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David Stewart, Maximal subalgebras of exceptional Lie algebras

1880 Sophus Lie, Math. Ann. "Theorie des transformationsgruppen"
Asks for a classification of primitive actions of Lie groups
$\rightsquigarrow$ maximal subalgebras of Lie algebras
Included classification of all Lie algebras in dim's 1, 2, 3

1950s Dynkin classified the maximal subalgebras of simple Lie algebras over $\mathbb{C}$. Two papers
- exceptional types (longer paper)
- classical types

In classical types one considers the action of a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ on the natural module, V for $\mathfrak{g}$. If $V|_{\mathfrak{h}}$ is reducible then the problem is easy. Eventually reduce to the problem of finding $\mathfrak{h} \subset \mathrm{Iso}(V)$ s.t. $\mathfrak{h}$ is simple & $V|_{\mathfrak{h}}$ is irreducible.

Example $\mathfrak{sl}_2$ has irreducible rep's of all (finite) dimensions.

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{so}_{2n+1}$$
$$\mathfrak{sl}_2 \longrightarrow \mathfrak{sp}_{2n}$$

Always a maximal subalgebra except $\mathfrak{sl}_2 \to \mathfrak{so}_7$ where we have $\mathfrak{sl}_2 \subset G_2 \subset \mathfrak{so}_7$

1987, 1991, 2004: Seitz / Liebeck-Seitz / Testerman
Memoirs AMS

Classification of positive-dimensional maximal subgroups of simple algebraic groups. Extra: e.g. $p=3$, $\tilde{A}_2 \leq G_2 \leq SO_7$ irreducible.

Problem Let $\mathfrak{g} = \mathrm{Lie}(G)$ where G is a simple algebraic group over $k = \bar{k}$, char $k = p > 0$.
Determine the maximal subalgebras of $\mathfrak{g}$.

From now on let G be of exceptional type, p good prime for G. Since p is good, ~~we have~~ it is at least 5 so we have a classification of simple modular Lie algebras (Premet-Strade). e.g., Witt algebra $W_1 = \mathrm{Der}\left(k[X]/(X^p) \right)$

For $p \geq 5$ W_1 is simple

W_1 has basis $\{ \partial_X, X\partial_X, \ldots, X^{p-1}\partial_X \}$ $\quad \partial = \partial_X$

$$[X^i\partial, X^j\partial] = (i-j)X^{i+j-1}\partial$$

We can generalize this in two directions:

$$\begin{tikzcd}
W_1 \arrow[r, squiggly, "\text{add variables}"] \arrow[d, squiggly, "\substack{\text{work with}\\ \text{divided}\\ \text{powers}}"'] & W(n, \underline{1}) \arrow[d, squiggly] \\
W(1,n) \arrow[r, squiggly] & W(n, \underline{m})
\end{tikzcd}$$

Zassenhaus algebras

Preserving differential forms gives further simple Lie algebras H, K, S and we need to take filtered deformations.

We also have Melikyan algebras in characteristic 5.

<u>Theorem</u> (Herpel - S.) Let $\mathfrak{g} = \mathrm{Lie}(G)$ be simple of exceptional type. Let $\mathfrak{h}$ be a ~~(maximal)~~ p-subalgebra of $\mathfrak{g}$, and assume $\mathfrak{h}$ is simple of type W_1.

Then ∂ is represented by a nilpotent element, regular in a Levi subalgebra $\mathfrak{l}$ with connected root system
$p = h(\mathfrak{l}) + 1$ where $h(\mathfrak{l})$ is the Coxeter number of $\mathfrak{l}$ (or $[\mathfrak{l}, \mathfrak{l}] \cong \mathfrak{sl}_p$)

$\mathfrak{h}$ is maximal $\iff$ ∂ is regular in $\mathfrak{g}$ except $W_1 < F_4 < E_6$

Moreover, there is one conjugacy class of such.

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Ideas: we need $\partial \in \mathrm{im}(\mathrm{ad}\, \partial)^{p-1}$ & $\partial^{[p]} = 0$

Theorem (Herpel - S.)
If h is simple in G then either $h \cong W_1$ or h is classical (A - G types).

Proof used information about W_1 subalgebras of the ~~contact~~ Hamiltonian algebras H.

Theorem (Morozov) If $\mathfrak{g}$ is a simple Lie algebra $/\mathbb{C}$ & h is a maximal non-semisimple subalgebra then h is a parabolic subalgebra.
Initially proved in 1943, later a cleaned up version appeared in 1956.

The proof goes wrong if you attempt to replicate it for exceptional Lie algebras $\mathfrak{g}$ in positive characteristic.

Theorem (Premet) The conclusion of Morozov holds for very good characteristic.

- Uses Weisfeiler filtrations, Block's theorems
- GIT
- representations of Hamiltonians
- classifies all p-balanced elements in $\mathfrak{g}$ (using Kac coordinates, sheets of $\mathfrak{g}$ in good characteristic (Premet - S.))

Rk All the previous theorems are incorrect in bad characteristic
Premet found a subalgebra $E_8 \supset h \cong W_2 + (\mathcal{O}_2/_{k}1)^*$
Premet's student Purslow found a subalgebra $W(1;2) \subset E_8$ $p=5$ as well as a 124-dimensional subalgebra of E_8 when $p=2$.

Kubiesa found a 26-dimensional simple maximal subalgebra of F_4 when $p=3$. This may be a new simple Lie algebra.

Direct sums of simples can be handled by an argument of Hespel/ Bate-Martin-Röhrle.

All centralizers of subgroup schemes of G are smooth in very good characteristic. Also use results of Liebeck-Seitz.

<u>Theorem</u> (Premet-S.) $\mathfrak{g} = \mathrm{Lie}(G)$, G exceptional, $\mathfrak{h} \subset \mathfrak{g}$. p good.

Let $\mathfrak{h}$ be simple, maximal of classical type

Then $\mathfrak{h} = \mathrm{Lie}(H)$ for H a maximal connected closed subgroup of G.

<u>Theorem</u> (Premet-S.) Let $\mathfrak{h}$ be semisimple with unique minimal ideal ($\mathfrak{h} \overset{\subset}{\cong} S \otimes \mathcal{O}(m,n) + 1 \otimes W(m,n)$, which turns out to be semisimple because $W(m,n)$ acts by derivations on $\mathcal{O}(m,n)$). Then have a classification of such maximal subalgebras $(\mathfrak{g}, p) = (E_7, 5)$ $(E_7, 7)$ or $(E_8, 7)$

We know the isomorphism types of $\mathfrak{h}$.