Urchin Graphs
Degenerate bar-and-joint circuits

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LMS Workshop on Rigidity of Frameworks and Applications
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Outline

1. Bar & Joint Frameworks
2. 3-dimensional examples
3. Generic flatness and urchins
4. Varieties of urchins
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1. Bar & Joint Frameworks
2. 3-dimensional examples
3. Generic flatness and urchins
4. Varieties of urchins
1. Bar & Joint Frameworks
   - Generic frameworks and matroids
   - Degenerate frameworks and matroids
   - Flatness
   - Flatness and symmetry
   - Flatness and $X$-replacement
Let $d \geq 1$ be an integer. A \textit{d-dimensional framework} is a pair $(G, p)$ where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$.

- The map $p$ is a \textit{realisation} of $G$ in $\mathbb{R}^d$. It assigns each vertex a position in $\mathbb{R}^d$.
- Edge lengths are considered fixed. We have mechanical and \textit{infinitesimal} notions rigidity and flexibility.

A realisation $p$ is \textit{generic} if the multi-set of all $d|V|$ coordinates used to position the vertices of $G$ in $\mathbb{R}^d$ is algebraically independent over $\mathbb{Q}$. Say that a framework $(G, p)$ is \textit{generic} if $p$ is generic.
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Generic frameworks

Rigidity matroids

Each $d$-dimensional framework $(G, P)$ has an associated rigidity matroid $\mathcal{R}_d(G, p)$. 

Generic rigidity matroids

For all generic realisations $p$ of $G$ in $\mathbb{R}^d$, the rigidity matroids $\mathcal{R}_d(G, p)$ are identical. We refer to this matroid as $\mathcal{R}_d(G)$, the generic $d$-dimensional rigidity matroid of $G$. $\mathcal{R}_d(G)$ depends only on the graph $G$ (and the dimension $d$ of the space we realise it in).
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What we know already

- $R_1(G)$ and $R_2(G)$ are well understood combinatorially in terms of the structure of $G$. Good algorithms etc.
- $R_d(G)$ is not yet well understood combinatorially for $d \geq 3$. This open problem seems very difficult, even for the case when $d = 3$.
- There are many applications, particularly for $d = 3$. 
Generic frameworks

The story so far...

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1 Bar & Joint Frameworks

- Generic frameworks and matroids
- Degenerate frameworks and matroids
- Flatness
- Flatness and symmetry
- Flatness and $X$-replacement
If $p$ is not generic, sometimes $\mathcal{R}_d(G, p) \neq \mathcal{R}_d(G)$.

Both topologically, and measure-theoretically, such realisations $p$ are extremely rare. Nonetheless, there are of course a great many of them(!)

Understanding $\mathcal{R}_d(G, p)$ for arbitrary $p$ seems to be very difficult (even for $d = 2$, despite our fairly thorough understanding of $\mathcal{R}_2(G)$).
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Bolker & Roth, 1980

The graph $K_{4,6}$ is generically rigid (and independent) in $\mathbb{R}^3$. However, if realised with the 4-class of the bipartition positioned coplanar in $\mathbb{R}^3$, it produces a flexible (and dependent) framework.

We will return to this example later.
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Suppose $d \geq 3$ is an integer, $p : V \rightarrow \mathbb{R}^d$, and $U \subseteq V$ has $2 \leq |U| \leq d + 1$. If $p$ was generic, the dimension of the affine span of $p(U)$ would be $(|U| - 1)$.

**Definition: Flatness**

Say that a realisation $p$ is $U$-flat if the dimension of the affine span of $p(U)$ is at most $(|U| - 2)$. Say a framework $(G, p)$ is $U$-flat if $p$ is $U$-flat.

Thus, for example, flatness can consist of:

- two coincident vertices,
- three collinear vertices,
- four coplanar vertices,
- if $d \geq 4$, five co-$\mathbb{R}^3$-ar vertices, etc...
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- Flatness and symmetry
- Flatness and X-replacement
Flatness is one of the most elementary types of degeneracy, in the sense that other types of degeneracy often induce a great deal of flatness. This is the case with symmetric frameworks.

