Support Varieties for Linear Algebraic groups

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HAPPY BIRTHDAY, SASHA!!

Support Varieties

Strategy: Given a finite group G, associate to each kG-module M a geometric object $M \mapsto V(G)_M$ which reflects some of the properties of extensions in the abelian category of G-modules. Here, k is an algebraically closed field of characteristic p dividing the order of G.

Two quite different constructions for finite groups which lead to same theory:

- Cohomological varieties:
  $V^{coh}(G) = \text{Spec}\, H^\bullet(G,k), \quad V^{coh}(G)_M \subset V^{coh}(G)$
  is the subvariety of the annihilator of $Ext^*_G(M,M)$.
- π-points spaces:
  $\Pi(G) = \{[\alpha] : \alpha : k[t]/t^p \to kG \ \pi\text{-point}\}$,
  $\Pi(G)_M = \{[\alpha] : \alpha^*(M) \ \text{is not free}\}$.

Earlier "applications"

Study extensions and cohomology of finite group schemes, NOT irreducibility.

Examples of elementary abelian p-groups. Category of kE-modules is wild. Carlson rank varieties. Vector bundles of projective spaces

Examples of restricted Lie algebras. Kac-Weisfeiler conjecture proved by Premet.

Arbitrary finite group schemes. Classification of thick, tensor-closed subcategories. Modules of constant Jordan type.

Linear Algebraic Groups

We consider a linear algebraic group G, a reduced, irreducible affine group scheme of finite type over an algebraically closed field k of characteristic $p > 0$.

A rational G-module is a comodule for the coalgebra $k[G]$.

1-parameter subgroup of $G \equiv$ homomorphism $\mathbb{G}_a \to G$.

$k[G] \equiv$ coordinate algebra of G.

$\mathfrak{g} \equiv Lie(G)$, a (restricted) Lie algebra.

$kG \equiv$ "group algebra" (i.e., algebra of distributions at id).

$G_{(r)}$, the r-th Frobenius kernel $ker\{F^r : G \to G^{(r)}\}$.

$kG_{(1)}$ is the restricted enveloping algebra of $\mathfrak{g}$.

Infinitesimal 1-parameter subgroups

A third construction of support varieties due to [SFB] using infinitesimal 1-parameter subgroups:
$V(G_{(r)}) = \{\mu : G_{a(r)} \to G_{(r)}\}$, affine scheme.

$V(G_{(r)})_M = \{\mu : (\mu_* \circ \epsilon_r)^* M$ is not free as a $k[t]/t^p$-*module*$\}$.

Consider $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \subset \mathcal{N}_p(\mathfrak{g})^{\times r}$, the variety of r-tuples of p-nilpotent, pairwise commuting elements of $\mathfrak{g}$.

For G classical, have $\psi : k[\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g}))] \to H^\bullet(G_{(r)}, k)$ which is a p-isogeny.

Challenge: Extend theory to linear algebraic groups

We use a mixture of the approaches of cohomology, 1-parameter subgroups, and π-points. There are issues to overcome.

- If G is a simple algebraic group, then its rational cohomology is trivial.
- There is no appropriate connection between $V(G_{(r)})_M$ and $V(G_{(r+1)})_M$ for a $G_{(r+1)}$-module M.
- The map $\epsilon_r : k[t]/t^p \to kG_{a(r)}$ depends upon r.
- The category of rational G-modules has no projectives; all injectives are infinite dimensional.

Applications for representation theory of G

(•) Extend theory of support varieties to linear algebraic groups.

(•) Introduce interesting classes of rational G-modules.

(•) Inform computations of the rational cohomology of unipotent algebraic groups.

(•) Construct "Geometric slices" of the category of rational representations.

