LINNIK’S PROOF OF THE WARING-HILBERT THEOREM
FROM HUA’S BOOK
(with a correction)

Notes by Tim Jameson

For integers \( s \geq 1, \ k \geq 2 \) and \( n \geq 0 \), let \( r_s^{(k)}(n) \) denote the number of solutions \((n_1, \ldots, n_s)\) of the equation \( n_1^k + \cdots + n_s^k = n \) with \( n_1, \ldots, n_s \geq 0 \).

The fact that \( r_4^{(2)}(n) > 0 \) for all \( n \) was probably already suspected by Diophantus (c.250). It was stated explicitly by Bachet in 1621, and later Fermat claimed to have a proof. The first accepted proof was by Lagrange in 1770, building on work of Euler.

Also in 1770, Waring wrote a letter to Euler in which he asserted that every natural number is a sum of 4 squares, 9 cubes, 19 biquadrates “and so on”. By this it is usually assumed that he meant that for each \( k \) there exists an \( s \) such that \( r_s^{(k)}(n) > 0 \) for all \( n \). We let \( g(k) \) denote the least such value of \( s \). The problem of showing that \( g(k) < \infty \) for all \( k \) has become known as Waring’s problem. It was first solved by Hilbert in 1909, by a complicated method [Hil]. Hardy and Littlewood [HL] gave a more elegant proof by the “circle method” in 1919, which was then refined and simplified by Vinogradov [Vin].

An “elementary” proof, using only number-theoretic methods, was given by Linnik in 1943 [Lin]. This method is presented in [Hua], but with one serious mistake. Here I give a version with this mistake corrected. A more general version, with \( x^k \) replaced by a polynomial, is given in [Nath].

Linnik’s method depends on the notion of Shnirelman density, defined as follows. Let \( A \) be a set of non-negative integers (possibly including 0). For each \( n \geq 1 \), let \( A(n) \) be the number of \( a \in A \) with \( 1 \leq a \leq n \). The Shnirelman density is

\[
\sigma(A) = \inf_{n \geq 1} \frac{A(n)}{n}.
\]

The application to Waring’s problem is via the following Lemma.

**LEMMA 1.** If \( A \) contains 0 and has positive Shnirelman density, then there exists \( h \) such that every positive integer is expressible as the sum of \( h \) members of \( A \).

We omit the proof of this Lemma, which is quite straightforward. See [Nath, Theorem 11.4] or [Hua, Theorem 19.2.3].
So it will be enough to prove that for given $k$, there exists $s$ such that \( \{ n : r_s^{(k)}(n) > 0 \} \) has positive Shnirelman density.

**LEMMA 2.** Let

\[
q(n; h_1, h_2) = \sum_{m_1, m_2 = -M \atop h_1 m_1 + h_2 m_2 = n} 1.
\]

For \( (h_1, h_2) \neq (0, 0) \) we have \( q(n; h_1, h_2) = 0 \) unless \( g = (h_1, h_2) | n \), say \( h_1 = ga_1 \), \( h_2 = ga_2 \) (so \( (a_1, a_2) = 1 \)). Then

\[
q(n; h_1, h_2) \leq \frac{2M}{\max(|a_1|, |a_2|)} + 1.
\]

**Proof.** Say \( n = gf \). Then the equation becomes

\[
a_1 m_1 + a_2 m_2 = f.
\]

On writing \( a_1 \bar{a}_1 + a_2 \bar{a}_2 = 1 \), we may write this as

\[
a_1 (m_1 - f \bar{a}_1) + a_2 (m_2 - f \bar{a}_2) = 0.
\]

Thus the general solution has the form

\[
m_1 = f \bar{a}_1 + ka_2,
\]

\[
m_2 = f \bar{a}_2 - ka_1.
\]

