

# ELEMENTARY LIE ALGEBRAS AND LIE A-ALGEBRAS

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## 1 Introduction

Throughout this paper  $L$  will denote a finite-dimensional Lie algebra over a field  $F$ . The Frattini ideal of  $L$ ,  $\phi(L)$ , is the largest ideal of  $L$  contained in all maximal subalgebras of  $L$ . The Lie algebra  $L$  is called  *$\phi$ -free* if  $\phi(L) = 0$ , and *elementary* if  $\phi(B) = 0$  for every subalgebra  $B$  of  $L$ . Elementary Lie algebras were introduced by Stitzinger [24] and Towers [27] by analogy to the definition of an elementary group given earlier by Bechtel [3]. An interesting property of an elementary Lie algebra is that it splits over each of its ideals, see [27].

The class of elementary Lie algebras is closely related to the class of Lie algebras all whose nilpotent subalgebras are abelian (called *A*-algebras) and to the class of Lie algebras  $L$  such that  $\phi(B) \leq \phi(L)$  for all subalgebras  $B$  of  $L$  (called *E*-algebras). *A*-algebras have been studied by Drensky [7], Sheina [23], Premet and Semenov [21] and Dallmer [6]. Since the Frattini ideal of a nilpotent Lie algebra  $L$  is just the derived subalgebra of  $L$ , every elementary Lie algebra is an *A*-algebra. *E*-algebras were introduced by Stitzinger in [24]. He proved that  $L$  is an *E*-algebra if and only if  $L/\phi(L)$  is elementary. A Lie algebra  $L$  is called *strongly solvable* if  $L^2$  is nilpotent. Stitzinger also

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proved in [24] that if  $L$  is strongly solvable then  $L$  is an  $E$ -algebra. In this paper it is shown that over a perfect field the converse also holds

For algebraically closed fields of characteristic zero, elementary Lie algebras were determined by Towers in [27]. This classification is shown to remain true for any algebraically closed field of characteristic different from two or three.

Following Jacobson [15], we say that a linear Lie algebra  $L \leq \mathfrak{gl}(V)$  is *almost algebraic* if  $L$  contains the nilpotent and semisimple Jordan components of its elements. Every algebraic Lie algebra is almost algebraic. An abstract Lie algebra  $L$  is called almost algebraic if  $\text{ad}L \leq \mathfrak{gl}(L)$  is almost algebraic. Recently, Zhao and Lu have proved in [29] that every almost-algebraic  $A$ -algebra is elementary, whenever the ground field is algebraically closed of characteristic zero. In this paper we prove that every elementary Lie algebra is almost algebraic, provided that  $\text{char}(F) = 0$ .

The final section of the paper is devoted to classifying the real elementary simple Lie algebras.

We will denote algebra direct sums by  $\oplus$ , direct sums of the vector space structure alone by  $\dot{+}$ , and semidirect products by  $\rtimes$ . The *nilradical* of  $L$  will be denoted by  $N(L)$ , whilst  $\text{Asoc}(L)$  will denote the sum (necessarily direct) of the minimal abelian ideals of  $L$ .

## 2 The solvable case

Over a field of characteristic zero every solvable Lie algebra is strongly solvable, by Lie's Theorem. This fails in characteristic  $p$  for every  $p > 0$  (see [22], page 96). However, we have the following result.

**Proposition 2.1** *Let  $L$  be an elementary solvable Lie algebra over a perfect field  $F$ . Then  $L$  is strongly solvable.*

*Proof.* Let  $L$  be a minimal counter-example. As the hypotheses are sub-algebra closed, every proper subalgebra of  $L$  is strongly solvable, and so  $L$  has the structure described in Theorem 4 of [5]. Thus,  $L = A \rtimes B$ , where  $A$  is the unique minimal ideal of  $L$ ,  $\dim A \geq 2$ ,  $A^2 = 0$ ,  $B = M \dot{+} Fx$  with  $M^2 = 0$ , and either  $M$  is a minimal ideal of  $B$ , or  $B$  is the three-dimensional Heisenberg algebra.

Pick any  $m \in M$  and put  $C = A + Fm$ . Then  $C$  is  $\phi$ -free, so  $A \subseteq N(C) = \text{Asoc}(C)$  by Theorem 7.4 of [26], and  $A$  is completely reducible as

an  $Fm$ -module. Write  $A = \bigoplus_{i=1}^r A_i$ , where  $A_i$  is an irreducible  $Fm$ -module for  $1 \leq i \leq r$ . Then the minimum polynomial of the restriction of  $\text{ad } m$  to  $A_i$  is irreducible for each  $i$ , and so  $\{(\text{ad } m)|_A : m \in M\}$  is a set of commuting semisimple operators. Let  $\Omega$  be the algebraic closure of  $F$  and put  $A_\Omega = A \otimes_F \Omega$ , and so on. As  $F$  is perfect,  $\{(\text{ad } m)|_{A_\Omega} : m \in M\}$  is a set of simultaneously diagonalizable linear maps. So, we can decompose  $A_\Omega$  into

$$(A_\Omega)_{\alpha_i} = \{a \in A_\Omega : [a, m] = \alpha_i(m)a \quad \forall m \in M\},$$

where  $1 \leq i \leq s$ .