**Reflectional symmetry in $\mathbb{R}^3$ inducing flatness**

If a 3-dimensional framework has a plane of reflectional (mirror) symmetry, then any two vertices on one side of the plane, together with their mirror images, are coplanar.
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Rotational symmetry in $\mathbb{R}^3$ inducing flatness

- If a 3-dimensional framework has a rotational symmetry of order 4 or more, then any vertex and any 3 of its images under this symmetry form a set of four coplanar vertices.
- If a 3-dimensional framework has rotational symmetry of any order, and three vertices are forced to lie on the axis of rotation, then they are necessarily collinear.
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Flatness and $X$-replacement

**$X$-replacement**

The graph operation of $X$-replacement adds a new vertex with $d + 2$ edges, and removes 2 edges, as shown below:

![Diagram of $X$-replacement](image)

**$X$-replacement conjecture (Tay, Whiteley)**

The operation of $X$-replacement preserves independence (rigidity, isostaticity) in $\mathcal{R}_3(G)$. (It can fail to in $\mathcal{R}_d(G)$ when $d \geq 4$)
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There is a strong relationship between $X$-replacement and flatness.

**$X$-replacement with four coplanar vertices**

If $d \geq 3$ is an integer, and $E(G)$ is independent (or rigid, isostatic) in $\mathcal{R}_d(G, p)$, and $p(v_1), p(v_2), p(v_3), p(v_4)$ are coplanar in $\mathbb{R}^d$, then an $X$-replacement removing edges $v_1v_2$ and $v_3v_4$ preserves independence (or rigidity, isostaticity).
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2 3-dimensional examples

- Four coplanar vertices
- Three collinear vertices
- Two coincident vertices
The graph $K_{4,5}$ is generically independent in $\mathbb{R}^3$. However, if realised with the 4-class of the bipartition coplanar, it is dependent.

In fact, this example extends to all dimensions $d \geq 3$ if we replace $K_{4,5}$ by $K_{4,d+2}$ (with the 4-class coplanar in $\mathbb{R}^d$).
3-dimensional examples

Four coplanar vertices II

\( K_{4,4} + e + f \) in \( \mathbb{R}^3 \)

The graph \( K_{4,4} + e + f \), is generically independent (isostatic in fact) in \( \mathbb{R}^3 \). However, if realised with the modified 4-class coplanar, it is dependent.

Also works in all dimensions \( d \geq 3 \) if we replace \( K_{4,4} + e + f \) by \( K_{4,d+1} + e + f \) (with the 4-class coplanar in \( \mathbb{R}^d \)).
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The graph $K_{3,5}$ is generically independent in $\mathbb{R}^3$. However, if realised with the 3-class collinear, it is dependent.

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The graph $K_{3,4} + e$ is generically independent in $\mathbb{R}^3$. However, if realised with the 3-class collinear, it is dependent.

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2. 3-dimensional examples
   - Four coplanar vertices
   - Three collinear vertices
   - Two coincident vertices
The graph $K_{2,4}$ is generically independent in $\mathbb{R}^3$. However, if realised with the 2-class coincident, it is dependent.

This example extends to all dimensions $d \geq 3$ if we replace $K_{2,4}$ by $K_{2,d+1}$ (with the 2-class coincident in $\mathbb{R}^d$).
3 Generic flatness and urchins

- Generically flat rigidity matroids
- Urchins
Let $G = (V, E)$ be a graph, and $d \geq 3$ be an integer. We make the following definition:

### Generically-$U$-flat realisations

For $U = \{u_1, ..., u_k\} \subseteq V$ with $2 \leq k \leq d + 1$, say that $p : V \rightarrow \mathbb{R}^d$ is generically-$U$-flat if:

1. $p|_{V - \{u_k\}}$ is generic, and
2. there exists a set of $(k - 2)$ real numbers $\lambda_1, ..., \lambda_{k-2}$ which is algebraically independent over the field generated over $\mathbb{Q}$ by the $d(|V| - 1)$ coordinates of $p(V - \{u_k\})$, such that $p(u_k) = \sum_{i=1}^{k-1} \lambda_i p(u_i)$, where $\lambda_{k-1} = 1 - \sum_{i=1}^{k-2} \lambda_i$. 
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Generically-\(U\)-flat rigidity matroids

- It follows that any two generically-\(U\)-flat realisations of \(G\) in \(\mathbb{R}^d\) have identical rigidity matroids.
- Thus there is a generically-\(U\)-flat rigidity matroid \(\mathcal{R}_d^U(G)\) which describes the rigidity of almost all \(U\)-flat frameworks realising \(G\) in \(\mathbb{R}^d\).
- This allows us to ignore the particulars of any specific \(U\)-flat realisation, and explore the associated graph theory, analogous to the study of \(\mathcal{R}_d(G)\).
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3 Generic flatness and urchins
   - Generically flat rigidity matroids
   - Urchins
We finally define the objects this talk is intended to be about:

**Urchin graphs**

Say that a graph $G = (V, E)$ is a $d$-dimensional urchin if there is a subset $U = \{u_1, \ldots, u_k\} \subseteq V$ with $2 \leq k \leq d + 1$ such that:

1. Every vertex $v \in V - U$ has $N(v) = U$, and
2. $E$ is independent in $R_d(G)$ and a circuit in $R^U_d(G)$.

- We refer to the subgraph $G[U]$ as the shell of the urchin $G$, and the vertices in $V - U$ as the spikes of $G$. 

Adam Watson — Urchin Graphs
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These are some of the graphs we have already discussed, and in fact are all urchins. Their shells are the ringed parts, and the remaining vertices are spikes.
Urchins
Why higher dimensional space?

- The higher the dimensional of the space, the more urchins there are.
- Six urchins occur in $\mathbb{R}^3$.
- These six are important for understanding other types of degeneracy in $\mathbb{R}^3$, including symmetry.
- They make much more sense when we consider them in the broader context of higher dimensions, with many striking patterns emerging. There are several distinctive families of urchin.
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Outline

1. Bar & Joint Frameworks
2. 3-dimensional examples
3. Generic flatness and urchins
4. Varieties of urchins
4 Varieties of urchins

- Urchins with \((d + 2)\) spikes
- Urchins with \((d + 1)\) spikes
- Urchins with \(d\) spikes
- Urchins with fewer spikes
- New urchins from old
Let $E_k$ denote the empty graph on $k$ vertices.

**Empty-shelled urchins**

Let $d, k$ be integers, with $d \geq 3$, and $2 \leq k \leq d + 1$. Then $E_k$ is the shell of a $d$-dimensional urchin with $(d + 2)$ spikes (or $(d + 1)$ spikes if $k = 2$).

The different number of spikes when $k = 2$ can be explained by membership of our second family of urchins, the split-shells.
4 Varieties of urchins

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Urchins with \((d + 1)\) spikes

Split-shelled urchins

Let \(d, k, i\) be integers, with \(d \geq 3\), and \(2 \leq k \leq d + 1\), and \(1 \leq i \leq \lfloor k/2 \rfloor\). Then the disjoint union of \(K_i\) and \(K_{k-i}\) is the shell of a \(d\)-dimensional urchin with \(d + 1\) spikes (if \((d, k, i) \neq (3, 4, 1))\).
Urchins with \((d + 1)\) spikes

Split-shelled urchins - examples and comments

Some split-shells we’ve already seen

- \(K_{4,4} + e + f\) in \(\mathbb{R}^3\) with four coplanar vertices.
- \(K_{3,4} + e\) in \(\mathbb{R}^3\) with three collinear vertices.
- \(K_{2,4}\) in \(\mathbb{R}^3\) with two coincident vertices.

