An inequality for the gamma function conjectured by D. Kershaw


1. Introduction

It has long been known that \( \Gamma(x) + \Gamma(1/x) \geq 2 \); a quick proof is by convexity of \( \Gamma(x) \) and \( \Gamma(1/x) \). Gautschi [3] generalized this statement by showing that the harmonic mean of \( \Gamma(x) \) and \( \Gamma(1/x) \) is not less than 1, which of course implies also that \( \Gamma(x)\Gamma(1/x) \geq 1 \). Since then, inequalities concerning \( \Gamma(1/x) \) have been investigated in many further articles. Alzer [1] extended Gautschi’s result to the power means \( M_r[\Gamma(x), \Gamma(1/x)] \). Kershaw and Laforgia [5] showed that \( [\Gamma(1+1/x)]^x \) is decreasing, while \( x[\Gamma(1+1/x)]^x \) is increasing. Giordano and Laforgia [4] extended Gautschi’s product inequality by proving that

\[
\frac{1}{e} \Gamma \left( 1 + x + \frac{1}{x} \right) \leq \Gamma(x) \Gamma \left( \frac{1}{x} \right) \leq \Gamma \left( 1 + x + \frac{1}{x} \right).
\]

In connection with these results, Donald Kershaw, in private communication, formulated the following conjecture: for all \( x > 0 \),

\[
\Gamma(x) + \Gamma \left( \frac{1}{x} \right) \leq \Gamma \left( 1 + x + \frac{1}{x} \right),
\]

(1)

with equality only at \( x = 1 \). In a recent article [2], Alzer obtained the following result in this direction (alongside a further generalization of Gautschi’s result on harmonic means): \( \Gamma(x) + \Gamma(1/x) \leq b\Gamma(x + 1/x) \), where \( b \approx 2.098 \). Since \( x + 1/x \geq 2 \), this implies a version of (1) with an intervening factor \( b/2 \) (which, of course, fails to reproduce equality at 1). The methods of [2] rely on a considerable number of specific values of the gamma function and higher derivatives.

Here we will prove that Kershaw’s conjecture is true, without any extra factor.

Since (1) is unchanged when \( x \) is replaced by \( 1/x \), it is enough to prove it for \( x > 1 \). We use an assortment of different methods on different parts of the domain. The inequality (indeed, a rather stronger one) is obtained quite easily for all \( x \geq 2 \). For \( \frac{5}{4} \leq x \leq 2 \), we use convexity of \( \Gamma(x) \) and \( \Gamma(1/x) \) to derive linear bounds for the two sides of (1), which then only need to be compared at the end points; we do this on two shorter intervals. Only a few specific values of \( \Gamma(x) \) are needed, to no great degree of accuracy.

The most interesting part of the problem is for \( x \) close to 1. Both sides of (1) have derivative 0 at 1, so there is no longer any chance of deducing the result from linear upper


and lower bounds. However, after substituting $1 + x$ for $x$, a lower bound for the right-hand side of the form $2 + c(x^2 - x^3)$ can still (as before) be derived from the tangent to $\Gamma(x)$ at 3. To estimate the left-hand side, we now use the power series for $\Gamma(1 + x)$. We apply the exact values of the first three coefficients, together with a bound for the remaining ones, to establish (1) for $1 < x \leq \frac{5}{4}$.

2. Proof of (1) for $x \geq 2$

Note that $\Gamma(1/x) < x$ for $x > 1$. For $x \geq 2$, we prove (1) with $\Gamma(1/x)$ replaced by $x$.

Case $x \geq 3$. Since $\Gamma(x) > \Gamma(3) = 2$ for $x > 3$, we have

$$\Gamma \left( 1 + x + \frac{1}{x} \right) - \Gamma(x) > \Gamma(1 + x) - \Gamma(x) = (x - 1)\Gamma(x) \geq 2(x - 1) > x.$$

LEMMA 1. Let $a > 0$, and let $P_a(x) = \Gamma(1 + x + a) - \Gamma(x) - x$. Then $P_a(x)$ is increasing for $x \geq 2$.

