

C-SECTIONS OF LIE ALGEBRAS

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Abstract

Let M be a maximal subalgebra of a Lie algebra L and A/B a chief factor of L such that $B \subseteq M$ and $A \not\subseteq M$. We call the factor algebra $M \cap A/B$ a c -section of M . All such c -sections are isomorphic, and this concept is related those of c -ideals and ideal index previously introduced by the author. Properties of c -sections are studied and some new characterizations of solvable Lie algebras are obtained.

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1 Preliminary results

Throughout L will denote a finite-dimensional Lie algebra over a field F . We denote algebra direct sums by ‘ \oplus ’, whereas vector space direct sums will be denoted by ‘ $\dot{+}$ ’. If B is a subalgebra of L we define B_L , the *core* (with respect to L) of B to be the largest ideal of L contained in B . In [9] we defined a subalgebra B of L to be a c -ideal of L if there is an ideal C of L such that $L = B + C$ and $B \cap C \subseteq B_L$.

Let M be a maximal subalgebra of L . We say that a chief factor C/D of L supplements M in L if $L = C + M$ and $B \subseteq C \cap M$; if $B = C \cap M$ we say that C/D complements M in L . In [10] we defined the *ideal index* of a maximal subalgebra M of L , denoted by $\eta(L : M)$, to be the well-defined dimension of a chief factor C/D where C is an ideal minimal with respect to supplementing M in L . Here we introduce a further concept which is related to the previous two.

Let M be a maximal subalgebra of L and let C/D be a chief factor of L with $D \subseteq M$ and $L = M + C$. Then $(M \cap C)/D$ is called a *c-section* of M in L . The analogous concept for groups was introduced by Wang and Shirong in [13] and studied further by Li and Shi in [3].

We say that L is *primitive* if it has a maximal subalgebra M with $M_L = 0$. First we show that all c-sections of M are isomorphic.

Lemma 1.1 *For every maximal subalgebra M of L there is a unique c-section up to isomorphism.*

Proof. Clearly c-sections exist. Let $(M \cap C)/D$ be a c-section of M in L , where C/D is a chief factor of L , $D \subseteq M$ and $L = M + C$. First we show that this c-section is isomorphic to one in which $D = M_L$. Clearly $D \subseteq M_L \cap C \subseteq C$, so either $M_L \cap C = C$ or $M_L \cap C = D$. If the former holds, then $C \subseteq M_L$, giving $L = M$, a contradiction. In the latter case put $E = C + M_L$. Then $E/M_L \cong C/D$ is a chief factor and $(M \cap E)/M_L$ is a c-section. Moreover,

$$\frac{M \cap E}{M_L} = \frac{M_L + M \cap C}{M_L} \cong \frac{M \cap C}{M_L \cap C} = \frac{M \cap C}{D}.$$

So suppose that $(M \cap C_1)/M_L$ and $(M \cap C_2)/M_L$ are two c-sections, where $C_1/M_L, C_2/M_L$ are chief factors and $L = M + C_1 = M + C_2$. Then L/M_L is primitive and so either $C_1 = C_2$ or else $C_1/M_L \cong C_2/M_L$ and $C_1 \cap M = M_L = C_2 \cap M$, by [12, Theorem 1.1]. In the latter case both c-sections are trivial. \square

Given a Lie algebra L with a maximal subalgebra M we define $Sec(M)$ to be the Lie algebra which is isomorphic to any c-section of M ; we call the natural number $\eta^*(L : M) = \dim Sec(M)$ the *c-index* of M in L .

The relationship between c-ideals and c-sections, and between ideal index and c-index, for a maximal subalgebra M of L is given by the following lemma.

Lemma 1.2 *Let M be a maximal subalgebra of a Lie algebra L . Then*

(i) M is a c -ideal of L if and only if $\text{Sec}(M) = 0$; and

(ii) $\eta^*(L : M) = \eta(L : M) - \dim(L/M)$.

Proof.

(i) Suppose first that M is a c -ideal of L . Then there is an ideal C of L such that $L = M + C$ and $M \cap C \subseteq M_L$. Then $M \cap C = M_L \cap C$ is an ideal of L . Let K be an ideal of L with $M \cap C \subset K \subseteq C$. Then $K \not\subseteq M$, so $L = M + K$ and $M \cap C = M \cap K$. This yields that $\dim L = \dim M + \dim K - \dim(M \cap K) = \dim M + \dim C - \dim(M \cap C)$, so $K = C$ and $C/(M \cap C)$ is a chief factor of L . It follows that $\text{Sec}(M) = 0$.

The converse is clear.