Comments:

- \(E_2\) is the shell of both a split-shelled and empty-shelled urchin. With \((d + 1)\) spikes rather than \((d + 2)\). This is the only urchin whose shell has size 2.
- The case where \((k, i) = (4, 1)\) is fine when \(d \geq 4\), only for \(d = 3\) is the graph not Laman.
- \((d + 1)\)-spiked urchins can have hugely variable numbers of edges in their shells (more on that to come...).
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Urchins with \((d + 1)\) spikes
Star-shelled and bi-star-shelled urchins

Let \(d, k\) be integers with \(d \geq 3\), and \(4 \leq k \leq d + 1\). Then the complete bipartite graphs \(K_{1,k-1}\) and \(K_{2,k-2}\) are the shells of \(d\)-dimensional urchins with \((d + 1)\) spikes (when Laman).

Note the absence of \(k = 3\). Also, \(K_{3,k-3}\) is not a shell.
4 Varieties of urchins

- Urchins with \((d + 2)\) spikes
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- Urchins with fewer spikes
- New urchins from old
Let $d, k$ be integers with $d \geq 3$ and $3 \leq k \leq d$. Then $K_{k-1} + e$ is the shell of a $d$-dimensional urchin with $d$ spikes.

We don’t include $k = d + 1$ because the resulting graph would not be Laman. Notice the case $k = 3$ (explains last slide).
Wheel-shelled urchins

Let \( d, k \) be integers with \( d \geq 5 \), and \( 5 \leq k \leq d + 1 \) Then the graph \( K_k - E(K_{k-2}) \) is the shell of a \( d \)-dimensional urchin with \( d \) spikes.
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The following construction produces urchins with progressively fewer spikes.

**Stripping off spikes**

Suppose $d, k, i$ are integers, with $d \geq 3$, $3 \leq k \leq d + 1$, and $0 \leq i \leq k - 2$. Then the graph $K_k - E(K_{k-i})$ is the shell of an urchin with $d + 2 - i$ spikes (when Laman).
Looking at this from the other side, we can urchins with very few spikes:

Let $s, d$ be integers with $s \geq 5$ and $d \geq s - 2$. Then $K_{d+1} - E(K_{s-1})$ is the shell of an urchin with $s$ spikes.
The shells of these urchins have $d + 1$ of vertices.

- When $d = s - 2$, we get an empty-shelled urchin.
- When $d = s - 1$, we get a star-shelled urchin.
- When $d = s$, we get a wheel-shelled urchin.
- When $d \geq s + 1$, we get our first explicit examples of urchins with less than $d$ spikes.
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Crossing shells I: Keeping spikes

Let $G = (V, E)$ be a graph on $k$ vertices and $E = E_1 \cup E_2$, with $E_1 \neq E_2$ and $e \in E_1 \cap E_2$. Suppose $G_i = (V, E_i)$ is the shell of an urchin with $s_i$ spikes, for $i = 1, 2$. Then $G - e$ contains the shell $G'$ of an urchin with at most $\max(s_1, s_2)$ spikes, and $V(G') = V$.

This works for simple matroidal reasons. New urchins discovered this way can have as many spikes as their spikiest parent did, although sometimes they have less.
Crossing shells I: Keeping spikes

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Crossing shells II: Reducing spikes

Let $G = (V, E)$ be a graph on $k$ vertices and $E = E_1 \cup E_2$, with $E_1 \neq E_2$. Suppose $G_i = (V, E_i)$ is the shell of an urchin with $s_i$ spikes, for $i = 1, 2$. Then $G$ contains the shell $G'$ of an urchin with at most $s - 1$ spikes, where $s = \max(s_1, s_2)$, and $V(G') = V$.

Again, this works for simple matroidal reasons. New urchins discovered this way have at least one less spike than their spikiest parent.
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Future work

- Can we find nice ways to prove all of the families of urchins we have described?
- Can we characterise all urchins?
- Can we relate urchins to symmetric and other degenerate frameworks?
- What happens if we flatten even more vertices into a small subspace?
- What happens if we flatten multiple sites of a framework?
- There are other frameworks which cause dependence when flattened (not built from bipartite graphs). Is it true that their motions are only infinitesimal, and don’t extend to finite motions?
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A few shells to look at

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Adam Watson — Urchin Graphs