Representation theory and symplectic singularities

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New and old trends in PI theory

Claudio Procesi:

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Polynomial identities are at the crossroad between

Non commutative Algebra

Representation Theory

Algebraic Geometry

A bird's eye view

I will point out some of the main aspects of the theory, also as advertisement of a forthcoming book:

Rings with polynomial identities and finite dimensional representations of algebras

E. Aljadeff, A. Giambruno, C. Procesi, A. Regev

Symbolic calculus

A basic tool in algebra, which is also the base of most algorithms and computer programs is

Symbolic calculus

In such a calculus we are given some operations and using some formalism, as for instance parentheses one establishes a given set of *rules, or identities* as the symbolic calculus of elementary algebra

operations $+, \ \times, \ -,$ rules $x + y = y + x, \ x \times y = y \times x,$ etc..

Polynomial identities

In this talk I will discuss only non commutative associative algebras for which we need to use non commutative polynomials.

Look at 2×2 matrices, Wagner identity

A matrix always satisfies its characteristic polynomial, this is the Cayley–Hamilton Theorem.

- For a 2×2 matrix X this means

$$X^2 - tr(X)X + \det(X) = 0.$$

- If $tr(X) = 0$ then $X^2 = -\det(X)$ is a scalar matrix so $[X^2, Y] = X^2 Y - YX^2 = 0$.
- It follows that for any 3, 2×2 matrices, X, Y, Z we have

$$\text{Wagner identity} \quad [[X, Y]^2, Z] = 0.$$

Polynomial identities

We have just seen that there is a non zero (non commutative) polynomial, which vanishes identically when computed on 2×2 matrices. This then suggests the

Definition

A non zero non commutative polynomial $f(x_1, \dots, x_m) \in F\langle X \rangle$ is a *polynomial identity* for an algebra R if it vanishes identically when computed in R.

From this definition several questions arise.

Basic questions

1. Which algebras satisfy polynomial identities?
2. In other words what are the implications of the existence of some polynomial identities on an algebra?
3. Which are the polynomial identities of a given algebra? For instance a matrix algebra.
4. What about the resulting symbolic calculus?

As we shall see each of this questions corresponds to a large piece of theory and the last two questions have only rather implicit answers.

The theory was prompted by Kurosh problem

The Burnside problem

posed by William Burnside in 1902: Is every finitely generated torsion group G finite?
Kurosh is every finitely generated algebraic algebra A, over a field F, finite dimensional?
Both solved in the negative In 1964, Golod and Shafarevich.

Bounded problems

In the bounded Burnside problem one assumes that every element $x \in G$ satisfies $x^n = 1$ for *fixed* n.

In the bounded Kurosh problem one assumes that every element $x \in A$ satisfies some polynomial $x^n + a_1 x^{n-1} + \ldots + a_1 x = 0,\ a_i \in F$ for *fixed* n.

A surprising answer

the bounded Burnside problem has sometimes a positive sometimes a negative answer

The bounded Kurosh problem has a positive answer

An algebraic algebra of bounded degree satisfies a polynomial identity (Jacobson).
A finitely generated algebraic algebra A which satisfies a polynomial identity is finite dimensional. (Levitzki and Kaplansky).

Examples

The commutative law $xy - yx$ is a polynomial identity, so:
all commutative algebras satisfy a polynomial identity.
It is easy to see that all finite dimensional algebras satisfy some polynomial identity.

A mixed theory

The theory of polynomial identities mixes methods of commutative algebra with methods of finite dimensional algebras.
The blending agent is representation theory and invariant theory.

The example of matrices

The algebra $M_n(A)$ of $n \times n$ matrices over a commutative ring A is the basic example. The identity of minimal degree is given by the

Theorem

[Amitsur–Levitzki] The algebra of $n \times n$ matrices over any commutative ring A satisfies the standard polynomial St_{2n}

$$St_{2n} := \sum_{\sigma \in S_{2n}} \epsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)}$$

Generic matrices

The free algebra $F\langle x_1, \dots, x_m\rangle$ modulo the ideal J_n of polynomial identities of $n \times n$ matrices should be thought of as the algebra of polynomial functions of matrix variables.

$$F\langle x_1, \dots, x_m\rangle / J_n = F\langle \xi_1, \dots, \xi_m\rangle$$

it has a rich and complex structure.

- $F\langle \xi_1, \ldots, \xi_m \rangle$ is an integral domain (Amitsur).
- $F\langle \xi_1, \ldots, \xi_m \rangle$ has a quotient division algebra of fractions $D(m, n)$ of dimension n^2 over its center $Z(m, n)$ (Amitsur).
- $Z(m, n)$ is the field of rational functions on the space $M_n(F)^m$ of m–tuples of matrices, invariant under conjugation (Procesi).