Wlog we may suppose \( a_1 \geq a_2 \geq 0 \). This implies \( a_1 > 0 \) and

\[
q(n; h_1, h_2) = \sum_{k : -M - f \bar{a}_1 \leq ka_2 \leq M - f \bar{a}_1 \atop -M + f \bar{a}_2 \leq ka_1 \leq M + f \bar{a}_2} 1.
\]

By the second condition, \( k \) is constrained to lie in an interval of length \( 2M/a_1 \). Such an interval contains at most \( 2M/a_1 + 1 \) integers, hence the statement. Note this all works fine if \( n = 0 \). \( \square \)

**LEMMA 3.** Let

\[
q(n) = \sum_{h_1, h_2 = -H \atop h_1, h_2 \neq 0} \sum_{m_1, m_2 = -M \atop h_1 m_1 + h_2 m_2 = n} 1.
\]

Then

\[
q(n) \leq \begin{cases} 
20HM \sigma_{-1}(n) & (n \neq 0, \ H \leq M), \\
20H^2 M & (n = 0).
\end{cases}
\]
Proof. We have

\[ q(n) = \sum_{h_1, h_2 = -H}^{H} q(n; h_1, h_2) \]

\[ = 4 \sum_{h_1, h_2 = 1}^{H} q(n; h_1, h_2) \]

\[ \leq 4H^2 + 8M \sum_{g|n} \sum_{1 \leq a_1 \leq H/g} \sum_{0 \leq a_2 < a_1} \frac{1}{\max(a_1, a_2)}. \]

We could have said \( a_2 \geq 1 \) here, but have allowed \( a_2 = 0 \) also (since it naturally leads to no worse an estimate and) so that the following applies to the variant of this lemma required for Lemma 7b:

\[ q(n) \leq 4H^2 + 8M \sum_{g|n} \sum_{1 \leq a_1 \leq H/g} \sum_{0 \leq a_2 < a_1} \frac{1}{a_1} + \sum_{1 \leq a_2 \leq H/g} \sum_{1 \leq a_1 \leq a_2} \frac{1}{a_2} \]

\[ = 4H^2 + 16M \sum_{g|n} \sum_{1 \leq a \leq H/g} 1 \]

\[ \leq 4H^2 + 16HM \sum_{g|n} \frac{1}{g}, \]

and the result follows. \( \square \)

Although we have thrown away a lot in the case \( n = 0 \) (we could give the result as \( 4H^2 + 16HM \log(eH) \)) this will have little effect in the application.

The following lemmas are of some interest in their own right.

**Lemma 4.** We have

\[ \sum_{a^2b^2 \leq H} \frac{1}{a^2b^2} = \frac{5}{2}. \]

*Proof.* Denote the sum by \( S \). We have

\[ \sum_{d,e=1}^{\infty} \frac{1}{d^2e^2} = \zeta(2)^2. \]

Now write \((d, e) = g\) and \(d = ga, e = gb\), so that \((a, b) = 1\). Then

\[ \sum_{d,e=1}^{\infty} \frac{1}{d^2e^2} = \sum_{g,a,b=1}^{\infty} \frac{1}{g^4a^2b^2} = \zeta(4)S. \]
Hence
\[ S = \frac{\zeta(2)^2}{\zeta(4)} = \frac{\pi^4/36}{\pi^4/90} = \frac{5}{2} \]

\[ \square \]

**LEMMA 5.** We have
\[ \sum_{d,e=1}^{\infty} \frac{1}{de[d,e]} = \frac{5}{2} \zeta(3). \]

**Proof.** Denote the sum by \( C \). With \( g, a, b \) as above, we have \([d, e] = gab\), hence
\[ C = \sum_{\substack{g,a,b \geq 1 \\ (n,h)=1}} \frac{1}{g^3a^2b^2} = \zeta(3)S. \]
\[ \square \]