Suppose first that  $M$  is a minimal ideal of  $B$ . Then  $M_\Omega$  has a basis  $m_1, \dots, m_t$  of eigenvectors of  $\text{ad } x$  with corresponding eigenvalues  $\beta_1, \dots, \beta_t$ . Let  $0 \neq a_i \in (A_\Omega)_{\alpha_i}$ . Then

$$[x, a_i] = \sum_{k=1}^s a'_k \quad \text{where } a'_k \in (A_\Omega)_{\alpha_k}.$$

But now

$$\begin{aligned} 0 &= [[a_i, m_j], x] + [[m_j, x], a_i] + [[x, a_i], m_j] \\ &= \alpha_i(m_j)[a_i, x] + \beta_j[m_j, a_i] + \sum_{k=1}^s [a'_k, m_j] \\ &= -\sum_{k=1}^s \alpha_i(m_j)a'_k - \beta_j \alpha_i(m_j)a_i + \sum_{k=1}^s \alpha_k(m_j)a'_k \end{aligned}$$

Hence

$$\beta_j \alpha_i(m_j)a_i = \sum_{k=1}^s (\alpha_k(m_j) - \alpha_i(m_j))a'_k$$

This yields that  $\beta_j \alpha_i(m_j)a_i = 0$  and therefore either  $[x, m_j] = 0$  or  $[m_j, a_i] = 0$  for all  $1 \leq i \leq r$ . The former is impossible, since it implies that 0 is a characteristic root of  $(\text{ad } x)|_M$ , whence  $B$  is two-dimensional abelian and  $L^2$  is nilpotent. The latter is also impossible, since then  $[M, A] = 0$  and  $L^2 \subseteq A \oplus M$ , which is abelian.

Hence  $B$  is the three-dimensional Heisenberg algebra. But then  $B$  is a non-abelian nilpotent subalgebra of  $L$  and hence not  $\phi$ -free. This contradiction establishes the result.

A Lie algebra  $L$  is called an  $E$ -algebra if  $\phi(B) \leq \phi(L)$  for all subalgebras  $B$  of  $L$ . Groups with the analogous property are called  $E$ -groups by Bechtell. Stitzinger in [24] proved that a Lie algebra is an  $E$ -algebra if and only if  $L/\phi(L)$  is elementary. He also proved that every strongly solvable Lie algebra over an arbitrary field is an  $E$ -algebra. Next, we prove the converse of this result, provided that the ground field is perfect.

**Corollary 2.2** *Let  $F$  be perfect. Then every solvable  $E$ -algebra is strongly solvable.*

*Proof.* Let  $L$  be a solvable  $E$ -algebra. If  $L$  is  $\phi$ -free, then  $L$  is elementary, by [24], and  $L^2$  is nilpotent, by Proposition 2.1. So suppose that  $\phi(L) \neq 0$ . Then  $L/\phi(L)$  is a solvable elementary Lie algebra, and so  $L^2/\phi(L) = (L/\phi(L))^2$  is nilpotent. But then  $L^2$  is nilpotent, by Theorem 5 of [2].

**Lemma 2.3** *Let  $L$  be a Lie algebra over any field  $F$  and let  $A$  be a minimal ideal of  $L$  with  $[L^2, A] = 0$ . Then  $A \subseteq \text{Asoc}(C)$  for every subalgebra  $C$  of  $L$  containing  $A$ .*

*Proof.* We have  $A^3 = [A^2, A] = 0$ . Minimality of  $A$  implies that  $A$  is abelian. Moreover, since  $[L^2, A] = 0$ , we have that  $(\text{adx})|_A$  is  $C$ -linear for every  $x \in L$ . This implies that the sum of the irreducible  $C$ -submodules of  $A$  is invariant under  $L$ , and thus that it coincides with  $A$ . The result follows.

If  $A$  is a subset of  $L$  we denote by  $C_L(A)$  the centraliser of  $A$  in  $L$ . Now we give a construction of elementary solvable Lie algebras.

