Premet 60
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Tilting modules for algebraic groups and applications
(Henning Haahr Andersen)

(joint with C. Stroppel and D. Tubbenhauer).

G connected reductive algebraic group $/ k = \bar{k}$
$T \subset B \subset G$ max'l torus, Borel
$\Phi = \Phi(G, T) \supset \Phi^+$ roots, positive roots
$X = X(T) \supset X^+$ weights, dominant weights.

Then in the category of f.d. G-modules we have for each $\lambda \in X^+$

$$\begin{tikzcd} & L(\lambda) \text{ simple} \arrow[dr, hook] & \\ \Delta(\lambda) \arrow[ur, two heads] \arrow[rr] \arrow[dr, hook] & & \nabla(\lambda) \\ \text{Weyl} & T(\lambda) \arrow[ur, two heads] & \text{dual Weyl} \end{tikzcd}$$

indec. tilting

Here a G-module Q is tilting if it has both a Δ-filtration and a ∇-filtration.

THM (AST) Q tilting $\Rightarrow$ $\mathrm{End}_G(Q)$ is a cellular algebra

Remark If char $k = 0$ then all G-modules are s.s. Hence in that case $\Delta(\lambda) = L(\lambda) = \nabla(\lambda) = T(\lambda)$ for all λ and all modules are tilting, all $\mathrm{End}_G(Q)$'s are s.s.

The cellular basis of $\mathrm{End}_G(Q)$ is obtained as follows

$$\mathcal{P} = \{\lambda \in X^+ \mid (Q : \Delta(\lambda)) \neq 0\}, \text{ poset (wrt } \leq)$$
$$m_\lambda = (Q : \Delta(\lambda)), \quad I^\lambda = \{1, 2, \dots, m_\lambda\}, \quad \lambda \in \mathcal{P}.$$

Choose a basis $\{g_1^\lambda, \dots, g_{m_\lambda}^\lambda\}$ for $\mathrm{Hom}_G(\Delta(\lambda), Q)$ (note that $\dim \mathrm{Hom}_G(\Delta(\lambda), Q) = (Q : \nabla(\lambda)) = (Q : \Delta(\lambda)) = m_\lambda$) and set $f_j^\lambda = (g_j^\lambda)^* \in \mathrm{Hom}_G(Q, \nabla(\lambda))$. Then we have homomorphisms $\tilde{g}_i^\lambda$ and $\tilde{f}_j^\lambda$ with commutative diagrams

$$\begin{tikzcd}
 & \Delta(\lambda) \arrow[d, hook] \arrow[dr, "g_i^\lambda"] & \\
Q \arrow[r, dashed, "\tilde{f}_j^\lambda"] \arrow[dr, "f_j^\lambda"'] & T(\lambda) \arrow[r, dashed, "\tilde{g}_i^\lambda"] \arrow[d] & Q \\
 & \nabla(\lambda) &
\end{tikzcd}$$

(because $\mathrm{Ext}_G^1(\Delta, \nabla) = 0$)

Set $c_{ij}^\lambda = \tilde{g}_i^\lambda \circ \tilde{f}_j^\lambda \in \mathrm{End}_G(Q)$. These constitute a cellular basis for $\mathrm{End}_G(Q)$.

<u>Examples</u> 1. $\mathrm{End}_{GL(V)}(V^{\otimes d})$ is cellular.
By Schur-Weyl duality this algebra is a quotient of $k\mathcal{S}_d$ for all d and isomorphic to $k\mathcal{S}_d$ if $d \leq \dim V$.
In particular, $k\mathcal{S}_d$ is cellular.
By quantizing: $\mathcal{H}_q(\mathcal{S}_d)$ is cellular (for all $q \in k^\times$)

2. $\mathrm{End}_{E_8}(St_n^{\otimes d})$ is cellular $(St_n = L((p^n-1)\rho))$

3. Let V be the natural module for $G \in \{Sp(V), SO(V)\}$. Then $\mathrm{End}_G(V^{\otimes d})$ is cellular for all d (in case G has type B_n we need to exclude char $k = 2$).

Brauer algebras

Let $\delta \in \mathbb{Z}$, $d \in \mathbb{N}$. The Brauer algebra $B_d(\delta)$ over $\mathbb{Z}$ with d strands and parameter δ is the free $\mathbb{Z}$-module with basis all elements of the form

$(d = 5)$ (Brauer 1937)

and product given by: if A and B are two basis elements then $A \cdot B = \delta^m C$ where C is the diagram obtained by gluing B on top of A and removing all connected components, $m = \#$ components.

We set $B_d^k(\delta) = B_d(\delta) \otimes_{\mathbb{Z}} k$.

THM (Schur-Weyl duality for Sp)
Let V be a v.sp. of $\dim V = 2n$. Then there is a surjective homomorphism $\varphi : B_d^k(-2n) \longrightarrow \mathrm{End}_{Sp(V)}(V^{\otimes d})$ ~~with~~ and φ is an isomorphism if $n \geq d$.

