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Minimal-dimensional modules for reduced enveloping algebras for $\mathfrak{gl}_n$

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$k = \bar{k}$ char $k = p > 0$, G cnctd red. alg. gp. $/ k$

p good, $\mathfrak{g} = \mathrm{Lie}(G)$.

$e \in \mathfrak{g}$ nilpotent

$\chi \in \mathfrak{g}^*$ dual to e (assume existence of G-inv. bijection $\mathfrak{g} \to \mathfrak{g}^*$

$d_\chi = \frac{1}{2} \dim G \cdot \chi$

$\mathfrak{g} = \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}(r)$ good grading for e (or χ)

$e \in \mathfrak{g}(2)$ $\mathfrak{g}^e \subseteq \sum_{i \geq 0} \mathfrak{g}(i) =: \mathfrak{p}$

$\mathfrak{h} = \mathfrak{g}(0)$. $\mathfrak{t}$ will be a maximal toral algebra in $\mathfrak{g}$

e.g. $e \in \mathfrak{gl}_7$ with Jordan type $(4, 2, 1)$

$$\begin{array}{|c|c|c|c|} \hline 1 & & & \\ \hline 2 & 4 & & \\ \hline 3 & 5 & 6 & 7 \\ \hline \end{array} \qquad \mathfrak{g}(r) = \langle e_{ij} : 2(\mathrm{col}(j) - \mathrm{col}(i)) = r \rangle$$

$e = \sum_{\boxed{i}\boxed{j}} e_{ij} = e_{24} + e_{35} + e_{56} + e_{67}$ in our case

Here $\mathfrak{p} = \begin{pmatrix} * & * \\ & \boxed{*} \\ & & \boxed{*} \\ & & & \boxed{*} \end{pmatrix}$ $\mathfrak{h} \cong \mathfrak{gl}_3 \oplus \mathfrak{gl}_2 \oplus k^2$

$\mathfrak{t}^* \longleftrightarrow$ fillings of a pyramid with elements of k (shifted by e)

Thm (Goodwin–Topley) $\mathfrak{g} = \mathfrak{gl}_n(k)$ & assume there is an even good grading for e. Let $V \in U_\chi(\mathfrak{g})$-mod, s.t. $\dim V = p^{d_\chi}$.

Then $V = U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p})} k_\lambda$ for some $\lambda \in \mathfrak{z}(\mathfrak{h})^*_0$

$\mathfrak{z}(\mathfrak{h})^*_0 = \{\mu \in \mathfrak{z}(\mathfrak{h}^*) : \mu(x^{[p]}) = \mu(x)^p \ \forall x \in \mathfrak{z}(\mathfrak{h})\}$

$$Z_p(\mathfrak{g}) = \langle x^p - x^{[p]} : x \in \mathfrak{g} \rangle \subseteq Z(\mathfrak{g}) = \langle U(\mathfrak{g})^G, Z_p(\mathfrak{g}) \rangle$$

$$U_\chi(\mathfrak{g}) = U(\mathfrak{g}) \Big/ U(\mathfrak{g})\{x^p - x^{[p]} - \chi(x)^p : x \in \mathfrak{g}\}$$

$$\dim U_\chi(\mathfrak{g}) = p^{\dim \mathfrak{g}}$$

<u>Conjecture (Kac-Weisfeiler)</u> $\rightsquigarrow$ Thm (Premet)

$V \in U_\chi(\mathfrak{g})$-mod then $p^{d_\chi} \mid \dim V$

<u>Qu/conjectures</u> (Kac/Humphreys)

- Is there a module $V \in U_\chi(\mathfrak{g})$-mod with dimension p^{d_χ}?
- Can we classify / construct these?