**Proposition 2.4** *Let  $F$  be an arbitrary field. Let  $A$  be a vector space of finite dimension and let  $B$  be an abelian completely reducible subalgebra of  $\text{gl}(A)$ . Then the semidirect product  $A \rtimes B$  is an elementary almost-algebraic Lie algebra.*

*Proof.* Put  $L = A \rtimes B$ . Then  $L$  is strongly solvable and hence an  $E$ -algebra. But  $A \leq \text{Asoc}(L) \leq C_L(A) = A$ , so  $L$  is  $\phi$ -free by Theorem 7.3 of [26]. It follows that  $L$  is elementary, and  $A = A_1 \oplus \cdots \oplus A_n$  where  $A_i$  is a minimal ideal of  $L$  for  $1 \leq i \leq n$ .

So suppose now that  $A \not\subseteq M$ . We may assume that  $A_1 \not\subseteq M$ . But then  $L = A_1 \dot{+} M$ , and  $M \cong L/A_1 \cong A_2 \oplus \cdots \oplus A_n \dot{+} B$ . Since  $A$  is a faithful  $B$ -module, we have that  $A_2 \oplus \cdots \oplus A_n = \text{Asoc}(A_2 \oplus \cdots \oplus A_n \dot{+} B)$ , whence  $\phi(M) = 0$ .

In order to prove that  $L$  is almost algebraic, let  $x \in L$ ,  $x \notin A$ . By Lemma 2.3 we have that  $A = \text{Asoc}(A + Fx)$  and so the action of  $x$  on  $A$  is semisimple. Let  $A = E_1 \oplus \cdots \oplus E_k$ , where  $E_i$  is an irreducible  $Fx$ -module. Then  $[E_i, x] = 0$  or  $E_i$  for each  $1 \leq i \leq k$ , so we can write  $A = C \oplus D$ , where

$[C, x] = 0$ ,  $[D, x] = D$ . Put  $x = c + d + b$ , where  $c \in C, d \in D, b \in B$ . Then  $[D, b] = [D, x] = D$ , so there is a  $y \in D$  such that  $[y, b] = d$ . Consider the automorphism  $e^{\text{ad}y}$  of  $L$ . We have  $e^{\text{ad}y}(b) = (1 + \text{ad}y)(b) = b + [y, b] = b + d$ , so  $b + d \in e^{\text{ad}y}(B)$ . Since  $e^{\text{ad}y}(B)$  is abelian, it follows that  $\text{ad}(b + d)$  is semisimple. Now  $(\text{ad}c)^2L = [[L, c], c] \subseteq A^2 = 0$ , whence  $\text{ad}c$  is nilpotent. This yields that  $L$  is almost algebraic. The proof is complete.

An elementary Lie algebra which can be constructed as in Proposition 2.4 will be called of *type I*. Next, we show that every elementary solvable Lie algebra over a perfect field can be constructed as an algebra direct sum of an abelian Lie algebra and a Lie algebra of type I.

We say that  $L$  is *metabelian* if  $L^2$  is abelian. We denote the centre of  $L$  by  $Z(L)$ .

**Theorem 2.5** *Let  $F$  be perfect. For a solvable Lie algebra  $L$ , the following statements are equivalent:*

1.  $L$  is elementary,
2.  $L$  is  $\phi$ -free and strongly solvable,
3.  $L$  is  $\phi$ -free and metabelian,
4.  $L = \text{Asoc}(L) \rtimes B$ , where  $B$  is an abelian subalgebra of  $L$ ,
5.  $L \cong A \oplus E$ , where  $A$  is an abelian Lie algebra and  $E$  is an elementary Lie algebra of type I.

*Proof.* (1) $\Rightarrow$ (2): This follows from Proposition 2.1.

(2) $\Rightarrow$ (3): Let  $L$  be  $\phi$ -free and strongly solvable. By Theorems 7.3 and 7.4 of [26], we have that  $L = \text{Asoc}(L) \rtimes B$ , where  $B$  is a subalgebra of  $L$ , and that  $\text{Asoc}(L)$  is precisely the largest nilpotent ideal of  $L$ . As  $L$  is strongly solvable, we have  $L^2 \leq \text{Asoc}(L)$ . This yields that both  $L^2$  and  $B$  are abelian.

(3) $\Rightarrow$ (4): This is clear from Theorems 7.3 and 7.4 of [26] as above.

(4) $\Rightarrow$ (5): Decompose  $\text{Asoc}(L) = Z(L) \oplus K$ , where  $K$  is an ideal of  $L$ . Put  $E = K \dot{+} B$ . We have

$$C_E(K) \cap B \leq Z(E) \leq Z(L) \cap E = 0,$$

so that  $B \lesssim \mathfrak{gl}(K)$ . Moreover, since  $K \leq \text{Asoc}(L)$  and  $L = \text{Asoc}(L) \rtimes B$ , it follows that  $K$  is completely reducible as a  $B$ -module. Hence  $E$  is of type I.