This suggests to study the ring of polynomial functions $F : M_n(F)^m \to M_n(F)$ which are equivariant under conjugation.

$$f(gX_1g^{-1}, \dots, gX_ng^{-1}) = gf(X_1, \dots, X_n)g^{-1}, \ \forall g \in GL(n, F).$$

We can thus consider

1. The ring T of invariants generated by all the coefficients of the characteristic polynomials of elements f of $F\langle \xi_1, \dots, \xi_m \rangle$, in fact generated just when f is a *primitive monomial*.
2. The ring $T\langle \xi_1, \dots, \xi_m \rangle$ generated by $F\langle \xi_1, \dots, \xi_m \rangle$ and T.
3. We have natural inclusions

$$F\langle \xi_1, \dots, \xi_m \rangle \subset T\langle \xi_1, \dots, \xi_m \rangle \subset D(m, n).$$

Example $m = 2,\ n = 2$ (like quaternions)

In this case T is the polynomial ring in 5 variables

$$\boxed{tr(x),\ tr(y),\ \det(x),\ \det(y),\ tr(xy)}.$$

$$T\langle x, y\rangle$$

is a free module of rank 4 over T with basis $1, x, y, xy$.

$$x^2 - tr(x)x + \det(x) = y^2 - tr(y)y + \det(y) = 0,$$

$$xy + yx - tr(x)y - tr(y)x + tr(x)ty(y) - tr(xy) = 0.$$

A MAIN THEOREM is that

$T\langle \xi_1, \dots, \xi_m \rangle$ equals the algebra of polynomial functions $F : M_n(F)^m \to M_n(F)$ which are equivariant under conjugation.

This is a finitely generated module over the ring $T(m, n) := T$ of polynomial functions $F : M_n(F)^m \to F$ which are invariant under conjugation.

This is more or less classical in characteristic 0 and it follows from a deep Theorem of Donkin in all characteristics.

The spectrum of $T(m, n)$

The ring of invariants T parametrizes equivalence classes of n–dimensional semi–simple representations of the free algebra in m variables.

Except for the trivial case $n = 1$ or $n = m = 2$, its smooth part parametrizes irreducible n–dimensional representations.

We call this smooth variety $X(m, n)$.

The spectrum of a PI (polynomial identity) algebra

For PI algebras one has several theorems which resemble the theorems of commutative algebra, the main difference is that the spectrum is divided into natural strata, let us see it in a special case.

Nullstellensatz Let $R = F[a_1, \ldots, a_m]$ be a a finitely generated algebra over F algebraically closed and satisfying a PI of degree $2d$ then.

Theorem (Nullstellensatz Procesi–Razmyslov)

If M is a maximal ideal of R then $R/M \sim M_k(F),\ k \leq d$.
$\cap_{M \text{ maximal ideal}} M$ is a maximal nilpotent ideal.

The spectrum of generic matrices

Denote the spectrum of m generic $n \times n$ matrices by $Y(m, n)$ then

$$Y(m, n-1) \subset Y(m, n), \quad Y(m, n) \setminus Y(m, n-1) = X(m, n)$$

recall $X(m, n)$ is a smooth variety parametrizing irreducible n–dimensional representations of the free algebra in m variables. As a consequence for the spectrum we have:

$$Y(m, n) = \cup_{i=1}^{n} X(m, i).$$

Azumaya algebras

An Azumaya algebra R over its center Z of fixed rank n^2 is a *non split form of matrices* that is an algebra which, under a faithfully flat (even étale) extension of its center $Z \subset B$, becomes matrices

$$R \otimes_Z B = M_n(B).$$

It should be though of geometrically as a *a principal* $PGL(n, F)$ *bundle* over $Spec(Z)$.
Related to *Severi–Brauer varieties*.

The Theorem of Mike Artin

We have a very strong statement in case in some sense the spectrum is made of matrices of the same size.

Theorem (Artin)

Assume that R is an algebra which satisfies all polynomial identities of $n \times n$ matrices *for some n.*
Assume further that there is no quotient R/I which satisfies a polynomial identity of $n-1 \times n-1$ matrices which is not an identity of $n \times n$ matrices.
Then R is a rank n^2 Aumaya algebra over its center Z.

The theorem was extended to R any ring by Procesi.

The lack of Noetherian property T–ideals

It is easily seen that finitely generated PI algebras are almost never Noetherian, the Hilbert basis theorem does not hold even for generic matrices.

T–ideals

In the free algebra the ideals J of polynomial identities of some algebra R are very special, and are called *T–ideals.*

Characterization of T–ideals

T–ideals are characterized by the fact of being stable when one substitutes the variables with elements of the free algebra, that is they are stable under endomorphisms of the free algebra.

