \mathbb{M}_2)^+_{\mathrm{inv}}$.

It suffices to show these actions have no common fixed point.

Suppose Θ_x^+ and Θ_y^+ have a common fixed point, say $q(s)$ for some positive invertible $s = (s_n) \in \ell_\infty \otimes \mathbb{M}_2$. A little work shows that

- $\lim_{n\in J} \operatorname{dist}(s_n, \operatorname{Fix}(\theta_x^+)) = 0$, for all $\mathsf{J} \in \mathcal{F}$;
- $\lim_{n\in K} \operatorname{dist}(s_n, \operatorname{Fix}(\theta_y^+)) = 0$, for all $\mathsf{K} \in \mathcal{G}$.

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But our choice of x and y turns out to force

$$\inf_n \operatorname{dist}(s_n, \operatorname{Fix}(\theta_x^+)) + \operatorname{dist}(s_n, \operatorname{Fix}(\theta_y^+)) = \delta > 0.$$

From these we get $\mathsf{X}, \mathsf{Y} \subseteq \mathbb{N}$, with $\mathsf{X} \cup \mathsf{Y} = \mathbb{N}$, such that

$$|\mathsf{X} \cap \mathsf{J}| < \infty \quad \text{for all } \mathsf{J} \in \mathcal{F} \quad , \quad |\mathsf{Y} \cap \mathsf{K}| < \infty \quad \text{for all } \mathsf{K} \in \mathcal{G}.$$

This contradicts the gap condition. So no such s exists. □

We found non-unitarizable reps $(\theta, \Gamma, \mathcal{Q}(\mathsf{H}))$, but required uncountable Γ.
Recall: Ozawa's **original question** was

might there be a non-unitarizable $(\theta, \mathbb{Z}, \mathcal{Q}(\mathsf{H}))$?

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might there be a non-unitarizable $(\theta, \mathbb{Z}, \mathcal{Q}(\mathsf{H}))$?

Answer: **NO.**

Theorem (C.–Farah–Ozawa, *ibid.*)

Let Γ be a **countable** *amenable group. Then every bounded representation $(\theta, \Gamma, \mathcal{Q}(\mathsf{H}))$ is unitarizable.*

The same is true if we replace the Calkin algebra by certain other algebras, e.g. $\prod_n \mathbb{M}_n / \bigoplus \mathbb{M}_n$ or $(\ell^\infty / c_0) \otimes \mathbb{M}_n$ or ultraproducts of a sequence of C^*-algebras.

We need to find a fixed point of the action $\theta^+ : \Gamma \curvearrowright A_{\mathrm{inv}}^+$.

$$\theta^+(x)(h) := \theta(x)h\theta(x)^* \qquad (x \in \Gamma, h \in A_{\mathrm{inv}}^+).$$

A standard theme: when looking for a fixed point of a (semi)group action, try to take an "average over an orbit".

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So now suppose Γ has a Følner **sequence** (F_n). Put

$$h_n = \frac{1}{|F_n|} \sum_{y \in F_n} \theta(y)\theta(y)^* \ .$$

Then for any $x \in \Gamma$,

$$\|\theta(x)h_n\theta(x)^* - h_n\| \leq |F_n|^{-1} |xF_n \triangle F_n| \, \|\theta\|^2 \to 0 \, ,$$

so (h_n) is an "asymptotically invariant" sequence in A^+_{inv}.

The key point

If A has a certain "countable saturation property", tools from the metric model theory of C^*-algebras allow us to construct the desired h from the sequence (h_n).

(These tools are an axiomatic version of ideas used by G. K. Pedersen to study derivations from separable C^*-algebras into corona algebras.)

Theorem (C., 2013)

Let A be a closed, commutative subalgebra of a **finite** *von Neumann algebra. If A is amenable, then it is isomorphic to $C_0(X)$ for some X.*

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Theorem (Marcoux–Popov, 2013 preprint)

Let A be a closed, commutative subalgebra of $\mathcal{B}(\mathsf{H})$. If A is amenable, then it is isomorphic to $C_0(X)$ for some X.

In both papers, one uses amenability to show that the Gelfand transform $A \to C_0(\Phi_A)$ is bounded below. (From there the rest is a standard application of Sheinberg's theorem.)

Question.

Let A be a **separable** closed subalgebra of $\mathcal{B}(\mathrm{H})$. If A is amenable, must it be isomorphic to a C^*-algebra?

Question.

What about amenable subalgebras of e.g. the CAR algebra?

The final question was suggested to me by S. A. White.

Question.

Let A be a **weak*-closed**, "Connes-amenable" subalgebra of $\mathcal{B}(\mathrm{H})$. Must it be isomorphic to a von Neumann algebra?