**LEMMA 6.** We have
\[ \sum_{n \leq x} \sigma_{-1}(n)^2 \leq \frac{5}{2} \zeta(3)x. \]

**Proof.** This sum is
\[
\sum_{n \leq x} \sum_{d,e|n} \frac{1}{de} = \sum_{d,e \leq x} \frac{1}{de} \sum_{n \leq x \mod [d,e]} 1 \\
\leq \sum_{d,e \leq x} \frac{x}{de[d,e]} \\
\leq Cx,
\]
where \( C \) is as in Lemma 5.
\[ \square \]

A few further estimates show that in fact
\[ \sum_{n \leq x} \sigma_{-1}(n)^2 = \frac{5}{2} \zeta(3)x + O(\log^2 x). \]

**LEMMA 7 (for the inductive step in Theorem 1).** For \( H \leq M \) we have
\[ \sum_{h_1,\ldots,h_4=\pm H}^{H} \sum_{m_1,\ldots,m_4=\pm M}^{M} 1 \leq 5250(HM)^3. \]

**Proof.** The LHS is
\[
\sum_{n=-2HM}^{2HM} q(n)^2 \leq 20^2 \left( H^4M^2 + 2H^2M^2 \sum_{n=1}^{2HM} \sigma_{-1}(n)^2 \right) \\
\leq 20^2 \left( H^4M^2 + 2H^2M^2 \cdot \frac{5}{2} \zeta(3) \cdot 2HM \right) \\
= 20^2 \left( H^4M^2 + 10\zeta(3)H^3M^3 \right) \\
\leq 20^2(1 + 10\zeta(3))(HM)^3.
\]
Calculation shows that $20^2(1 + 10\zeta(3)) \approx 5208.$

**LEMMA 7b** (irritating variant needed for Lemma 8). *For* $2H \leq M$ *we have*

$$\sum_{h_1, \ldots , h_4 = -H}^{H} \sum_{m_1, \ldots , m_4 = -M}^{M} 1 \leq 162M^4 + 5250(HM)^3.$$  

*Proof.* Let

$$Q(n) = \sum_{h_1, h_2 = -H}^{H} \sum_{m_1, m_2 = -M}^{M} q(n; h_1, h_2) \quad = q(n; 0, 0) + 4 \sum_{1 \leq h_1 \leq H} q(n; h_1, h_2).$$

For $n \neq 0$ we have $q(n; 0, 0) = 0$ and obtain the same bound for $Q(n)$ as that for $q(n)$ given by Lemma 2: In the working of Lemma 2 the $8H^2$ is replaced by $8H(H + 1)$, but since $H < M$ we can still say $8H(H + 1) \leq 8HM$.

However, in the case $n = 0$ we have an additional term $q(n; 0, 0) = (2M + 1)^2 \leq 9M^2$. Thus we have

$$Q(0) \leq 9M^2 + 20H^2M,$$

and so

$$Q(0)^2 \leq 2(9M^2)^2 + 2(20H^2M)^2 \quad = 162M^4 + 20^2H^2M^2 \cdot 2H \quad \leq 162M^4 + 20^2(HM)^3.$$  

The working of Lemma 7 now gives

$$\sum_{h_1, \ldots , h_4 = -H}^{H} \sum_{m_1, \ldots , m_4 = -M}^{M} 1 = \sum_{n = -2HM}^{2HM} Q(n)^2 \leq 162M^4 + 5250(HM)^3,$$

as required.  

**LEMMA 8** (case $k = 2$ of Theorem 1). *Let*

$$f(n) = a_2n^2 + a_1n$$
where $a_2, a_1$ are integers with

$$0 < |a_2| \leq c_2, \quad |a_1| \leq c_1N.$$ 

Then for $N \geq 1$ we have

$$\int_0^1 \left| \sum_{n=0}^N e(\alpha f(n)) \right|^8 d\alpha \leq CN^6$$

where

$$C = 162(2c_2 + c_1)^4 + 5250(2c_2 + c_1)^3.$$ 

In particular $C = 44592$ when $f(n) = n^2$, $c_2 = 1$, $c_1 = 0$.