Proof. Since $\Gamma'(x)$ is increasing for all $x > 0$, so is $\Gamma(x + a) - \Gamma(x)$. Also, $\Gamma(x)$ is increasing for $x \geq 2$. The statement follows, since

$$P_a(x) = (x + a)\Gamma(x + a) - \Gamma(x) - x = x(\Gamma(x + a) - \Gamma(x)) + a\Gamma(x + a) + x\Gamma(x) - \Gamma(x) - x = x(\Gamma(x + a) - \Gamma(x)) + a\Gamma(x + a) + (\Gamma(x) - 1)(x - 1) - 1. \quad \square$$

Case $2^\frac{1}{3} \leq x \leq 3$. Then $\Gamma(1 + x + \frac{1}{x}) \geq \Gamma(1 + x + \frac{1}{3})$. Our inequality follows by Lemma 1 provided that $P_{1/3}(2^\frac{1}{3}) > 0$. We verify this:

$$P_{1/3}(2^\frac{1}{3}) = \frac{2}{3}\Gamma(\frac{5}{3}) - \Gamma(\frac{5}{3}) - \frac{7}{3} \approx 4.012 - 1.191 - 2.333 = 0.488 > 0.$$

Case $2 \leq x \leq 2^\frac{1}{4}$. We now have $\frac{1}{2} \geq \frac{3}{7}$ on the interval, so we verify

$$P_{3/7}(2) = \frac{17}{5}\Gamma(\frac{17}{7}) - 1 - 2 \approx 3.074 - 3 > 0.$$

3. Proof of (1) for $\frac{5}{4} \leq x \leq 2$

LEMMA 2. $\Gamma(1/x)$ is a convex function of $x$ for $x > 0$.

Proof. Let $0 < x_1 < x_2$, and write $y_j = 1/x_j$ ($j = 1, 2$). Choose $a, b$ so that $\Gamma(1 + y_j) = ay_j + b$ for $j = 1, 2$. Since $\Gamma$ is convex, $\Gamma(1 + y) \leq ay + b$ for $y_2 \leq y \leq y_1$. Also,
since \( \Gamma(1 + y) = y \Gamma(y) \), we have \( \Gamma(y) \leq a + b/y \) for \( y_2 \leq y \leq y_1 \), with equality at \( y_1 \) and \( y_2 \). So \( \Gamma(1/x) \leq a + bx \) for \( x_1 \leq x \leq x_2 \), with equality at \( x_1 \) and \( x_2 \), so that \( a + bx \) is the linear function agreeing with \( \Gamma(1/x) \) at these points. \( \square \)

(In general, if a function \( f \) is convex and increasing, then \( f(1/x) \) is convex, but this is not true for decreasing \( f \).)

**Note.** Let \( G(x) = \Gamma(x) + \Gamma(1/x) \). It is shown in [2, Lemma 2] that \( G(x) \) is decreasing on \((0, 1]\). This follows at once from our Lemma 2, since \( G(x) \) is convex and \( G'(1) = 0 \). Of course, the inequality \( G(x) \geq 2 \) (for all \( x \)) follows.

**LEMMA 3.** For all \( x > 0 \),

\[
\Gamma \left( 1 + \frac{x + 1}{x} \right) \geq 2 + (3 - 2\gamma) \frac{(x - 1)^2}{x}.
\]

**Proof.** By convexity of the gamma function,

\[
\Gamma(3 + y) \geq \Gamma(3) + y \Gamma'(3) = 2 + (3 - 2\gamma)y
\]

for all \( y > 0 \). The statement follows, since

\[
1 + x + \frac{1}{x} = 3 + \frac{(x - 1)^2}{x}.
\]

**Proof of (1) for \( \frac{5}{4} \leq x \leq 2 \).** We consider the intervals \([\frac{5}{4}, \frac{3}{2}]\) and \([\frac{3}{2}, 2]\) separately. Write \( \Gamma(x) + \Gamma(1/x) = G(x) \). By Lemma 2, \( G(x) \) is convex. Using Lemma 3, we define a linear function \( F(x) \) that is a lower bound for \( \Gamma(1 + x + 1/x) \) on the interval in question. The statement then follows on verification that \( F(x) > G(x) \) at the end points.

Let \( h(x) = (x - 1)^2/x \). Then \( h'(x) = 1 - 1/x^2 \). Hence \( h(x) \) is convex, and \( h(x) \geq h(x_0) + (x - x_0)h'(x_0) \) for any \( x, x_0 > 0 \). For the interval \([\frac{5}{4}, \frac{3}{2}]\), take \( x_0 = \frac{5}{4} \). We find that \( h(x) \geq h_1(x) \) on the interval, where

\[
h_1(x) = \frac{1}{20} + \frac{9}{25} \left( x - \frac{5}{4} \right).
\]

Our linear lower bound is \( F_1(x) = 2 + (3 - 2\gamma)h_1(x) \). Note that \( h_1(\frac{3}{2}) = \frac{7}{50} \). The values are

\[
F_1(\frac{5}{4}) \approx 2.092, \quad G(\frac{5}{4}) = \Gamma(\frac{5}{4}) + \Gamma'(\frac{5}{4}) \approx 0.906 + 1.164 = 2.070.
\]

\[
F_1(\frac{3}{2}) \approx 2.258, \quad G(\frac{3}{2}) = \Gamma(\frac{3}{2}) + \Gamma'(\frac{3}{2}) \approx 0.886 + 1.354 = 2.240.
\]