Cor $B_d^k(\delta)$ is cellular for all $\delta (\in \mathbb{Z})$.

Proof: As $\delta = \delta + p$ in k we may assume $\delta \in [0, p-1]$.
If δ is even choose a s.t. $n = ap - \frac{\delta}{2} \geq d$. Then
$$B_d^k(\delta) = B_d^k(-2n) \simeq \mathrm{End}_{Sp(V)}(V^{\otimes d})$$

~~pos?~~

If δ is odd choose a s.t. $n = ap + \frac{p-\delta}{2} \geq d$.
Then $$B_d^k(\delta) = B_d^k(-2n) \simeq \mathrm{End}_{Sp(V)}(V^{\otimes d})$$

$(p = 2, \delta = 1$?? $)$.

$(p>2)$

<u>THM</u> (Schur-Weyl duality for SO)

i) $\dim V = 2n \Rightarrow \varphi: B_d^k(2n) \longrightarrow End_{SO(V)}(V^{\otimes d})$ is injective for $d \leq 2n$ and iso for $n > d$

ii) $\dim V = 2n+1 \Rightarrow \varphi: B_d^k(2n+1) \longrightarrow End_{SO(V)}(V^{\otimes d})$ is surjective for all d, n and an iso for $2n > d$.

<u>Corollary</u> $B_d^k(\delta)$ is semisimple iff one of the following holds

1) $\operatorname{char} k = 0$
  a) $\delta \neq 0$: $d \leq |\delta| + 1$
  b) $\delta = 0$: $d \in \{1, 3, 5\}$

2) $\operatorname{char} k = 2$: $d = 1$

3) $\operatorname{char} k = p > 2$
  a) δ_p odd : $d \leq \min\{\delta_p + 1, \frac{p - \delta_p + 2}{2}\}$
  b) $\delta_p \neq 0$ even : $d \leq \min\{\delta_p + 1, p - \delta_p + 3, p - 1\}$
  c) $\delta_p = 0$: $d \in \{1, 3, 5\} \cap [1, p-1]$

<u>Remark</u>: proved by W. P. Brown 1956 ($\operatorname{char} k = 0$)
(H. Wenzl 1988?, –"– $\delta \notin \mathbb{Z}$)
Rui 2005 $\operatorname{char} k$ arbitrary.

<u>Proposition</u> $End_G(Q)$ is s.s. iff Q is s.s iff $Q = \bigoplus_{\lambda \in P} \Delta(\lambda)$ with $\Delta(\lambda) = L(\lambda)$ for all $\lambda \in P$.

<u>Proof of corollary</u>

• char p $\Rightarrow$ char 0 : B^k s.s. for ∞ many char $p > 0$
$\Rightarrow B^K$ s.s (K of char 0).

• Consider 3 a) : $B_d^k(\delta) = B_d^k(\delta_p - p)$. Then $p - \delta_p = 2n$ and $B_d^k(\delta) \underset{\uparrow \text{iff } d \leq \frac{p-\delta_p}{2}}{\simeq} End_{Sp(V)}(V^{\otimes d})$, $\dim V = 2n$ (borderline : $d = \frac{p - \delta_p + 2}{2}$).

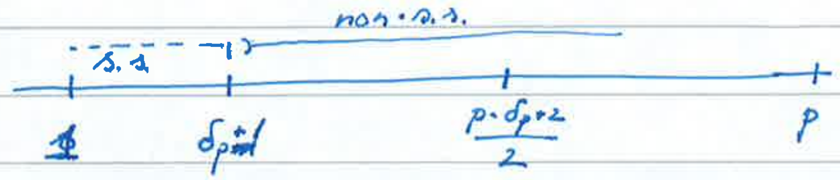
If $\Delta(\lambda)$ occur in $V^{\otimes d}$ then $\lambda \leq d\omega_1$ and $\underset{d + p - \delta_p - 1}{\langle \lambda + \rho, \alpha_0^\vee \rangle} \leq p$

Hence $\Delta(\lambda) = L(\lambda)$ by Strong Linkage.

<u>Obs</u> <u>(3a)</u> Let V have large even dimension with $\frac{1}{2}\dim V \equiv \delta_p$. Then $B_d(\delta) \simeq \mathrm{End}_{SP(V)}(V^{\otimes d})$ and $B_{d+2}(\delta) \simeq \mathrm{End}_{SP(V)}(V^{\otimes (d+2)})$.

Note that $\Delta(0)$ is a Weyl factor of $V^{\otimes 2}$. Hence if $\Delta(\lambda)$ is a Weyl factor of $V^{\otimes d}$ it is also a Weyl factor of $V^{\otimes (d+2)}$.

Hence $B_d(\delta)$ non-s.s $\Rightarrow$ $B_{d+2}(\delta)$ non-s.s.



Note: Q is non-s.s iff there exist $\Delta(\lambda) \neq L(\lambda)$ with $(Q:\Delta(\lambda)) \neq 0$. To find such λ we use the Jantzen sum formula (to check whether $\Delta(\lambda) \neq L(\lambda)$).