Linear algebraic groups of exponential type

Definition

A structure of exponential type on a linear algebraic group G is a morphism of schemes

$$\mathcal{E} : \mathcal{N}_p(\mathfrak{g}) \times \mathbb{G}_a \ \to \ G, \quad (B, s) \mapsto \mathcal{E}_B(s)$$

satisfying

1. Each $\mathcal{E}_B : \mathbb{G}_a \to G$ is a 1-parameter subgroup.
2. $[\mathcal{E}_B(s), \mathcal{E}_{B'}(s')] = 1$ if $(B, B') = 0$.
3. $\mathcal{E}_{\alpha \cdot B}(s) = \mathcal{E}_B(\alpha \cdot s)$.
4. Every 1-parameter subgroup can be written as a finite product $\mathcal{E}_{\underline{B}} = \prod_{s=0}(\mathcal{E}_{B_s} \circ F^s)$.
5. $\mathcal{C}_r(\mathcal{N}_p(\mathfrak{g})) \stackrel{\sim}{\to} V(G_{(r)})$.

Consequences

- If a structure of exponential type for G exists, then it is unique up to automorphism of $\mathcal{N}_p(\mathfrak{g})$.
- $V_r(G) = \{\prod_{s=0}^{r-1}(\mathcal{E}_{B_s} \circ F^s)\} \subset V(G)$ is an exhaustive filtration.
- $V_r(G) \stackrel{\sim}{\to} V(G_{(r)}), \quad \mathcal{E}_{\underline{B}} \mapsto \mu_{\underline{B}} \equiv \mathcal{E}_{\underline{B}} \circ i_r$.
- Spec $H^\bullet(G_{(r)}, k) \simeq V_r(G)$.

[SFB] Any classical simple group or a "standard" parabolic subgroup of such a simple group or the unipotent radical of a standard parabolic is a group of exponential type

[Sobaje] If G is reductive and if $p > h(G)$ (perhaps "separably good" suffices), then G, its standard parabolic subgroups, and their unipotent radicals admit a structure of exponential type.

Definition of $M \mapsto V(G)_M$

Assume that G is equipped with a structure of exponential type $\mathcal{E} : \mathbb{G}_a \times \mathcal{N}_p(\mathfrak{g}) \to G$ and let M be a rational G-module.

Definition

For $\underline{B} = (B_0, \ldots) \in V(G)$, define

$$\alpha_{\underline{B}} : k[t]/t^p \to kG, \quad t \ \mapsto \ \sum_{s \geq 0} (\mathcal{E}_{B_s})_*(u_s)$$

(where $u_s : \mathbb{G}_a = k[T] \to k, \ u_s(T^n) = \delta_{p^s,n}$). We define

$$V(G)_M \ \equiv \ \{\underline{B} : \alpha^*_{\underline{B}} M \text{ not free as a } k[t]/t^p\textit{-module}\} \ \subset \ V(G).$$

If M finite dimensional, same information as $V(G_{(r)})_M$ for $r >> 0$ (through a subtle twist).

Properties

Let G be a linear algebraic group of exponential type and M, N, M_i be rational G-modules.

- $V(G)_M \subset V(G)$ is $G(k)$-stable.
- If M is finite dimensional, the $V(G)_M \subset V(G)$ is closed.
- $V(G)_{M \otimes N} = V(G)_M \cap V(G)_N$.
- if $0 \to M_1 \to M_2 \to M_3 \to 0$ be exact. Then for any permutation σ of $\{1, 2, 3\}$, $V(G)_{M_{\sigma(2)}} \subset V(G)_{M_{\sigma(1)}} \cup V(G)_{M_{\sigma(3)}}$.
- If M is rationally injective, then $V(G)_M = 0$.
- If $M = k$, then $V(G)_M = V(G)$.

NOTE: this theory can be "refined" by taking into account t he Jordan types of the $k[t]/t^p$-modules $\alpha^*_{\underline{B}}(M)$.

Relationship to $V(G_{(r)})_M$

Key Observation: For $\underline{B} = (B_0, \ldots, B_{r-1})$, the π-points

$$\alpha_{\underline{B}} : k[t]/t^p \to kG_{(r)}, \quad \mu_{\Lambda_r(\underline{B})} \circ \epsilon_r : k[t]/t^p \to kG_{(r)}$$

are equivalent, where $\Lambda_r(B_0, \ldots) = (B_{r-1}, B_{r-2} \ldots, B_0)$.

Theorem

If M is a rational G-module and $r > 0$ such that $\Delta_M : M \to M \otimes k[G] \to M \otimes k[\mathcal{N}_p(\mathfrak{g})] \otimes k[\mathbb{G}_a]$ projects to $k[\mathbb{G}_a]_{\leq p^r}$, then

$$V(G)_M = \Lambda_r^{-1} V(G_{(r)})_M).$$

For any finite dimensional M, there exists such an r.