Proof. We have

$$\int_0^1 \left| \sum_{n=0}^N e(\alpha f(n)) \right|^8 d\alpha = \sum_{n_1, \ldots, n_8=0}^N \int_0^1 e(\alpha(f(n_1) + \cdots + f(n_4) - f(n_5) - \cdots - f(n_8))) d\alpha$$

$$= \sum_{n_1, \ldots, n_8=0}^N 1.$$

We may write the equation here as

$$\sum_{i=1}^4 (f(n_i) - f(n_{i+4})) = \sum_{i=1}^4 (a_2(n_i^2 - n_{i+4}^2) + a_1(n_i - n_{i+4}))$$

$$= \sum_{i=1}^4 h_i m_i$$

$$= 0,$$

where

$$h_i = n_i - n_{i+4}$$

$$m_i = a_2(n_i + n_{i+4}) + a_1.$$

Note that $(h_i, m_i)$ uniquely determines $(n_i, n_{i+4})$ since

$$\begin{pmatrix} 1 & -1 \\ a_2 & a_2 \end{pmatrix} \begin{pmatrix} n_i \\ n_{i+4} \end{pmatrix} = \begin{pmatrix} h_i \\ m_i - a_1 \end{pmatrix}$$

has the inverse

$$\begin{pmatrix} n_i \\ n_{i+4} \end{pmatrix} = \frac{1}{2a_2} \begin{pmatrix} a_2 & 1 \\ -a_2 & 1 \end{pmatrix} \begin{pmatrix} h_i \\ m_i - a_1 \end{pmatrix}$$

for $a_2 \neq 0$. Clearly we have

$$|h_i| \leq N.$$
and

$$|m_i| \leq M, \text{ where } M = (2c_2 + c_1)N.$$  

Noting that $M \geq 2N$ since $c_2 \geq |a_2| \geq 1$, the result follows from Lemma 6b.

**THEOREM 1.** Let $k \geq 2$ and

$$f(n) = a_k n^k + \cdots + a_1 n$$

where $a_1, \ldots a_k$ are integers with $a_k \neq 0$ and

$$|a_j| \leq c_{j,k} N^{k-j}.$$ 

Then for $N \geq 1$ we have

$$\int_0^1 \left| \sum_{n=0}^N e(\alpha f(n)) \right|^{8^{k-1}} d\alpha \ll_{k,c_{1,k},\ldots,c_{k,k}} N^{8^{k-1}-k}.$$ 

**Proof.** The statement is more than we need for the application. It has been elaborated to make its proof by induction on $k$ work. The case $k = 2$ is given by Lemma 8. We will omit the suffices in the $\ll$ notation. Suppose the statement is true with $k-1$ in place of $k$. We have

$$\left| \sum_{n=0}^N e(\alpha f(n)) \right|^2 = \sum_{m,n=0}^N e(\alpha f(m) - \alpha f(n)) = N + 1 + \sum_{h=0}^{N} b_h, \quad (1)$$

where

$$b_h = \sum_{m,n=0}^N e(\alpha f(m) - \alpha f(n)) = \sum_{n=\max(0,-h)}^{\min(N,N-h)} e(\alpha h \phi(n,h)),$$

where

$$\phi(n,h) = \frac{1}{h} (f(n+h) - f(n))$$

$$= \frac{1}{h} \sum_{j=1}^k a_j ((n+h)^j - n^j)$$

$$= \sum_{j=1}^k a_j \sum_{r=0}^{j-1} \binom{j}{r} h^{j-r-1} n^r$$

$$= \sum_{r=0}^{k-1} \left( \sum_{j=r+1}^k \binom{j}{r} a_j h^{j-r-1} \right) n^r$$

is a degree $k-1$ polynomial in $n$. From the definition we see that $\phi(n,h) \ll N^{k-1}$. The coefficient of $n^{k-1}$ in $\phi(n,h)$ is

$$\left( \binom{k}{k-1} a_k h^{k-(k-1)-1} = ka_k \neq 0, \right.$$
and the coefficient of \( n^r \) is

\[
\sum_{j=r+1}^{k} \binom{j}{r} a_j h^{j-r-1} \ll \sum_{j=r+1}^{k} \binom{j}{r} N^{k-j} N^{j-r-1} \ll N^{k-r-1}.
\]