For the interval \([\frac{3}{2}, 2]\), take \( x_0 = \frac{3}{2} \), giving

\[
h_2(x) = \frac{1}{6} + \frac{5}{9} \left( x - \frac{3}{2} \right),
\]

For the interval \([\frac{3}{2}, 2]\), take \( x_0 = \frac{3}{2} \), giving
with corresponding $F_2(x)$. Clearly, $F_2\left(\frac{3}{2}\right) > F_1\left(\frac{3}{2}\right)$. Also, $h_2(2) = \frac{4}{9}$, and we find

$$F_2(2) \approx 2.820, \quad G(2) = \Gamma(2) + \Gamma\left(\frac{1}{2}\right) \approx 2.772.$$

4. The power series for $\Gamma(1 + x)$

We write the power series for $\Gamma(1 + x)$ in the form $\sum_{n=0}^{\infty}(-1)^n a_n x^n$, since (as we now show) the coefficients alternate in sign. Note that $a_0 = \Gamma(1) = 1$. Now

$$\Gamma(n)(x) = \int_0^\infty t^{x-1} e^{-t} (\log t)^n \, dt,$$

hence

$$a_n = \frac{(-1)^n}{n!} \Gamma(n)(1) = \frac{1}{n!} \int_0^\infty e^{-t} (- \log t)^n \, dt.$$

The following bound is not optimal, but it is adequate for our purposes.

**Lemma 4.** With this notation, we have $0 < a_n \leq m$ for $n \geq 4$, where $m \leq \frac{13}{12}$.

**Proof.** We have

$$\int_0^1 e^{-t} (- \log t)^n \, dt < \int_0^1 (- \log t)^n \, dt = \int_0^\infty u^n e^{-u} \, du = n!.$$

At the same time, this integral is greater than $e^{-1} n!$. Also, since $\log t < t^{1/2}$ for $t > 1$,

$$\int_1^\infty e^{-t} (\log t)^n \, dt < \int_1^\infty e^{-t^{n/2}} \, dt < \Gamma\left(\frac{n}{2} + 1\right) = \frac{n!}{n(n-1)} \leq \frac{n!}{12}$$

for $n \geq 4$. The statement follows. $\square$

**Note.** Using the series expansion for $e^{-t}$, one finds that

$$\frac{1}{n!} \int_0^1 e^{-t} (- \log t)^n \, dt = 1 - \frac{1}{2!2^n} + \frac{1}{3!3^n} - \cdots.$$

One can deduce that $\lim_{n \to \infty} a_n = 1$ and $a_n < 1$ for all $n$. The authors are grateful to Pascal Sebah for these observations, and for the calculated values of $a_n$ for $n \leq 20$.

Meanwhile, explicit values for the first few coefficients can be derived more pleasantly as follows. We use the power series (convergent for $|x| < 1$)

$$\frac{\Gamma'(1 + x)}{\Gamma(1 + x)} = \sum_{n=0}^{\infty} (-1)^{n+1} c_n x^n,$$

where $c_0 = \gamma$ and $c_n = \zeta(n+1)$ for $n \geq 1$ [6, p. 12]. Now equating coefficients in the identity

$$\sum_{n=0}^{\infty} (-1)^{n+1}(n + 1) a_{n+1} x^n = \left(\sum_{n=0}^{\infty} (-1)^n c_n x^n\right) \left(\sum_{n=0}^{\infty} (-1)^n a_n x^n\right)$$

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we see that

\[(n + 1)a_{n+1} = c_n a_0 + c_{n-1} a_1 + \cdots + c_0 a_n\]

for all \(n \geq 1\). In particular, \(a_1 = c_0 = \gamma\),

\[
a_2 = \frac{1}{2}(c_1 a_0 + c_0 a_1) = \frac{1}{2} (\zeta(2) + \gamma^2) \approx 0.9891, \\
a_3 = \frac{1}{3}(c_2 a_0 + c_1 a_1 + c_0 a_2) = \frac{1}{6} (2\zeta(3) + 3\zeta(2)\gamma + \gamma^3) \approx 0.9075.
\]

5. Proof of \((1)\) for \(1 \leq x \leq \frac{5}{4}\)

We now substitute \(1 + x\) for \(x\), so that \((1)\) becomes

\[
\Gamma(1 + x) + \Gamma \left( \frac{1}{1 + x} \right) \leq \Gamma \left( 2 + x + \frac{1}{1 + x} \right) \tag{2}
\]