Let $\mathfrak{k} \subseteq \mathfrak{g}(-1)$ be a lagrangian subspace for $\langle x, y \rangle = \chi([x,y])$

$\mathfrak{m} = \mathfrak{k} \oplus \mathfrak{g}(\leq -2)$ $\quad \mathfrak{m}_\chi = \{x - \chi(x) : x \in \mathfrak{m}\} \subseteq U(\mathfrak{m})$

$$Q_0(\mathfrak{g}, \chi) = U_\chi(\mathfrak{g}) \otimes_{U(\mathfrak{m})} k_\chi \cong U_\chi(\mathfrak{g}) \Big/ U_\chi(\mathfrak{g})\mathfrak{m}_\chi$$

$$U_0(\mathfrak{g}, \chi) = \mathrm{End}_{U_\chi(\mathfrak{g})}(Q_0(\mathfrak{g}, \chi))^{op}$$

<u>Thm</u> (Premet)

$$\begin{tikzcd} U_\chi(\mathfrak{g})\text{-mod} \arrow[r, shift left, "?^{\mathfrak{m}_\chi}"] \arrow[r, phantom, "\cong"] & U_0(\mathfrak{g}, \chi) \arrow[l, shift left, "Q_0(\mathfrak{g},\chi) \otimes_{U_0(\mathfrak{g},\chi)} ?"] \end{tikzcd}$$

$$\dim Q_0(\mathfrak{g}, \chi) \otimes_{U_0(\mathfrak{g},\chi)} V = p^{d_\chi} \dim V$$

<u>Idea of proof</u>

$M \leq G$ s.t. Lie $M = \mathfrak{m}$

$Q(\mathfrak{g}, \chi) = U(\mathfrak{g}) \big/ U(\mathfrak{g})\mathfrak{m}_\chi$ $\quad$ non-reduced version of Q_0

$U(\mathfrak{g}, \chi) = \left(U(\mathfrak{g}) \big/ U(\mathfrak{g})\mathfrak{m}_\chi \right)^{\mathrm{Ad}\, M}$ $\quad$ non-reduced version of $U_0(\mathfrak{g}, \chi)$

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Thm (Goodwin–Topley)
There is a grading of $U(\mathfrak{g}, \chi)$ s.t.
$$\mathrm{gr}(U(\mathfrak{g},\chi)) \cong k[e+v] \cong S(\mathfrak{g}^e)$$

(and $U(\mathfrak{g},\chi)$ does not depend on choice of k or the good grading)

Moreover: · if $U(\mathfrak{g}_{\mathbb{C}},\chi)_{\mathbb{Z}}$ is a "nice" $\mathbb{Z}$-form of $U(\mathfrak{g}_{\mathbb{C}},\chi)$ then
$$U(\mathfrak{g}_{\mathbb{C}},\chi) \otimes_{\mathbb{Z}} k = U(\mathfrak{g},\chi)$$

· $U(\mathfrak{g},\chi) \twoheadrightarrow U_0(\mathfrak{g},\chi)$ "quotient by p-centre"

From now on $\mathfrak{g} = \mathfrak{gl}_n(k)$ & we have an even good grading.

$$U(\mathfrak{g},\chi) = \{u \in U(\mathfrak{p}) : m\cdot u - u \in U(\mathfrak{g})\mathfrak{m}_\chi \ \forall m \in M\} \leq U(\mathfrak{p})$$

$$\begin{array}{ccc} U(\mathfrak{g},\chi) & \searrow & \\ \wedge\!\!| & & \\ U(\mathfrak{p}) & \longrightarrow\!\!\!\!\!\rightarrow & U_0(\mathfrak{p}) \end{array}$$
image is $U_0(\mathfrak{g},\chi)$

$\lambda \in \mathfrak{z}(\mathfrak{h})^* \rightsquigarrow k_\lambda \in U(\mathfrak{g},\chi)$-mod via restriction
$$U(\mathfrak{g},\chi) \hookrightarrow U(\mathfrak{p}) \twoheadrightarrow U(\mathfrak{h})$$

Brundan–Kleshchev: $U(\mathfrak{g}_{\mathbb{C}},\chi)$ is a truncated shifted Yangian

We can check $U(\mathfrak{g},\chi)^{ab}$ has the same description as $U(\mathfrak{g}_{\mathbb{C}},\chi)^{ab}$ & deduce that $\{k_\lambda : \lambda \in \mathfrak{z}(\mathfrak{h})^*\}$ is all 1-dim'l modules for $U(\mathfrak{g},\chi)$. We then show that k_λ factors to $U_0(\mathfrak{g},\chi) \iff \lambda \in \mathfrak{z}(\mathfrak{h})_0^*$