Examples

- $G = GL_n$, and M a polynomial representation of G of degree d. If r is sufficiently large that $p^r > (p-1)d$, then $V(GL_n)_M \ = \ \Lambda_r^{-1} V(GL_{n(r)})_M)$.
- If M is rationally injective, then $V(G)_M = \{0\}$.
- if $H \subset G$ is a normal algebraic subgroup and the projection $q : G \to G/H$ is a map of linear algebraic groups of exponential type, then for any rational G-module M, $V(G)_{q^*M} = q^{-1}(V(G/H)_M)$.

Question

For $H \subset G$ a map of linear algebraic groups of exponential type and N a rational H-module, what can we say about $V(G)_M$ for $M = ind_H^G(N)$?

Filtrations

Study rational G-modules by considering their restrictions to sub-coalgebras of $k[G]$ (contrast with restriction to Frobenius kernels which are given by subalgebras of kG.)

Definition

For any $d \geq 0$, the sub-coalgebra $(k[G])_{[d]} \subset k[G]$ consists of functions $f \in k[G]$ whose restrictions along all 1-parameter subgroups $\mathcal{E}_B = \mathcal{E}_{(B,0,0,\dots)} : \mathbb{G}_a \to G,\ B^{[p]} = 0$ have degree $\leq d$; i.e., $(k[G])_{[d]}$ is the pre-image of $k[G] \otimes k[T]_{\leq d}$ for

$$(1 \otimes \mathcal{E}_B) \circ \Delta : k[G] \to k[G] \otimes k[G] \to k[G] \otimes k[T]).$$

For any rational G-module M, define $M_{[d]} \subset M$ to be maximal $k[G]_{[d]}$ sub-comodule of M.

Mock Injectives

Injectivity Criterion: M is injective if and only if $M_{[d]}$ is an injective $(k[G])_{[d]}$-comodule for all $d \geq 0$.

Definition

A rational G-module M is said to be mock injective if $V(G)_M = 0$; this is equivalent to the condition that M is $G_{(r)}$-injective for all $r > 0$.

Surprise! Even for $G = \mathbb{G}_a$, there are mock injective which are not injective. These are difficult to write down explicitly.

Question

Construct a theory of "Picard groups": mock injectives which are sub-modules of $k[G]$.

Possible refinement

Let $\hat{B} = (B_0, \ldots, B_n, \ldots)$ be an infinite sequence of p-nilpotent elements $B_n \in \mathfrak{g}$ which are pair-wise commuting. Then for any rational G-module M, the sum $(\sum_{s=0}^{\infty} (\mathcal{E}_{B_s})_*(u_s))(m)$ is finite for any $m \in M$, determining a p-nilpotent operator on M.

Define $\hat{V}(G) = \varprojlim_s V_s(G)$, and define $\hat{V}(G)_M$ to be the subset of $\hat{V}(G)$ of those $\hat{B}$ such that the action of $(\sum_{s=0}^{\infty} (\mathcal{E}_{B_s})_*(u_s))$ on M is a sum of blocks of size p.

Question

Does $\hat{V}(G)_M = 0$ imply that M is injective?

New Classes of Rational G-modules

Definition

M is said to have exponential degree $\leq d$ if $M = M_{[d]}$.

If M is finite dimensional, then $M = M_{[d]},\ d >> 0$.

If M has exponential degree $\leq p^r - 1$, then $V(G)_M$ is determined by $V(G_{(r)})_M$.

Definition

M is said to be mock trivial if $\mathcal{E}_{\underline{B}}^* M$ is a trivial $\mathbb{G}_a$-module for all $\mathcal{E}_{\underline{B}} \in V(G)$. In other words, if $M = M_{[0]}$.

Observation If $G \neq U_p(G)$, then $k[G]_{[0]} \subset k[G]$ is a sub-Hopf algebra which is a non-trivial indecomposable mock trivial module.