We have to prove \((2)\) for \(0 \leq x \leq \frac{1}{4}\). We continue to use Lemma 3. In the new notation, this says

\[
\Gamma \left( 2 + x + \frac{1}{1 + x} \right) \geq 2 + (3 - 2\gamma) \frac{x^2}{1 + x}.
\]

For \(0 < x < 1\), we have \(1/(1 + x) > 1 - x\), hence

\[
\Gamma \left( 2 + x + \frac{1}{1 + x} \right) \geq 2 + (3 - 2\gamma)(x^2 - x^3). \tag{3}
\]

**Lemma 5.** For \(0 \leq x \leq \frac{1}{4}\),

\[
\Gamma(1 + x) \leq 1 - \gamma x + a_2 x^2 + b_3 x^3, \tag{4}
\]

where \(b_3 = -a_3 + \frac{4}{15} m \approx -0.619\).

**Proof.** Since the terms of the power series alternate in sign,

\[
\Gamma(1 + x) \leq 1 - \gamma x + a_2 x^2 - a_3 x^3 + m(x^4 + x^6 + \cdots),
\]

and for \(0 \leq x \leq \frac{1}{4}\),

\[
x^4 + x^6 + \cdots = \frac{x^4}{1 - x^2} = x^3 \frac{x}{1 - x^2} \leq \frac{4}{15} x^3. \quad \square
\]

**Lemma 6.** For \(0 \leq y \leq \frac{1}{5}\),

\[
\Gamma(1 - y) \leq 1 + \gamma y + a_2 y^2 + c_3 y^3, \tag{5}
\]

where \(c_3 = a_3 + \frac{4}{15} m \approx 1.178\).
Proof. We have \( \Gamma(1-y) = 1 + \gamma y + \sum_{n=2}^{\infty} a_n y^n \), and for \( 0 \leq y \leq \frac{1}{5} \),
\[
a_4 y^4 + a_5 y^5 + \cdots \leq m \frac{y^4}{1-y} = my^3 \frac{y}{1-y} \leq \frac{1}{4} my^3.
\]
\[\square\]

**Lemma 7.** For \( 0 \leq x \leq \frac{1}{4} \),
\[
\Gamma \left( \frac{1}{1+x} \right) \leq 1 + \gamma x + d_2 x^2 + d_3 x^3, \tag{6}
\]
where
\[
d_2 = a_2 - \frac{4}{5} \gamma, \quad d_3 = c_3 - \frac{36}{25} a_2 \approx -0.246.
\]

**Proof.** Note that \( \frac{1}{1+x} = 1 - x/(1+x) \). We apply Lemma 6, with \( y = x/(1+x) \), using the following estimates derived from convexity of \( 1/(1+x) \) and \( 1/(1+x)^2 \): for \( 0 \leq x \leq \frac{1}{4} \),
\[
\frac{1}{1+x} \leq 1 - \frac{4}{5} x, \quad \frac{1}{(1+x)^2} \leq 1 - \frac{36}{25} x.
\]
We obtain
\[
\Gamma \left( \frac{1}{1+x} \right) \leq 1 + \gamma x (1 - \frac{4}{5} x) + a_2 x^2 (1 - \frac{36}{25} x) + c_3 x^3
\]
\[
= 1 + \gamma x + (a_2 - \frac{4}{5} \gamma) x^2 + (c_3 - \frac{36}{25} a_2) x^3. \quad \square
\]

Proof of (2) for \( 0 \leq x \leq \frac{1}{4} \). By (3), (4), (6), for \( 0 \leq x \leq \frac{1}{4} \),
\[
\Gamma \left( 2 + x + \frac{1}{1+x} \right) - \Gamma(1+x) - \Gamma \left( \frac{1}{1+x} \right) \geq A_2 x^2 + A_3 x^3,
\]
where
\[
A_2 = 3 - \frac{6}{5} \gamma - 2a_2 \approx 0.329, \quad A_3 = -(3-2\gamma) - b_3 - d_3 \approx -1.845 + 0.619 + 0.246 = -0.980.
\]
Clearly, \( A_2 x^2 + A_3 x^3 > 0 \) for \( 0 \leq x \leq \frac{1}{4} \). \[\square\]

**Remark.** The power series shows clearly that we cannot replace \( \Gamma[1/(1+x)] \) by \( \Gamma(1-x) \) in (2), illustrating the closeness of our inequality. Indeed, for small \( x > 0 \), we have the slightly stronger reverse inequality \( \Gamma(1+x) + \Gamma(1-x) > \Gamma(3+x^2) \), since
\[
\Gamma(1+x) + \Gamma(1-x) = 2 + 2a_2 x^2 + O(x^4), \quad \Gamma(3+x^2) = 2 + (3-2\gamma)x^2 + O(x^4),
\]
and \( 3-2\gamma < 2a_2 \).
References