(ii) Let C/D be a chief factor such that $L = M + C$ and C is minimal in the set of ideals supplementing M in L . Then $\eta(L : M) = \dim(C/D)$, by the definition of ideal index. Thus,

$$\begin{aligned} \eta(L : M) &= \dim(C/D) = \dim C - \dim D \\ &= \dim C - \dim C \cap M + \dim C \cap M - \dim D \\ &= \dim L - \dim M + \dim(C \cap M/D) \\ &= \dim(L/M) + \eta^*(L : M). \end{aligned}$$

□

Lemma 1.3 *Let A/B be an abelian chief factor of L . Then any maximal subalgebra of L that supplements A/B must complement A/B .*

Proof. Let M supplement A/B , so $L = A + M$ and $B \subseteq M$. Then $[L, M \cap A] = [A + M, M \cap A] \subseteq B + M \cap A = M \cap A$. So $M \cap A$ is an ideal of L and $M \cap A = B$. □

The following lemma will also be useful.

Lemma 1.4 *Let $B \subseteq M \subseteq L$, where M is maximal in L and B is an ideal of L . Then $\text{Sec}(M) \cong \text{Sec}(M/B)$.*

Proof. Clearly M/B is a maximal subalgebra of L/B . Let $(C/B)/(D/B)$ be a chief factor of L/B such that $D/B \subseteq M/B$ and $C/B + M/B = L/B$.

Then C/D is a chief factor of L such that $L = C + M$ and $D \subseteq M$. Hence $Sec(M) \cong C \cap M/D \cong Sec(M/B)$. \square

In [12] it was shown that a primitive Lie algebra can be one of three types: it is said to be

1. *primitive of type 1* if it has a unique minimal ideal that is abelian;
2. *primitive of type 2* if it has a unique minimal ideal that is non-abelian; and
3. *primitive of type 3* if it has precisely two distinct minimal ideals each of which is non-abelian.

If M is a maximal subalgebra of L , then L/M_L is clearly primitive; we say that M is of type i if L/M_L is primitive of type i for $i = 1, 2, 3$. Then we have the following result.

Lemma 1.5 *Let L be a Lie algebra over a field F and let M be a maximal subalgebra of L .*

- (i) *If M is of type 1 or 3 then $Sec(M) = 0$.*
- (ii) *If F has characteristic zero and M is of type 2 then $Sec(M) \cong M/M_L$.*

Proof.

- (i) This follows from [12, Theorem 1.1 3(a),(c)].
- (ii) Let A/B be a nonabelian chief factor that is supplemented by M , so $L = A + M$ and $B = A \cap M_L$. Then L/M_L is simple, by [12, Theorem 1.7 2], which implies that $L = A + M_L$. Hence

$$\frac{M}{M_L} = \frac{M \cap (A + M_L)}{M_L} = \frac{M \cap A + M_L}{M_L} \cong \frac{M \cap A}{M_L \cap A} = \frac{M \cap A}{B} = Sec(M).$$

\square

2 Main results

First we can state Theorems 3.1, 3.2 and 3.3 of [9] in terms of c-sections as follows.

Theorem 2.1 *Let L be a Lie algebra over a field F . Then*

- (i) every maximal subalgebra M of L has trivial c -section if and only if L is solvable; and
- (ii) if F has characteristic zero, or is algebraically closed of characteristic greater than 5, then L has a maximal subalgebra with trivial c -section if and only if L is solvable.

Theorem 2.2 *Let L be a Lie algebra over a field F of characteristic zero. Then $\text{Sec}(M)$ is solvable for all maximal subalgebras M of L if and only if $L = R \dot{+} S$, where R is the (solvable) radical of L and S is a direct sum of simple algebras which are minimal non-abelian or isomorphic to $sl_2(F)$.*

Proof. Suppose first that $\text{Sec}(M)$ is solvable for all maximal subalgebras M of L , and let $L = R \dot{+} S$ be the Levi decomposition of L . Then $\text{Sec}(M)$ is solvable for all maximal subalgebras M of S , by Lemma 1.4. Let $S = S_1 \oplus \dots \oplus S_n$, where S_i is simple for each $1 \leq i \leq n$. If M contains all S_i apart from S_j , then $\text{Sec}(M) \cong M \cap S_j$, so every subalgebra of S_j is solvable. It follows from [8, Theorem 2.2 and the remarks following it] that S_j is minimal non-abelian or isomorphic to $sl_2(F)$ for each $1 \leq j \leq n$.

Suppose conversely that L has the claimed form and let M be a maximal subalgebra of L . Every chief factor of L is either abelian or simple, and so every c -section of M is either abelian or isomorphic to a proper subalgebra of one of the simple components of S . In either case $\text{Sec}(M)$ is solvable. \square

Corollary 2.3 *Let L be a Lie algebra over a field F and suppose that every maximal subalgebra has c -index k . Then*

- (i) if $k > 0$, L must be semisimple.