Raising (1) to the power \( 8^{k-2} \) using Hölder’s inequality gives

\[
\left| \sum_{n=0}^{N} e(\alpha f(n)) \right|^{2 \cdot 8^{k-2}} \ll N^{8^{k-2}} + \sum_{\substack{h=-N \\ h \neq 0}}^{N} \left| b_h \right|^{8^{k-2}}
\]

\[
\ll N^{8^{k-2}} + \left( \sum_{\substack{h=-N \\ h \neq 0}}^{N} 1 \right) \sum_{\substack{h=-N \\ h \neq 0}}^{N} \left| b_h \right|^{8^{k-2}}
\]

\[
\ll N^{8^{k-2}} + N^{8^{k-2}-1} \sum_{\substack{h=-N \\ h \neq 0}}^{N} \left| b_h \right|^{8^{k-2}}.
\]

Raising this to a further fourth power and integrating over \( \alpha \) then gives

\[
\int_{0}^{1} \left| \sum_{n=0}^{N} e(\alpha f(n)) \right|^{8^{k-1}} d\alpha \ll N^{4 \cdot 8^{k-2}} + N^{4 \cdot 8^{k-2}-4} \int_{0}^{1} \left( \sum_{\substack{h=-N \\ h \neq 0}}^{N} \left| b_h \right|^{8^{k-2}} \right)^{4} d\alpha.
\]

(2)

As a function of \( \alpha \), \( b_h \) has period \( 1/|h| \). Let \( |b_h|^{8^{k-2}} \) have the Fourier series

\[
|b_h|^{8^{k-2}} = \sum_{m=-\infty}^{\infty} A(m, h) e(\alpha m).
\]

This is finite really because

\[
A(m, h) \neq 0 \Rightarrow m \ll \max_{0 \leq n \leq N} |\phi(n, h)| \ll N^{k-1},
\]

so we may write the range for \( m \) as \( |m| \leq CN^{k-1} \) (where \( C \) is independent of \( h \)). The coefficients are given by

\[
A(m, h) = \int_{0}^{1} |b_h|^{8^{k-2}} e(-\alpha m) d\alpha
\]

\[
= \int_{0}^{1} \left| \sum_{n=\max(0, -h)}^{\min(N, N-h)} e((\text{sgn } h)\beta \phi(n, h)) (\text{sgn } h)\beta \phi(n, h)) \right|^{8^{k-2}} e((-\text{sgn } h)\beta m) d\beta
\]

\[
= \int_{0}^{1} \left| \sum_{n=\max(0, -h)}^{\min(N, N-h)} e(\beta \phi(n, h)) \right|^{8^{k-2}} e((-\text{sgn } h)\beta m) d\beta.
\]

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Thus

\[ |A(m, h)| \leq \int_0^1 \left| \sum_{n=\max(0,-h)}^{\min(N,N-h)} e(\beta \phi(n)) \right| d\beta \]

\[ \ll N^{8k^2-(k-1)}, \]

by the inductive hypothesis. (We have trivially translated the region of summation over \( n \). This should be written in as part of the official statement or perhaps done away with by restricting to \( h > 0 \) using a \( 2\Re \).) Now we have

\[ \int_0^1 \left( \sum_{h \neq 0}^N |b_h|^{8k^2} \right)^4 d\alpha = \int_0^1 \left( \sum_{h \neq 0}^N \sum_{|m| \leq CN^{k-1}} A(m, h)e(\alpha hm) \right)^4 d\alpha \]