The functor $(-)_{[d]}$

The functor

$$(-)_{[d]} : (G - Mod) \to (k[G]_{[d]}) - coMod)$$

is left exact, left adjoint to the exact, fully faithful the inclusion functor

$$\iota_{[d]} : (k[G]_{[d]}) - coMod) \to (G - Mod).$$

Proposition

The isomorphism of functors

$$H^0(G, -) \;\simeq\; Hom_{(k[G]_{[d]} - comod)}(k, -) \circ (-)_{[d]}$$

leads to Grothendieck spectral sequences converging to $H^*(G, M)$.

Projective spectrum for rational cohomology

Definition

We define $\operatorname{Proj} V^{coh}(G)$ as the image in $\operatorname{Proj} H^\bullet(G,k)$ of $\varinjlim_r \operatorname{Proj} H^\bullet(G_{(r)},k)$.

Theorem

For G a linear algebraic group of exponential type,

$$\Theta_G : \operatorname{Proj}' V(G) \to \operatorname{Proj} V^{coh}(G), \quad \mathcal{E}_{\underline{B}} \mapsto \ker\{\alpha^*_{\underline{B}}\}$$

is well defined, surjective, and factors through the co-invariants of the action of G on $\operatorname{Proj}' V(G)$.

Cohomological Support Varieties

Definition

We define $\mathrm{Proj}\, V^{coh}(G)_M \subset \mathrm{Proj}\, V^{coh}(G)$ to consist of those prime ideals $\mathfrak{p} \subset H^\bullet(G,k)$ which are of the form $\mathfrak{p} = \rho_r^{-1}(q_r)$ for homogeneous prime ideal $q_r \subset H^\bullet(G_{(r)},k)$ which contains the annihilator of $Ext^*_{G_{(r)}}(M,M)$.

Proposition

If G is a linear algebraic group of exponential type, then

$$\Theta_G : \mathrm{Proj}'\, V(G) \ \to \ \mathrm{Proj}\, V^{coh}(G)$$

restricts to the surjective map

$$\Theta_{G,M} : \mathrm{Proj}'\, V(G)_M \ \to \ \mathrm{Proj}\, V^{coh}(G)_M.$$

Example of $G = \mathbb{G}_a$

$\text{Proj}' \, V(\mathbb{G}_a) \;=\; \mathbb{A}^\infty$.

Observation: The restriction map $k[\mathbb{G}_a] \to k[\mathbb{G}_{a(r)}]$ is split as a map of coalgebras by $k[\mathbb{G}_a]_{<p^r} \subset k[\mathbb{G}_a]$.

Fact: A rational $\mathbb{G}_a$-module is a k-vector space V equipped with the structure of a $k[u_0, \ldots, u_n, \ldots]$-module such that $\forall v \in V, \; \exists r_v$ with u_s acting trivially on v for $s \geq r_v$.

In particular, $(\mathbb{G}_{a(r)} - Mod) \;\simeq (k\mathbb{Z}/p^r - Mod)$ embeds naturally in $(\mathbb{G}_a - Mod)$ (adjoint pairs, etc). This enables realization of many subspaces of $V(\mathbb{G}_a)$ as support varieties.

Proposition

For M a finite dimensional $\mathbb{G}_a$-module,

$$\Theta_{\mathbb{G}_a} : \text{Proj}' \, V(\mathbb{G}_a)_M \;\stackrel{\sim}{\to} \text{Proj} \, V^{coh}(\mathbb{G}_a)_M.$$

Coinvariants

Question

For which linear algebraic groups G of exponential type is it the case that $\Theta_G : \mathrm{Proj}\, V(G) \to \mathrm{Proj}^{coh}\, V(G)$ induces

$$? \ (\mathrm{Proj}'\, V(G))/G \ \stackrel{\sim}{\to} \mathrm{Proj}^{coh}\, V(G) \ ?$$

Remark

$$\Theta_{U_N} : \mathrm{Proj}'\, V(U_N) \to \mathrm{Proj}^{coh}\, V(U_N), \quad \mathcal{E}_{\underline{B}} \ \mapsto \ \ker\{\alpha^*_{\underline{B}}\}$$

appears to induce an isomorphism

$$(\mathrm{Proj}'\, V(U_N))/U_N \ \stackrel{\sim}{\to} \mathrm{Proj}^{coh}\, V(U_N).$$