Specht problem

The following Theorem of Kemer answers a basic question of Specht:

A Noetherian condition

Over a field of characteristic 0 all T–ideals of a free algebra are finitely generated *as T–ideals*.

This theorem is connected with a basic embedding problem.

The representability question

Problem

Can a PI algebra R be embedded into a ring $M_n(A)$ of $n \times n$ matrices over a commutative ring A? If the answer is positive we say that R is representable.

The answer is twofold, if R has no nil ideals then the answer is positive, also for Azumaya algebras it is positive otherwise, even if R is finitely generated the answer is in general negative!

So what makes a PI algebra embeddable into matrices?

This is the representability question.

A theorem of Kemer

Over a field of characteristic 0.

Theorem

If $R = F\langle X\rangle/J$ is the quotient of a free algebra in finitely many variables modulo a T–ideal J, then R is representable.

Notice that a T–ideal J defines what is called a *variety of algebras*, that is all algebras which satisfy the identities of J.
The algebra $R = F\langle X\rangle/J$ is a free object in the variables X in this variety and thus it is called a *relatively free algebra*.

A theorem of Kemer

The previous Theorem is equivalent to the following

Theorem

If R is a finitely generated PI algebra then R satisfies the same polynomial identities of a finite dimensional algebra.

A super theorem

By rather deep and hidden reasons, *super–algebras* play a role in the theory of polynomial identities. A superalgebra is just an algebra with a $\mathbb{Z}/(2)$ grading:

$$A = A_0 \oplus A_1, \quad A_0^2 \subset A_0,\ A_0 A_1 \subset A_1,\ A_1 A_0 \subset A_1,\ A_1^2 \subset A_0.$$

A basic super–algebra is the infinite Grassmann algebra $G = \bigwedge V$ where V is an infinite dimensional vector space.
Given a super algebra $A = A_0 \oplus A_1$.

Definition (Grassmann envelope)

The Grassmann envelope of a super algebra $A = A_0 \oplus A_1$ is the super algebra

$$G(A) = A_0 \otimes G_0 \oplus A_1 \otimes G_1.$$

A super theorem of Kemer

Theorem

If R is a PI algebra then R satisfies the same polynomial identities of the Grassmann envelope $G(A)$ of a finite dimensional super–algebra A.

The inverse Cayley–Hamilton Theorem char. 0

The role of trace

Theorems are much more precise when one introduces, in the symbolic calculus, also the operation of *trace*.

The characteristic polynomial $\chi_n(X)$ can be viewed as a formal expression in the variable X and the variables $tr(X^i)$ as for instance for $n = 2$.

$$\boxed{\chi_2(X) = X^2 - tr(X)X + \frac{1}{2}(tr(X)^2 - tr(X^2))}.$$

The fact that a matrix X satisfies its characteristic polynomial $\chi_n(X)$ can be understood as some kind of generalization of polynomial identity, a *trace identity*.

Using the same language of symbolic algebra one can define the

1. *category of algebras with trace.*
2. *The free algebras with trace,*
3. *and the notion of trace identity for an algebra.*

Cayley–Hamilton algebra

One can define an *n Cayley–Hamilton algebra* as an algebra with trace R, over $\mathbb{Q}$, such that each of its elements satisfies the formal Cayley–Hamilton identity $\chi_n(X) = 0$.

Trace identities of matrices and Young symmetrizers

1. *all trace identities of $n \times n$ matrices can be deduced from the Cayley–Hamilton identity (Procesi–Razmyslov).*
2. *The space of multilinear trace identities of $n \times n$ matrices of degree m can be identified with the two sided ideal of the group algebra of the symmetric group S_{m+1} sum of irreducible blocks with height $\geq n+1$.*

In positive characteristic one has to replace traces by determinants, one has an analogue (much more difficult) theorem by Zubkov.

A basic theorem (characteristic 0)

For Cayley–Hamilton algebras we have a canonical positive answer to the representability question:

Theorem

An n Cayley–Hamilton algebra R has a canonical embedding into $n \times n$ matrices over a universal commutative ring A compatible with the trace map.

A basic theorem

In fact the projective group $PGL(n, \mathbb{Q})$ acts on A and on $M_n(\mathbb{Q})$ so it acts on $M_n(A) = A \otimes M_n(\mathbb{Q})$ and

Theorem

An n Cayley–Hamilton algebra R under the universal embedding $i : R \to M_n(A)$ is isomorphic to $M_n(A)^{PGL(n,\mathbb{Q})}$.

Open question

Good filtrations

1. What are Cayley–Hamilton algebras in positive characteristic?
2. And, do we have a canonical positive answer to the representability question?
3. The study in characteristic $p > 0$ requires the theory of good filtrations.