(5) $\Rightarrow$ (1): In Towers [27], it is proved that a direct sum of elementary Lie algebras is elementary. So this follows from Proposition 2.4. This completes the proof.

**Corollary 2.6** *An elementary solvable Lie algebra over a perfect field is almost algebraic.*

In [13], Gein and Varea showed that solvability was a subalgebra lattice property, provided that  $L$  was at least three dimensional and the underlying field was perfect of characteristic different from 2, 3. We now have that the same is true for strong solvability.

**Corollary 2.7** *Let  $L$  be a strongly solvable Lie algebra over a perfect field of characteristic different from 2, 3, and let  $L^*$  be a Lie algebra that is lattice isomorphic to  $L$ . Then either*

1.  $L^*$  is three-dimensional non-split simple, or
2.  $L^*$  is strongly solvable and  $\dim L = \dim L^*$ .

*Proof.* Simply combine Theorem 2.5 with Theorem 3.3 of [13].

### 3 The non-solvable case

If  $C$  is a subalgebra of  $L$  we denote by  $R(C)$  the radical of  $C$ , and by  $N(C)$  the nilradical of  $C$ .

**Proposition 3.1** *Let  $F$  be perfect. Let  $L$  be an elementary Lie algebra which is neither solvable nor semisimple. Then,  $L = \text{Asoc}(L) \rtimes (B \dot{+} S)$ , where  $B$  is abelian,  $S$  is a semisimple subalgebra of  $L$  and  $[B, S] \leq B$ . If  $F$  has characteristic zero, then  $[B, S] = 0$ .*

*Proof.* As  $L$  is  $\phi$ -free,  $L = \text{Asoc}(L) \rtimes C$  for some subalgebra  $C$  of  $L$  and  $N(L) = \text{Asoc}(L)$ , by Theorem 7.3 of [26]. Let  $B = R(C)$ . Since  $C$  is elementary, it splits over  $B$ , by [27, Theorem 2.4]. So,  $C = B \dot{+} S$ , where  $S$  is semisimple. It is clear that  $R(L) = \text{Asoc}(L) \rtimes B$ . By Proposition 2.1, we have that  $R(L)$  is strongly solvable. So  $B^2 \leq R(L)^2 \cap B \subseteq N(L) \cap B = \text{Asoc}(L) \cap B = 0$ . If  $F$  has characteristic zero, the final assertion follows from [25, Theorem 4].

A Lie algebra all of whose proper subalgebras are abelian is called *semiabelian*.

**Example** Let  $S$  be a simple semiabelian Lie algebra over a field of characteristic  $p > 0$ , let  $B$  be a faithful finite-dimensional completely reducible  $L$ -module, and put  $L = B \dot{+} S$  where  $B^2 = 0$  and  $L$  acts on  $B$  under the given  $L$ -module action. Then  $L$  is elementary, but  $S$  is not an ideal of  $L$ .

### Notes

- Since elementary Lie algebras are Lie  $A$ -algebras it follows from Proposition 2 of [21] that over a field of cohomological dimension  $\leq 1$  every semisimple elementary Lie algebra is representable as a direct sum of simple ideals, each of which splits over some finite extension of the ground field into a direct sum of ideals isomorphic to  $\mathfrak{sl}(2, F)$ .
- Over a perfect field with non-trivial Brauer group there exists a finite-dimensional simple semiabelian Lie algebra (see Theorem 8.5 of [9]), and this is elementary.
- Let  $G$  be the algebra constructed by Gein in Example 2 of [12]. This is a seven-dimensional Lie algebra over a certain perfect field  $F$  of characteristic three. Every subalgebra of  $G$  of dimension greater than one is simple. So,  $G$  is elementary.

We finish this section by proving that the classification of elementary Lie algebras over an algebraically closed field of characteristic zero given by Towers in [27] remains true for any algebraically closed field of characteristic different from 2 or 3.

**Theorem 3.2** *Let  $L$  be a Lie algebra over an algebraically closed field  $F$  of characteristic  $\neq 2$  or 3. Then  $L$  is elementary if and only if*

1.  $L$  is isomorphic to a direct sum of copies of  $\mathfrak{sl}(2, F)$ , or
2. there is a basis  $\{a_1, \dots, a_m, b_1, \dots, b_n\}$  for  $L$  such that

$$[a_i, b_j] = -[b_j, a_i] = \lambda_{ij} a_i \text{ for all } 1 \leq i \leq m, 1 \leq j \leq n$$

all other products being zero, or

3.  $L \cong A \oplus B$ , where  $A$  is as in (1) and  $B$  is as in (2).

