# Commutative algebraic groups up to isogeny

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**Abstract**

Consider the abelian category $\mathcal{C}_k$ of commutative group schemes of finite type over a field $k$. By results of Serre and Oort, $\mathcal{C}_k$ has homological dimension 1 (resp. 2) if $k$ is algebraically closed of characteristic 0 (resp. positive). In this talk, we study the abelian category of commutative algebraic groups up to isogeny, defined as the quotient of $\mathcal{C}_k$ by the full subcategory $\mathcal{F}_k$ of finite $k$-group schemes. We show that $\mathcal{C}_k/\mathcal{F}_k$ has homological dimension 1, and we determine its projective or injective objects. We also obtain structure results for $\mathcal{C}_k/\mathcal{F}_k$, which take a simpler form in positive characteristics.

There has been much recent progress on the structure of algebraic groups over an arbitrary field; in particular, on the classification of pseudo-reductive groups (see [CGP15, CP15]). Yet commutative algebraic groups over an imperfect field remain somewhat mysterious, e.g., extensions with unipotent quotients are largely unknown; see [To13] for interesting results, examples, and questions.

In the preprint [Br16], we present a categorical approach to commutative algebraic groups up to isogeny, which bypasses the problems raised by imperfect fields, and brings forward some analogies with representation theory.

More specifically, denote by $\mathcal{C}_k$ the category with objects the group schemes of finite type over the ground field $k$, and with morphisms, the homomorphisms of $k$-group schemes (all group schemes under consideration will be assumed commutative). By a classical result of Grothendieck, $\mathcal{C}_k$ is an abelian category. We define the category of 'algebraic groups up to isogeny' as the quotient category of $\mathcal{C}_k$ by the Serre subcategory of finite group schemes; then $\mathcal{C}_k/\mathcal{F}_k$ is obtained from $\mathcal{C}_k$ by inverting all isogenies, i.e., all morphisms with finite kernel and cokernel.

It will be easier to deal with the full subcategory $\underline{\mathcal{C}}_k$ of $\mathcal{C}_k/\mathcal{F}_k$ with objects the smooth connected algebraic groups, since these categories turn out to be equivalent, and morphisms in $\underline{\mathcal{C}}_k$ admit a simpler description.

As a motivation for considering the 'isogeny category' $\underline{\mathcal{C}}_k$, note that some natural constructions involving algebraic groups are only exact up to isogeny; for example, the formations of the maximal torus or of the largest abelian variety quotient, both of which are not exact in $\mathcal{C}_k$. Also, some structure theorems for algebraic groups take a much simpler form when working up to isogeny. A classical example is the Poincaré complete reducibility theorem, which is equivalent to the semi-simplicity of the isogeny category of abelian varieties, i.e., the full subcategory $\underline{\mathcal{A}}_k$ of $\underline{\mathcal{C}}_k$ with objects abelian varieties. Likewise, the isogeny category of tori, $\underline{\mathcal{T}}_k$, is semi-simple.

We gather our main results in the following:

**Theorem.** (i) *The category $\underline{\mathcal{C}}_k$ is artinian and noetherian. Its non-zero simple objects are exactly the additive group $\mathbb{G}_{a,k}$, the simple tori, and the simple abelian varieties.*

(ii) *The product functor $\underline{\mathcal{T}}_k \times \underline{\mathcal{U}}_k \to \underline{\mathcal{L}}_k$ yields an equivalence of categories, where $\underline{\mathcal{U}}_k$ (resp. $\underline{\mathcal{L}}_k$) denotes the isogeny category of unipotent (resp. linear) algebraic groups.*

(iii) *If $\mathrm{char}(k) > 0$, then the product functor $\underline{\mathcal{S}}_k \times \underline{\mathcal{U}}_k \to \underline{\mathcal{C}}_k$ yields an equivalence of categories, where $\underline{\mathcal{S}}_k$ denotes the isogeny category of semi-abelian varieties. If in addition $k$ is locally finite, then the product functor $\underline{\mathcal{T}}_k \times \underline{\mathcal{A}}_k \to \underline{\mathcal{S}}_k$ yields an equivalence of categories as well.*

(iv) *The base change under any purely inseparable field extension $k'$ of $k$ yields an equivalence of categories $\underline{\mathcal{C}}_k \to \underline{\mathcal{C}}_{k'}$.*

(v) *The homological dimension of $\underline{\mathcal{C}}_k$ is* 1.

We also describe the projective or injective objects of the category $\underline{\mathcal{C}}_k$. Moreover, in characteristic 0, we obtain a structure result for that category, which turns out to be more technical than in positive characteristics.

Let us now compare the above statements with known results on $\mathcal{C}_k$ and its full subcategories $\mathcal{A}_k$ (resp. $\mathcal{T}_k$, $\mathcal{U}_k$, $\mathcal{L}_k$, $\mathcal{S}_k$) of abelian varieties (resp. tori, unipotent groups, linear groups, semi-abelian varieties).

About (i) (an easy result, mentioned by Serre in [Se60]): $\mathcal{C}_k$ is artinian and not noetherian. Also, every algebraic group is an iterated extension of 'elementary' groups; these are the simple objects of $\underline{\mathcal{C}}_k$ and the simple finite group schemes.