We have already recalled Kemer's theorem on the representability of the relatively free algebras in finitely many variables, that is algebras of the form $R = F\langle x_1, \ldots, x_k\rangle / I$ with I a T–ideal.
In fact these algebras have a deep structure

Theorem

1. *We have a filtration* $0 \subset K_0 \subset K_1 \subset \ldots \subset K_u = R$ *of* T *ideals.*
2. *Each* K_{i+1}/K_i *has a structure of a finitely generated module over a finitely generated commutative algebra* $\mathcal{T}_{\bar{R}_i}$ *associated to* $R_i := R/K_i$.

Relatively free algebras

Corollary (Belov)

If $R := F\langle X\rangle/I$ is a relatively free algebra in a finite number of variables X, its Hilbert series

$$H_R(\rho) := \sum_{k=0}^{\infty} \dim(R_i)\,\rho^i$$

is a rational function of the form

$$\frac{p(\rho)}{\prod_{j=1}^{N}(1-\rho^{h_j})}, \qquad h_j \in \mathbb{N}, \quad p(\rho) \in \mathbb{Z}[\rho]. \tag{1}$$

Codimension and growth

The codimension $cd_R(n)$ of an algebra R is the dimension of the space (of dimension $n!$) of multilinear polynomials in n variables modulo the polynomial identities.

Two basic theorems on codimensions are

Theorem (Regev)

There is some positive number ℓ such that $cd_R(n) \leq \ell^n$

Theorem (Giambruno–Zaicev)

$\lim_{n\to\infty} cd_R(n)^{\frac{1}{n}}$ *exists and it is a positive integer the exponent.*

There are more precise results on the *asymptotic behaviour* of $cd_R(n)$.

A basic result of Regev is for $R = M_k(F)$ (F a field of characteristic 0)

Theorem

$$cd_{M_k(F)}(n) \sim A_k n^{\frac{1-k^2}{2}} k^{2(n+1)}. \tag{2}$$

where the constant

$$A_k = \Big(\frac{1}{\sqrt{2\pi}}\Big)^{k-1} \Big(\frac{1}{2}\Big)^{\frac{(k^2-1)}{2}} \cdot 1!2! \cdots (k-1)! \cdot k^{\frac{k^2}{2}}. \tag{3}$$

Basic algebras

For an arbitrary finite dimensional algebra the asymptotics of the codimension are given via Kemer's Theory by a result of Eli Aljadeff, Geoffrey Janssens and Yakov Karasik.

It needs the notion of Kemer's index and Basic algebras.

The Kemer index

For every algebra R let Γ be its ideal of identities.

- We let $\beta(\Gamma)$ to be the greatest integer t such that, *for every* $\mu \in \mathbb{N}$, there exists a μ-fold t-alternating (in μ layers X_i with t elements) polynomial, not an identity of R:

$$f(X_1, \dots, X_\mu, Y) \notin \Gamma.$$

- We then let $\gamma(\Gamma)$ to be the maximum $s \in \mathbb{N}$ for which there exists, for all μ, a polynomial $f(X_1, \dots, X_\mu, Z_1, \dots, Z_s, Y) \notin \Gamma$, alternating in μ *small layers* X_i with $\beta(\Gamma)$ elements and in s *big layers* Z_j with $\beta(\Gamma) + 1$ elements.
- The pair $(\beta(\Gamma), \gamma(\Gamma))$ is the *Kemer index* of R.

Basic algebras

For a finite dimensional algebra R with radical J one has that the Kemer index is smaller or equal lexicographically to the pair d_R, s_R defined by

$$d_R = \dim R/J, \quad J^{s_R} \neq 0, \quad J^{s_R+1} = 0.$$

Basic algebras

A finite dimensional algebra R is *basic* if its Kemer index equals d_R, s_R

The role of Basic algebras

Two algebras R_1, R_2 are *PI equivalent* if they satisfy the same polynomial identities.

Basic theorem

A finite dimensional algebra R is PI equivalent to a direct sum of basic algebras.

Assume thus that R is basic with Kemer index d, s and semisimple part $\oplus_{i=1}^{q} M_{h_i}(F)$:

Theorem

There exists a constant C such that:

$$cd_R(n) \sim C n^{-\frac{d-q}{2}+s} d^n. \quad (4)$$

General PI algebras open problems

For general PI algebras the role of matrices is played by the Grassmann envelopes of the 3 classes of simple finite dimensional superalgebras.

$$M_n(F), \quad M_n(F[\eta]),\ \eta^2 = 1, \quad M_{h,k}(F).$$

The theory is rather incomplete

For instance a not well understood case is $M_n(G) = G(M_n(F[\eta]))$ the matrices over the Grassmann algebra.

THANK YOU

HAPPY BIRTHDAY SASHA

THE END