*Proof.* If  $\text{char}(F) = 0$ , then the result is Theorem 3.2 of [27]. Assume  $\text{char}(F) = p > 3$ . Let  $L (\neq 0)$  be elementary. If  $L$  is solvable, then  $L = \text{Asoc}(L) \rtimes B$ , by Theorem 2.5. Decompose  $\text{Asoc}(L) = A_1 \oplus \dots \oplus A_m$ , where  $A_i$  is a minimal ideal of  $L$ . As  $F$  is algebraically closed, we have  $\dim A_i = 1$ . Hence  $L$  is as in (2).

Now, let  $L$  be semisimple. Since  $L$  is an  $A$ -algebra, Proposition 2 of Premet and Semenov [21] applies and  $L$  is as in (1).

Then suppose that  $L$  is neither solvable nor semisimple. By Proposition 3.1 we have that  $L = A \rtimes (B \dot{+} S)$ , where  $0 \neq A = \text{Asoc}(L)$ ,  $B$  is abelian,  $0 \neq S$  is a semisimple subalgebra of  $L$  and  $[B, S] \leq B$ . From [21, Proposition 2] again it follows that  $S = S_1 \oplus \dots \oplus S_r$ , where  $S_i$  is an ideal of  $S$  and  $S_i \cong \mathfrak{sl}(2, F)$  for every  $1 \leq i \leq r$ . Put  $C_i = A \dot{+} S_i$ . We have that  $C_i$  is a  $\phi$ -free Lie algebra and so  $N(C_i) = \text{Asoc}(C_i)$ . This yields that  $A = \text{Asoc}(C_i)$  and therefore  $A$  is a completely reducible  $S_i$ -module. Let  $V$  be an irreducible  $S_i$ -submodule of  $A$ .

We claim that  $\dim_F V = 1$ . By general theory, there exists an element  $e \in S_i$  such that  $V$  is a cyclic  $Fe$ -module on which  $e$  acts nilpotently. This can be seen by looking at the representatives of the coadjoint orbits of  $\text{SL}(2)$  on  $\mathfrak{sl}(2, F)^*$  (see [11, §2]). Consequently,  $M = V \dot{+} Fe$  is a nilpotent subalgebra. This yields that  $M$  is abelian, whence  $[V, e] = 0$  and  $\dim_F V = 1$ .

Therefore, we have that  $[A, S] = 0$ . If  $B = 0$ , then we have that  $L$  is as in (3). Suppose then  $B \neq 0$ . We have that  $B \dot{+} S$  is an elementary Lie algebra, whence  $\text{Asoc}(B \dot{+} S) = N(B \dot{+} S) = B$ . As above, we obtain that  $[B, S] = 0$ , from which it follows that  $L = (A \dot{+} B) \oplus S$ . Since  $A \dot{+} B$  is a solvable elementary Lie algebra, it is as in (2), and therefore  $L$  is as in (3). This completes the proof in one direction. The converse is easily checked.

The above result does not hold in characteristic 2: over such a field, the three-dimensional simple Lie algebra with basis  $e_1, e_2, e_3$  and products  $[e_1, e_2] = e_3, [e_2, e_3] = e_1, [e_3, e_1] = e_2$  is elementary. The exclusion of characteristic 3 is used in order to invoke the result of Semenov and Premet,



which in turn relies on [20, Theorem 3]. This last result fails in characteristic 3, as is shown by the algebra  $G$  ([12, Example 2]) referred to earlier. However, we know of no counter-example to the above result in characteristic 3: if we pass to the algebraic closure, then  $G$  becomes  $\mathfrak{psl}(3)$ , which is no longer elementary.

## 4 The characteristic zero case

A subalgebra  $T$  of  $L$  is said to be a *toral* subalgebra of  $L$  if  $T$  is abelian and  $\mathrm{ad}_L t$  is semisimple for every  $t \in T$ . A Lie algebra  $L$  is said to be *ad-semisimple* if  $\mathrm{ad} x$  is semisimple for every  $x \in L$ .

**Proposition 4.1** *Let  $\mathrm{char}(F) = 0$ . For a solvable Lie algebra  $L$  the following statements are equivalent:*

1.  $L$  is elementary,
2.  $L$  is  $\phi$ -free and almost algebraic.

*Proof.* Assume that  $L$  is  $\phi$ -free and almost algebraic. Then  $L = N(L) \dot{+} T$ , where  $T$  is a toral subalgebra of  $L$  (see [1] or [26, Theorem 7.5]). Moreover, we have  $N(L) = \mathrm{Asoc}(L)$  by Theorem 7.4 of [26]. So,  $L$  is elementary by Theorem 2.5. The converse follows from Corollary 2.6.