About (ii): the product functor $\mathcal{T}_k \times \mathcal{U}_k \to \mathcal{L}_k$ yields an equivalence of categories if $k$ is perfect. But over an imperfect field, there exist non-zero extensions of unipotent groups by tori, and these are only partially understood (see [To13, §9]).

About (iii): the first assertion follows from recent structure results for algebraic groups (see [Br15b, §5]), together with a lifting property for extensions of such groups with finite quotients (see [Br15a] and [LA15]). The second assertion is a direct consequence of the Weil-Barsotti isomorphism (see e.g. [Oo66, §III.18]).

About (iv): this is a weak version of a result of Chow on abelian varieties, which asserts (in categorical language) that base change yields a fully faithful functor $\mathcal{A}_k \to \mathcal{A}_{k'}$ for any primary field extension $k'$ of $k$ (see [Ch55], and [Co06, §3] for a modern proof).

About (v), the main result of this article: recall that the homological dimension of an abelian category $\mathcal{D}$ is the smallest integer, $\mathrm{hd}(\mathcal{D})$, such that $\mathrm{Ext}^n_{\mathcal{D}}(A, B) = 0$ for all objects $A, B$ of $\mathcal{D}$ and all $n > \mathrm{hd}(\mathcal{D})$; these Ext groups are defined as equivalence classes of Yoneda extensions. In particular, $\mathrm{hd}(\mathcal{D}) = 0$ if and only if $\mathcal{D}$ is semi-simple.

It follows from work of Serre (see [Se60, 10.1 Thm. 1] together with [Oo66, §I.4]) that $\mathrm{hd}(\mathcal{C}_k) = 1$ if $k$ is algebraically closed of characteristic 0. Also, by a result of Oort (see [Oo66, Thm. 14.1]), $\mathrm{hd}(\mathcal{C}_k) = 2$ if $k$ is algebraically closed of positive characteristic. Building on these results, Milne determined $\mathrm{hd}(\mathcal{C}_k)$ when $k$ is perfect (see [Mi70, Thm. 1]); then the homological dimension may take arbitrary large values. In the approach of Serre and Oort, the desired vanishing of higher extension groups is obtained by constructing projective resolutions of elementary groups, in the category of pro-algebraic groups. The latter category contains $\mathcal{C}_k$ as a full subcategory, and has enough projectives.

In contrast, to show that $\mathrm{hd}(\underline{\mathcal{C}}_k) = 1$ over an arbitrary field $k$, we do not need to go to a larger category. We rather observe that tori are projective objects in $\underline{\mathcal{C}}_k$, and abelian varieties are injective objects there. This yields the vanishing of all but three extension groups between simple objects of $\underline{\mathcal{C}}_k$; two of the three remaining cases are handled directly, and the third one reduces to the known vanishing of $\mathrm{Ext}^2_{\mathcal{C}_k}(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$ when $k$ is perfect.

When $k$ has characteristic 0, the fact that $\mathrm{hd}(\mathcal{C}_k) \leq 1$ can be deduced from the similar result for the category of Laumon 1-motives up to isogeny (obtained by Mazzari in [Ma10, Thm. 2.5]), by using the fact that $\mathcal{C}_k$ is equivalent to a full subcategory of the latter category. Likewise, the fact that the category of Deligne 1-motives up to isogeny has homological dimension at most 1 (due to Orgogozo, see [Or04, Prop. 3.2.4]) implies the corresponding assertion for the isogeny category of semi-abelian varieties over an arbitrary field.

Abelian categories of homological dimension 1 are called hereditary. The most studied hereditary categories consist either of finite-dimensional modules over a finite-dimensional hereditary algebra, or of coherent sheaves on a weighted projective line (see e.g. [Ha01]). Such categories are $k$-linear and Hom-finite, i.e., all groups of morphisms are vector spaces of finite dimension over the ground field $k$. But this seldom holds for the above isogeny categories. More specifically, $\underline{\mathcal{A}}_k$ and $\underline{\mathcal{T}}_k$ are both $\mathbb{Q}$-linear and Hom-finite, but not $\underline{\mathcal{C}}_k$ unless $k$ is a number field. In fact, $\underline{\mathcal{C}}_k$ may be viewed as a mixture of $k$-linear and $\mathbb{Q}$-linear categories. When $k$ has characteristic 0, this is already displayed by the full subcategory $\underline{\mathcal{V}}_k$ with objects the vector extensions of abelian varieties: in fact, $\underline{\mathcal{V}}_k$ has enough projectives, and these are either the unipotent groups ($k$-linear objects), or the vector extensions of simple abelian varieties ($\mathbb{Q}$-linear objects).

In positive characteristic, one may also consider the quotient category of $\mathcal{C}_k$ by the Serre subcategory $\mathcal{I}_k$ of infinitesimal group schemes. This yields the abelian category of 'algebraic groups up to purely inseparable isogeny', which is equivalent to that introduced by Serre in [Se60]; as a consequence, it has homological dimension 1 if $k$ is algebraically closed. For any arbitrary field $k$, the category $\mathcal{C}_k/\mathcal{I}_k$ is again invariant under purely inseparable field extensions; its homological properties may be worth investigating.

# References


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