Suppose further that F has characteristic zero. Then

- (ii) every simple ideal of its Levi factor must have all of its maximal subalgebras of dimension k ;
- (iii) $k = 0$ if and only if L is solvable;
- (iv) $k = 1$ if and only if L is a direct sum of non-isomorphic three-dimensional simple ideals and $\sqrt{F} \not\subseteq F$; and
- (v) $k = 2$ if and only if L is a direct sum of non-isomorphic ideals each of which is a minimal non-abelian simple Lie algebra with all maximal subalgebras of dimension 2.

Proof.

- (i) If L has non-trivial radical, it has an abelian chief factor which is supplemented, and hence complemented, by Lemma 1.3, so $k = 0$.
- (ii) This is clear.
- (iii) This is Theorem 2.1 (i).
- (iv) Suppose that $k = 1$. Then L is semisimple and each simple component has all of its maximal subalgebras one dimensional, by (i) and (ii). It follows that they are three-dimensional simple and $\sqrt{F} \not\subseteq F$, by [11, Theorem 3.4]. If there are two that are isomorphic, say S and $\theta(S)$, where θ is an isomorphism, then the diagonal subalgebra $\{s + \theta(s) : s \in S\}$ is maximal in $S \oplus \theta(S)$. But this together with the simple components other than S and $\theta(S)$ gives a maximal subalgebra M of L with c -index 0 in L .
 Conversely, suppose that L is a direct sum of non-isomorphic three-dimensional simple ideals, $S_1 \oplus \dots \oplus S_n$, and $\sqrt{F} \not\subseteq F$. Let M be a maximal subalgebra of L with $S_i \not\subseteq M$ and $S_j \not\subseteq M$ for some $1 \leq i, j \leq n$ with $i \neq j$. Then $L = M + S_i = M + S_j$ which yields that $M \cap S_i$ and $M \cap S_j$ are ideals of L and hence are trivial. But then $S_i \cong L/M \cong S_j$, a contradiction. It follows that every maximal subalgebra contains all but one of the simple components and hence that $k = 1$.
- (v) This is similar to (iv), noting that there are no three-dimensional simple Lie algebras with all maximal subalgebras two dimensional.

□

Note that algebras as described in Corollary 2.3 do exist as the following example shows. This example was constructed by Gejn (see [2, Example 3.5]).

EXAMPLE 2.1 *Let L be the Lie algebra generated by the matrices*

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -E \\ 0 & E & 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & 0 \\ -E & 0 & 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 & -A & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$g_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -A \\ 0 & A & 0 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 0 & 2E \\ 0 & 0 & 0 \\ -A & 0 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} 0 & -2E & 0 \\ A & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, with respect to the operation $[\cdot, \cdot]$, over the rational numbers \mathbb{Q} . Then L is simple nonabelian (see [2, Example 3.5]), and the maximal subalgebras are $\mathbb{Q}f_i + \mathbb{Q}g_i$ for $i = 1, 2, 3$.

EXAMPLE 2.2 Gejn also goes on to construct simple minimal nonabelian Lie algebras over \mathbb{Q} of dimension $3k$ for $k \geq 1$ by putting

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 2 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

E as the $k \times k$ identity matrix and 0 as the $k \times k$ zero matrix (see [2, Example 3.6]). It is straightforward to check that in these every maximal subalgebra has c -index k .

The following corollary is straightforward.

Corollary 2.4 Let $L = R \dot{+} S$ be a Lie algebra over a field F of characteristic zero, where R is the radical and S is a Levi factor, and suppose that L has a maximal subalgebra with c -index k . Then

- (i) if $k > 0$ then $S \neq 0$;
- (ii) $k = 1$ if and only if S has a minimal ideal which is minimal non-abelian or isomorphic to $sl_2(F)$;
- (iii) $k > 1$ if and only if S has a minimal ideal with a maximal subalgebra of dimension k .

Let $(L_p, [p], \iota)$ be any finite-dimensional p -envelope of L . If S is a subalgebra of L we denote by S_p the restricted subalgebra of L_p generated by $\iota(S)$. Then the (absolute) toral rank of S in L , $TR(S, L)$, is defined by

$$TR(S, L) = \max\{\dim(T) : T \text{ is a torus of } (S_p + Z(L_p))/Z(L_p)\}.$$

This definition is independent of the p -envelope chosen (see [7]). We write $TR(L, L) = TR(L)$. A Lie algebra L is *monolithic* if it has a unique minimal ideal (the *monolith* of L). The *Frattini ideal*, $\phi(L)$, is the largest ideal contained in every maximal subalgebra of L . We put $L^{(0)} = L$, $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$ for $n \in \mathbb{N}$ and $L^{(\infty)} = \bigcap_{n=0}^{\infty} L^{(n)}$.