**Proposition 4.2** *Let  $L$  be an ad-semisimple Lie algebra over a field of characteristic zero. Then  $L$  is elementary*

*Proof.* Let  $L$  be a minimal counter-example. Then it suffices to show that  $L$  is  $\phi$ -free. But  $L = Z(L) \oplus S$ , where  $S$  is semisimple, by Levi's Theorem and Theorem 1 of [10]. It follows that  $\phi(L) = 0$ .

**Corollary 4.3** *Over the real field every compact semisimple Lie algebra is elementary.*

A Lie algebra  $L$  is said to be *reductive* if its adjoint representation is completely reducible; equivalently,  $L = S \oplus Z(L)$ , where  $S$  is a semisimple ideal of  $L$  and  $Z(L)$  is the centre of  $L$  (see [15]).

**Proposition 4.4** *Let  $\text{char}(F) = 0$ . Let  $L$  be a Lie algebra such that its radical  $R$  is elementary and  $L/R$  is ad-semisimple. Then,  $L$  is elementary and almost algebraic.*

*Proof.* Let  $S(L)$  be the largest semisimple ideal of  $L$ . Since  $S(L)$  is isomorphic to an ideal of  $L/R$ , by Proposition 4.2 it follows that  $S(L)$  is elementary. Since  $S(L)$  is a direct summand of  $L$  and since  $S(L)$  is almost algebraic, we may suppose without loss of generality that  $S(L) = 0$ . By Proposition 4.1 we have that  $R$  is almost algebraic. Then  $L$  is also almost algebraic by Corollary 3.1 of [1] (see also [18]).

Therefore  $L = N(L) \rtimes (B \dot{+} S)$  where  $B$  is a toral subalgebra of  $L$ ,  $S$  is a semisimple subalgebra of  $L$  and  $[B, S] = 0$ . We have  $N(R) = \text{Asoc}(R)$  since  $R$  is  $\phi$ -free. As  $\text{char}(F) = 0$ , we have that  $N(R)$  is a characteristic ideal of  $R$  and so  $N(R) \leq N(L)$ . It follows that  $N(L) = \text{Asoc}(R)$  and so  $N(L)$  is abelian. Put  $A = N(L)$ . Since every element of  $B$  acts semisimply on  $A$ , we have that  $A$  is completely reducible as a  $(B \oplus S)$ -module, see [15]. It follows that  $A = \text{Asoc}(L)$ . This yields that  $L$  is  $\phi$ -free. To prove that  $L$  is elementary it suffices to show that every maximal subalgebra  $M$  of  $L$  is also  $\phi$ -free.

Let us first consider the case when  $M$  does not contain  $R$ . Then  $L = R + M$ ,  $L/R \cong M/M \cap R$  and  $M \cap R$  is the radical of  $M$ . We have that  $M \cap R$  is elementary and  $M/M \cap R$  is ad-semisimple. By the above, we obtain that  $M$  is  $\phi$ -free.

Now suppose that  $R \leq M$ . Since  $M \cap S$  is ad-semisimple, by [10, Theorem 1] it follows that  $M \cap S = Z(M \cap S) \oplus (M \cap S)^2$  and  $(M \cap S)^2$  is semisimple. Put  $B^* = B \oplus Z(M \cap S)$ . We have that  $B^* \oplus (M \cap S)^2$  is reductive. Moreover, since every element of  $Z(M \cap S)$  acts semisimply on  $S$ , it acts also semisimply on  $A$  (see [15], page 101). It follows that every element of  $B^*$  acts semisimply on  $A$ . This yields that  $A$  is completely reducible as a  $(B^* \oplus (M \cap S)^2)$ -module (see [15]) and therefore  $A \leq \text{Asoc}(M)$ . On the other hand, we have  $R(M) = A \dot{+} B^*$  and  $A \leq N(M)$ . We claim that  $A = N(M)$ . Let  $x \in B^* \cap N(M)$ . We have that  $(\text{ad}x)|_A$  is nilpotent and semisimple. So,  $x \in C_L(A)$ . Moreover, we have that  $[x, B] \subseteq [S, B] = 0$ . This yields that  $x \in C_L(R)$ . Since  $C_L(R) \cap S$  is a semisimple ideal of  $L$ , we have  $C_L(R) \cap S = 0$ , whence  $C_L(R) = Z(R)$ . Decompose  $x = b + z$ ,  $b \in B$ ,  $z \in Z(M \cap S)$ . We have  $z \in R \cap S = 0$ . This yields that  $x \in B \cap Z(R) \leq Z(L) = 0$  and therefore  $A = N(M)$ , as claimed. Hence  $A = \text{Asoc}(M)$ . Since  $M$  splits over  $A$ , it follows that  $M$  is  $\phi$ -free. This completes the proof.

