Representations of Domestic Group Schemes

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Representation Theory and Symplectic Singularities
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- Let $d \in \mathbb{N}$. Then $\mathrm{mod}_{\Lambda}^{d}$ denotes the affine variety of d-dimensional Λ-modules.
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- The algebraic group $\mathrm{GL}_d(k)$ acts on mod_Λ^d. The orbits correspond to the isoclasses of Λ-modules.
- Let ind_Λ^d be the set of indecomposable modules of mod_Λ^d.

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Remarks. (1) If Λ is representation-finite, then there are only finitely many isoclasses of indecomposable Λ-modules (Brauer-Thrall II).
(2) If an algebra is wild, then its module category is at least as complicated as that of any other algebra (Drozd, 1977).

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3. The algebra $U_0(\mathfrak{sl}(2))$ is domestic.

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  - Only few tensor products were known.

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- Let M be a Λ-module, $\varphi \in \mathrm{Aut}(\Lambda)$.
- Twisted module $M^{(\varphi)}$: $\quad a\bullet m := \varphi(a).m \quad \forall\ a \in \Lambda, m \in M$.
- Special case: $\mathcal{G}$ algebraic group scheme with Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathcal{G})$.
    - $\mathrm{Ad} : \mathcal{G}(k) \longrightarrow \mathrm{Aut}(U_0(\mathfrak{g}))$. (adjoint representation)
    - $M^{(g)} := M^{(\mathrm{Ad}(g))},\ M \in \mathrm{mod}\, U_0(\mathfrak{g}),\ g \in \mathcal{G}(k)$.

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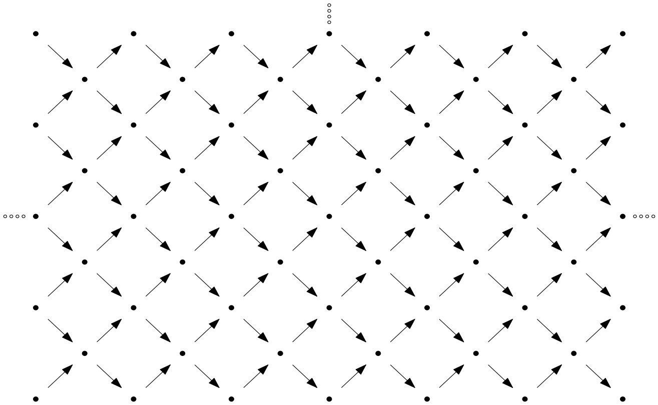

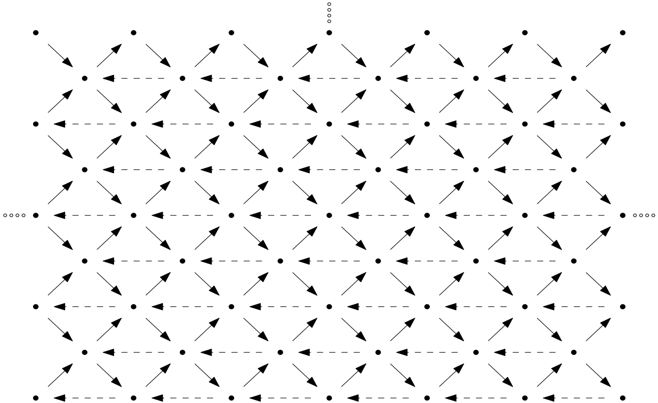

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Ranks of Tubes and Ramification

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$$e_x(\varphi) := \dim_k \mathcal{O}_{X,x}/\mathcal{O}_{X,x}\mathfrak{m}_{Y,\varphi(x)}$$

is the ramification index of φ at x.

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