$H^\bullet(U_{3(r)}, k)$

1st Ingredient
In the Lyndon-Hochschild-Serre spectral sequence for the central extension $1 \to Z \simeq \mathbb{G}_a \to U_3$ *to* $\overline{U}_3 \simeq \mathbb{G}_a^2 \to 1$, the Steenrod operation $\beta \circ \mathcal{P}^{p^j}$ applied to $(x_{1,3}^{(i)})^{p^j}$ equals

$$-(x_{1,2}^{(i)})^{p^{j+1}} \cdot x_{2,3}^{(i+1+j)} + (x_{2,3}^{(i)})^{p^{j+1}} \cdot x_{1,2}^{(i+1+j)}.$$

2nd Ingredient [SFB] provide a purely inseparable isogeny $\psi : H^\bullet(U_{3(r)}, k) \to k[V_r(U_3)]$ for any $r > 0$.

Conclusion Explicit computation of $H^\bullet(U_{3(r)}, k)_{red}$ compatible with $r \mapsto r+1$. Moreover, $(x_{1,3}^{(i)})^{p^j}$ represents a permanent cycle if and only if $i + j + 1 \geq r$.

$H^\bullet(U_3, k)$

Theorem

The commutative algebra $H^\bullet(U_3, k)_{red}$ is generated by classes $\{x_{1,2}^{(i)},\ x_{2,3}^{(i')},\ i, i' \geq 0\}$ modulo the relations

$$-(x_{1,2}^{(i)})^{p^{j+1}} \cdot x_{2,3}^{(i+1+j)} + (x_{2,3}^{(i)})^{p^{j+1}} \cdot x_{1,2}^{(i+1+j)}, \quad i, j \geq 0.$$

Thus, there is a natural closed embedding
$\mathrm{Proj}\, V^{coh}(U_3) \ \to\ \mathrm{Proj}\, V^{coh}(\mathbb{G}_a^{\times 2}) \ =\ \mathbb{A}^\infty \times \mathbb{A}^\infty.$

Corollary

$\Theta_{U_3} : \mathrm{Proj}'\, V(U_3) \ \to\ \mathrm{Proj}\, V^{coh}(U_3)$ *is the coinvariant map.*

Cohomological supports for rational U_3-modules

Example

Let N be a finite dimensional rational $\mathbb{G}_a^{\times 2}$ module.
Denote by M the "inflation" of N via $U_3 \twoheadrightarrow \overline{U}_3 \simeq \mathbb{G}_a^{\times 2}$.

Then $V(U_3)_M$ equals the pre-image of $V(\mathbb{G}_a^{\times 2})_N$ under the projection $V(U_3) \to V(\mathbb{G}_a^{\times 2})$.

Moreover, $\mathrm{Proj}\, V^{coh}(U_3)_M$ equals equals the intersection of

$$(\mathrm{Proj}'\, V(\mathbb{G}_a^{\times 2})_N \cap \mathrm{Proj}\, V^{coh}(U_3)) \;\subset\; \mathrm{Proj}'\, V(\mathbb{G}_a^{\times 2}).$$

Extension to U_N

Employ the T_N-equivariant Hochschild-Serre spectral sequence

$$E_2^{*,*} \ = \ H^*(U_N/\Gamma_{v-1},k)\otimes H^*(\Gamma_{v-1}/\Gamma_v,k) \ \Rightarrow \ H^*(U_N/\Gamma_v,k)$$

for terms of the descending central series for U_N.

Compute differentials such as

$$\beta\mathcal{P}^{p^j}(\textstyle\sum_{t=1}^{v-1}(x_{s,s+t}^{(i)})^{p^j}\otimes y_{s+t,s+v}^{(i+1+j)} - (x_{s+t,s+v}^{(i)})^{p^j}\otimes y_{s,s+t}^{(i+1+j)})$$

$$= \ \sum_{t=1}^{v-1}(x_{s,s+t}^{(i)})^{p^{j+1}}\otimes x_{s+t,s+v}^{(i+1+j)} - (x_{s+t,s+v}^{(i)})^{p^{j+1}}\otimes x_{s,s+t}^{(i+1+j)}$$

CONCLUSION: Computation for $V^{coh}(U_N)$ extending case $N = 3$.