**Corollary 4.5** *Let  $\text{char}(F) = 0$ . Let  $A$  be a vector space of finite dimension. Let  $K$  be a reductive subalgebra of  $\text{gl}(A)$  such that  $K^2$  is non-zero and ad-semisimple and every non-zero element of  $Z(K)$  is a semisimple transformation of  $A$ . Then, the semidirect product  $A \rtimes K$  is an elementary almost algebraic Lie algebra.*

*Proof.* Put  $L = A \rtimes K$ . We have that  $R(L) = A \dot{+} Z(K)$ . By Proposition 2.4, it follows that  $R(L)$  is an elementary Lie algebra. Also, we have  $L/R(L) \cong K^2$ . So that  $L/R(L)$  is ad-semisimple. The result follows from Proposition 4.4.

A Lie algebra which can be constructed as in the above corollary will be called of *type II*.

**Theorem 4.6** *Let  $\text{char}(F) = 0$ . A Lie algebra  $L$  is elementary if and only if  $L \cong A \oplus B \oplus S$ , where  $A$  is abelian,  $B$  is a Lie algebra of type I or of type II and  $S$  is an elementary semisimple Lie algebra.*

*Proof.* Let  $L$  be elementary. Then  $L$  splits over  $Z(L)$ , so  $L = Z(L) \oplus \hat{L}$ , where  $\hat{L}$  is a centerless Lie algebra. Now let  $S(\hat{L})$  be the largest semisimple ideal of  $\hat{L}$ . Then we have that  $\hat{L} = K \oplus S(\hat{L})$ , where  $K$  is a centerless Lie algebra which has no non-zero semisimple ideals. If  $K$  is solvable, then we find that  $K$  is an elementary Lie algebra of type I.

Then assume that  $K$  is not solvable. By Proposition 3.1 it follows that  $K = \text{Asoc}(K) \rtimes (B \oplus S)$ , where  $B$  is abelian,  $S$  is semisimple and  $[B, S] = 0$ . Let  $0 \neq s \in S$  such that  $\text{ad}_S s$  is nilpotent. Then we have that  $\text{Asoc}(K) + F s$  is a nilpotent subalgebra of  $L$ . Hence  $[s, \text{Asoc}(L)] = 0$ . This yields that  $s \in C_K(R(K)) \cap S = 0$ , which is a contradiction. Therefore  $S$  has no non-zero ad-nilpotent elements. As  $S$  is semisimple and  $\text{char}(F) = 0$ , it follows that  $S$  is ad-semisimple. Put  $C = C_K(\text{Asoc}(K)) \cap (B \oplus S)$ . We have that  $C$  is an ideal of the reductive Lie algebra  $B \oplus S$ , so  $C = (C \cap B) \oplus (C \cap S)$ . It follows that  $C \cap B \leq Z(K) = 0$  and that  $C \cap S = 0$  since it is a semisimple ideal of  $K$ . This yields that  $B \oplus S \lesssim \text{gl}(\text{Asoc}(K))$  and therefore  $K$  is of type II. This completes the proof in one direction.

The converse follows from Proposition 2.4 and Corollary 4.5.

**Corollary 4.7** *An elementary Lie algebra over a field of characteristic zero is almost algebraic.*

*Proof.* This follows from Theorem 4.6, Proposition 2.4 and Corollary 4.5.

**Corollary 4.8** *Let  $\text{char}(F) = 0$ . For a Lie algebra  $L$  without non-zero semisimple ideals, the following statements are equivalent:*

1.  $L$  is elementary.
2.  $L$  is almost algebraic and an  $A$ -algebra.
3.  $L$  is almost algebraic,  $N(L)$  is abelian and  $L/R(L)$  is ad-semisimple.
4.  $L$  is almost algebraic,  $\phi$ -free and  $L/R(L)$  is ad-semisimple.

*Proof.* (1) $\Rightarrow$ (2): This follows from Corollary 4.7.

From now on in this proof we assume that  $L$  is almost algebraic. Then we have that  $L = N(L) \dot{+} B \dot{+} S$ , where  $B$  is a toral subalgebra of  $L$ ,  $S$  is a semisimple subalgebra of  $L$  and  $[B, S] = 0$ .

(2) $\Rightarrow$ (3): Clearly,  $N(L)$  is abelian. It remains to prove that  $S$  is ad-semisimple. Let  $s \in S$  such that  $\text{ad}_S s$  is nilpotent. Then  $(\text{ad}_S)|_{N(L)}$  is nilpotent too. This yields that  $N(L) + Fs$  is a nilpotent subalgebra of  $L$  and therefore  $[N(L), s] = 0$ . Thus,  $s \in C_L(R(L)) \cap S = 0$  because  $L$  has no non-zero semisimple ideals. Hence  $S$  has no non-zero ad-nilpotent elements. As  $\text{char}(F) = 0$ , we have that  $S$  is ad-semisimple.