\[ = \sum_{h_1,\ldots,h_4=\neq 0}^N \sum_{|m_1|,\ldots,|m_4| \leq CN^{k-1}} \left( \prod_{i=1}^4 A(m_i, h_i) \right) \int_0^1 e \left( \alpha \sum_{i=1}^4 h_i m_i \right) d\alpha \]

\[ = N^4(8k^2-(k-1)) \sum_{h_1,\ldots,h_4=\neq 0}^N \sum_{|m_1|,\ldots,|m_4| \leq CN^{k-1}} \prod_{i=1}^4 A(m_i, h_i) \]

\[ \ll N^4(8k^2-(k-1)) N^{3k} \]

\[ = N^4(8k^2-2k+4), \]

by Lemma 7 (note that we may assume \( C \geq 1 \)). So finally (2) gives

\[ \int_0^1 \left( \sum_{n=0}^N e(\alpha f(n)) \right)^{8k^2-1} d\alpha \ll N^{48k^2-2} + N^{8k^2-4} N^{48k^2-2-k+4} \]

\[ \approx \ll N^{48k^2-2} + N^{8k^2-1}. \]

The fact that the second term dominates is equivalent to \( 8k^2 \geq 16k \), which is certainly true for \( k \geq 3 \) so completing the proof. \( \square \)

**Theorem 2.** Let \( k \geq 2 \) and \( s = 8k^2-1 \). Then the set \( A = \{ n \geq 1 : r_s^{(k)}(n) > 0 \} \) has positive Schnirelmann density.

**Proof.** Write \( r(n) \) for \( r_s^{(k)}(n) \). We have

\[ \sum_{n=0}^N r(n) = \sum_{m_1,\ldots,m_s \geq 0} \sum_{m_1^2+\ldots+m_s^2 \leq N} 1 \]

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\[ \geq \sum_{0 \leq m_1, \ldots, m_s \leq (N/s)^{1/k}} 1 \]
\[ \geq (N/s)^{s/k} \quad \gg_k N^{s/k}, \]

But on the other hand, for \( n \geq 1 \),
\[
\begin{align*}
\sum_{0 \leq m_1, \ldots, m_s \leq n^{1/k}} 1 &= \int_0^1 e(\alpha m_1 + \cdots + m_s n) \, d\alpha \\
&= \int_0^1 \left( \sum_{0 \leq m \leq n^{1/k}} e(\alpha m) \right)^s \, d\alpha \\
&\leq \int_0^1 \left| \sum_{0 \leq m \leq n^{1/k}} e(\alpha m) \right|^s \, d\alpha \\
&\ll_k \left( n^{1/k} \right)^{s-k} \quad \text{(by Theorem 1)} \\
&= n^{s/k-1},
\end{align*}
\]

so that
\[
\sum_{n=0}^N r(n) \ll_k 1 + N^{s/k-1} A(N).
\]

The statement clearly follows. \( \square \)

By Lemma 1, we can deduce at once:

THEOREM 3 (the Waring-Hilbert theorem). For each \( k \geq 2 \), there exists \( s(k) \) such that \( r_s(n) > 0 \) for all \( n \geq 0 \).

The mistake in [Hua] is the claim that \( Q(0) \ll \min(H^2 M, M^2 H) \ll (HM)^{3/2} \), which is wrong because there are \((2M+1)^2\) solutions with \( h_1 = h_2 = 0 \). This leads to the incorrect bound \( \ll (HM)^3 \) (regardless of the relative sizes of \( H \) and \( M \)) for the quantity in Lemma 6b. This is wrong if \( M \) is much bigger than \( H \) (as it is in the application to the inductive step in Theorem 1) since \( Q(0)^2 \) then dominates. The way I’ve corrected it is simply to note that \( h_1, \ldots, h_4 = 0 \) does not occur in Theorem 1 so we can use Lemma 7 instead.

References (added by Graham Jameson)