Theorem 2.5 *Let L be a Lie algebra over an algebraically closed field F of characteristic $p > 0$. Then $\text{Sec}(M)$ is nilpotent for every maximal subalgebra M of L if and only if L is solvable.*

Proof. Let L be a minimal non-solvable Lie algebra such that $\text{Sec}(M)$ is nilpotent for every maximal subalgebra M of L , and let R be the (solvable) radical of L . If L is simple then every maximal subalgebra of L is nilpotent, and no such Lie algebra exists over an algebraically closed field. So L has a minimal ideal A , and L/A is solvable. If there are two distinct minimal ideals A_1 and A_2 , then L/A_1 and L/A_2 are solvable, whence $L \cong L/(A_1 \cap A_2)$ is solvable, a contradiction. Hence L is monolithic with monolith A . If $A \subseteq R$ then again L would be solvable, so L is semisimple and $\phi(L) = 0$. Thus, there is a maximal subalgebra M of L such that $L = M + A$.

Put $C = M \cap A$ which is an ideal of M . If $\text{ad } a$ is nilpotent for all $a \in A$ then L is solvable, a contradiction. Hence there exists $a \in A$ such that $\text{ad } a$ is not nilpotent. Let $L = L_0 + L_1$ be the Fitting decomposition of L relative to $\text{ad } a$. Then $L_0 \neq L$ and $L_1 \subseteq A$, so that if P is a maximal subalgebra containing L_0 , we have $L = A + P$ and $a \in A \cap P$. We can, therefore, assume that $C \neq 0$.

Then C is nilpotent and $L/A \cong M/C$ is solvable, whence M is solvable. Now $[M, N_A(C)] \subseteq N_A(C)$, so $M + N_A(C)$ is a subalgebra of L . But $L = M + N_A(C)$ implies that C is an ideal of L , from which $C = A$ and L is solvable, a contradiction. It follows that $M = M + N_A(C)$, and so $N_A(C) = M \cap A = C$, and C is a Cartan subalgebra of A . Now C_p is a Cartan subalgebra of A_p , by [14, Lemma], and so there is a maximal torus $T \subseteq A_p$ such that $C_p = C_{L_p}(T)$ (see [5]).

Let $A_0(T) + \sum_{i \in \mathbb{Z}_p} A_{i\alpha}$ be a 1-section with respect to T . Then every element of C acts nilpotently on L_0 , the Fitting null-component relative to T , and thus so does every element of C_p . It follows that $L = L_0 + \sum_{i \in \mathbb{Z}_p} A_{i\alpha}$ so $L^{(\infty)} = A$ is simple with $TR(A) = 1$. We therefore have that

$$p \neq 2, \quad A \in \{sl_2(F), W(1 : \underline{1}), H(2 : \underline{1})^{(1)}\} \text{ if } p > 3$$

$$\text{and } A \in \{sl_2(F), psl_3(F)\} \text{ if } p = 3,$$

by [4] and [6]. But now, $\dim A_\alpha = 1$ (by [1, Corollary 3.8] for all but $psl_3(F)$, and this is straightforward to check) and $M = L_0 \subset L_0 + A_\alpha \subset L$, a contradiction. It follows that L is solvable.

The converse is clear. \square

A subalgebra U of L is *nil* if $\text{ad } u$ acts nilpotently on L for all $u \in U$. Notice that we cannot replace ‘nilpotent’ in Theorem 2.5 by ‘solvable’

or ‘supersolvable’ and draw the same conclusion, as $sl_2(F)$ is a counterexample. However, we can prove the same result with ‘nilpotent’ replaced by the stronger condition ‘nil’ without any restrictions on the field F .

Theorem 2.6 *Let L be a Lie algebra over any field F . Then $Sec(M)$ is nil for every maximal subalgebra M of L if and only if L is solvable.*

Proof. Let L be a minimal non-solvable Lie algebra such that $Sec(M)$ is nil for every maximal subalgebra M of L . If L is simple then every maximal subalgebra of L is nil. It follows that every element of L is nil and L is nilpotent, by Engel’s Theorem. Hence no such Lie algebra exists. So, arguing as in paragraphs 1 and 2 of Theorem 2.5 above, L is monolithic with monolith A , L/A is solvable, and there is a maximal subalgebra M of L such that $L = M + A$ with an element $a \in M \cap A$ such that $\text{ad } a$ is not nilpotent. But this is a contradiction, since $A \cap M = Sec(M)$ is nil.

Once again, the converse is clear. \square

Let $(L, [p])$ be a restricted Lie algebra. Recall that an element $x \in L$ is called p -nilpotent if there exists an $n \in \mathbb{N}$ such that $x^{[p]^n} = 0$. Then we have the following immediate corollary.

Corollary 2.7 *Let L be a restricted Lie algebra over a field F of characteristic $p > 0$. Then $Sec(M)$ is p -nilpotent for every maximal subalgebra M of L if and only if L is solvable.*

Proof. Simply note that that a p -nilpotent subalgebra is nil. \square

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