(3) $\Rightarrow$ (4): Since  $N(L)$  is a completely reducible  $(B \oplus S)$ -module and since  $N(L)$  is abelian, it follows that  $N(L) = \text{Asoc}(L)$ . Since  $L$  splits on  $N(L)$ , we have that  $L$  is  $\phi$ -free.

(4) $\Rightarrow$ (1): By Theorem 7.4 of [26] we have  $N(L) = \text{Asoc}(L)$ . It then follows from Theorem 2.5 that  $N(L) \dot{+} B$  is elementary. Since  $R(L) = N(L) \dot{+} B$ , Proposition 4.4 now gives that  $L$  is elementary.

## 5 The real field case

A subalgebra  $P$  of  $L$  is called *parabolic* if  $P \otimes_F \Omega$  contains a Borel subalgebra (that is, a maximal solvable subalgebra) of  $L \otimes_F \Omega$ , where  $\Omega$  is the algebraic closure of  $F$ . Over a field of characteristic zero, all maximal subalgebras of a reductive Lie algebra are reductive or parabolic (see [4], [16], [19]); it follows that a reductive Lie algebra is elementary if and only if its parabolic subalgebras are  $\phi$ -free.

For results concerning Lie algebras over the real field we refer the reader to the books by Helgason ([14]) and Knapp ([17]).

**Theorem 5.1** *Let  $F$  be the real field. For a simple Lie algebra  $L$ , the following statements are equivalent:*

1.  $L$  is elementary,
2.  $L$  is an  $A$ -algebra,
3.  $L$  is compact or isomorphic to one of the following Lie algebras:  $\mathfrak{sl}(2, \mathbb{R})$ ,  $\mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$ ,  $\mathfrak{so}(n, 1)$  ( $n > 3$ ).

*Proof.* (1) $\Rightarrow$ (2): This is clear.

(2) $\Rightarrow$ (3): Let  $L$  be a non-compact  $A$ -algebra. Suppose first that  $L = \bar{L}^{\mathbb{R}}$ , the realisation of the complex simple Lie algebra  $\bar{L}$ . If  $\bar{N}$  is a nilpotent subalgebra of  $\bar{L}$  then  $\bar{N}^{\mathbb{R}}$  is a nilpotent subalgebra of  $L$ , and hence abelian. It follows that  $\bar{N}$  is abelian and thus that  $\bar{L}$  is an  $A$ -algebra. The proof of Theorem 3.2 of [27] then shows that  $\bar{L} \cong \mathfrak{sl}(2, \mathbb{C})$ .

So assume now that  $L$  is a non-compact real form of a complex simple Lie algebra. The only such algebras for which the nilpotent subalgebra  $N$  in the Iwasawa decomposition of  $L$  is abelian are  $\mathfrak{sl}(2, \mathbb{R})$  and  $\mathfrak{so}(n, 1)$  ( $n > 3$ ).

(3) $\Rightarrow$ (1): Suppose now that  $L$  is one of the algebras described in (3). If  $L$  is compact it is elementary by Corollary 4.3, and  $\mathfrak{sl}(2, \mathbb{R})$  is clearly elementary.

Next suppose that  $L \cong \mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$ , and let  $S$  be a subalgebra of  $L$ . Then  $S \otimes \mathbb{C}$  is a subalgebra of  $L \otimes \mathbb{C}$ , which is elementary. This yields  $0 = \phi(S \otimes \mathbb{C}) = \phi(S) \otimes \mathbb{C}$ , by [8], whence  $\phi(S) = 0$  and  $L$  is elementary.

Finally, let  $L = \mathfrak{so}(n, 1)$  ( $n > 3$ ). We identify  $L$  with

$$\left\{ \begin{pmatrix} B & u \\ u^T & 0 \end{pmatrix} : u \in \mathbb{R}^n, B \in M_{n \times n}(\mathbb{R}), B^T = -B \right\}$$

From the remarks at the beginning of this section it suffices to show that the parabolic subalgebras of  $L$  are  $\phi$ -free. Now any such subalgebra is conjugate to a standard parabolic subalgebra  $P$  with Langlands decomposition  $P = (M \oplus A) \dot{+} N$ , where  $M \oplus A$  is reductive and  $N$  is an ideal of  $P$  contained in

$$\left\{ \begin{pmatrix} 0 & u & u \\ -u^T & 0 & 0 \\ u^T & 0 & 0 \end{pmatrix} : u \in \mathbb{R}^{n-1} \right\}.$$

Clearly  $N$  is abelian and every element of  $A$  acts semisimply on  $N$ , so  $P$  is  $\phi$ -free.

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