Subalgebras of graph C*-algebras [1]

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1. Introduction

I give a self-contained introduction to two novel classes of nonselfadjoint operator algebras, namely the generalised analytic Toeplitz algebras $\mathcal{L}_G$, associated with the "Fock space" of a graph G, and subalgebras of graph C*-algebras. These two topics are somewhat independent but in both cases I shall focus on fundamental techniques and problems related to classifying isomorphism types and to the recovery of underlying foundational structures, be they graphs or groupoids.

1.1. **Generalities on Operator Algebras.** Let us set the scene with a bird's eye view of how operator algebras "come about" and comment on their morphisms. I shall take the term *operator algebra* to mean a complex algebra of bounded linear operators on a separable complex Hilbert space. For example, the operator algebra $\mathcal{A}$ could be the set of all complex single variable polynomials in a given operator T ; that is

$$\mathcal{A} = \{p(T) = a_0 I + a_1 T + \cdots + a_n T^n : p(z) = a_0 1 + a_1 z + \cdots + a_n z^n\}.$$

Often, the operator algebras of interest are manufactured by specifying a set of generators (such as the set $\{I, T\}$ in the example) on a Hilbert space both of which (set and space) arise from a "foundational" mathematical structure, such as a group, or a graph, or a dynamical system. We might call this a *spatial* setting since the Hilbert space is in place at the outset and the operator algebra is taken to be the algebra generated by the given generators. The term "generated" may mean simply the unclosed complex algebra or it may refer to the closure of this algebra in some topology, usually either the operator norm topology or the weak operator topology. We do not assume that the generated algebra is self-adjoint, that is, closed under the conjugate transpose operation $X \to X^*$, although of course that will follow if the set of generators is a self-adjoint set.

Operator algebras are also constructed in a Hilbert space-free way, for example, as a particular operator algebra, within some huge general class of operator algebras, satisfying a *universal property* for (perhaps) a set of generators and relations. Alternatively the algebra $\mathcal{A}$ might be constructed in the category of normed algebras with the expectation that $\mathcal{A}$ is isometrically isomorphic to an operator algebra by virtue of the fact that the ingredients for $\mathcal{A}$ are operator algebras. For example $\mathcal{A}$ might be a Banach algebra direct limit of operator algebras, or, again, simply a quotient of operator algebras.

We shall focus on spatial viewpoints. However, let us note that the celebrated Gelfand Naimark theorem can bring us back into a spatial context. In truth, there are usually more direct ways of providing a Hilbert spaces on which an indirectly constructed operator algebra can sit as a set of bone fide operators.

Theorem 1.1. *Let $\mathcal{A}$ be a C^*-algebra (an involutive complete normed algebra with $\|ab\| \leq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ for all $a, b \in \mathcal{A}$). Then there is a Hilbert space $\mathcal{H}$ (a separable one is possible if A is separable) and an isometric star homomorphism $\mathcal{A} \to \mathcal{B}(\mathcal{H})$.*

A fundamental question for a class of operator algebras is:

When are two operator algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ isomorphic ?

The strongest sense of isomorphism is undoubtedly *unitary equivalence* whereby $\mathcal{A}_1 = U\mathcal{A}_2U^*$ for some isometric onto map U from the Hilbert space of $\mathcal{A}_2$ to that of $\mathcal{A}_1$. Here the operator algebras really are the same if only we would re-identify the Hilbert spaces. A weaker form of isomorphism which also takes account of how the operator algebra sits on the underlying Hilbert space is *star-extendible isomorphism.* This requires that there is a map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ which is the restriction of an adjoint respecting algebra isomorphism $\phi : C^*(\mathcal{A}_1) \to C^*(\mathcal{A}_2)$ between the generated C*-algebras. Weaker still, and now ignoring how the operator algebras are manifested, an *isometric algebra isomorphism* is simply an algebra isomorphism which is an isometric linear map. For nonselfadjoint operator algebras this form of isomorphism is usually the essential case to elucidate. In truth, while these forms of isomorphisms certainly are different in the case of operator algebras constructed from the *same* spatial scheme, as alluded to above, the resulting forms of *isomorphism type* are usually the same. By this I mean that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic in one of the three sense above then they are isomorphic in the other senses. Is there a metatheorem here I wonder ?

Let us also note a companion question to that above, which is generally deeper, concerning the symmetries of an operator algebra $\mathcal{A}$.

What is the isometric automorphism group of an operator algebra ?

Naturally one expects that when two instances of a foundational structure are isomorphic then this entails an isometric isomorphism between the associated operator algebras and indeed this is generally a routine verification. (We might more formally realise the association as a functor.) But how about the converse direction ? If $\mathcal{A}(S_1)$ and $\mathcal{A}(S_2)$ are the (norm closed say) operator algebras obtained from the foundation structures S_1 and S_2, and if $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric algebra isomorphism then does this somehow induce an isomorphism between S_1 and S_2 ? This would provide a satisfyingly definitive answer to the isomorphism question and it is in this connection that there is, as we say, clear blue water between the non-self-adjoint and the self-adjoint theory. Algebras of the former category seem to remember their foundations while the self-adjoint algebras need not. [2]

[2]For C*-algebras it is K-theory and associated invariants that often lead to classifying invariants. In general such invariants are insufficient as they are generally determined by the diagonal part $A \cap A^*$ of the operator algebra $\mathcal{A}$.

As an indication of the general landscape ahead here are the ingredients of several operator algebra contexts providing a hierarchy of Toeplitz algebras.

A: The classical context: The Hardy Hilbert space H^2 for the unit circle, the unilateral shift operator S, with $\dim(I - SS^*) = 1$, the disc algebra $A(\mathbb{D})$, the function algebra $H^\infty(\mathbb{D})$, realisations of $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$ as "analytic" Toeplitz algebras, and the Toeplitz C*-algebra $\mathcal{T}_{\mathbb{Z}_+}$.

Let us outline some important classical facts. The Hilbert space $\ell^2(\mathbb{Z}_+)$ is unitarily equivalent to the Hardy space H^2 of the Hilbert space $L^2(\mathbb{T})$ of square integrable functions on the circle. The basis matching unitary $U : \ell^2(\mathbb{Z}_+) \to H^2$ which does this (with $Ue_n := z^n$ for $n \geq 0$) effects a unitary equivalence between the unilateral shift S (with $Se_n = e_{n+1}, n \geq 0$) and the multiplication operator $T_z : f \to zf, f \in H^2$. That is

$$USU^* = T_z$$

More generally, the Toeplitz operator T_ϕ with symbol function ϕ in $C(\mathbb{T})$ or $L^\infty(\mathbb{T})$ is given by $T_\phi : f \to P_{H^2}\phi f$. A good exercise to do at least once is to show that the Toeplitz algebra $\mathcal{T}_{\mathbb{Z}_+} = C^*(I, T_z)$ (which is equal to $UC^*(I, S)U^*$) contains every compact operator K on H^2. The idea is to start with the rank one operator $I - T_zT_z^*$ and "move it around" with the shifts T_z, T_z^* to obtain every rank one operator of the form $f \to \langle f, z^k\rangle z^l$. Then we can take linear combinations to approximate any finite rank operator. Once this is done we obtain the following theorem.

Theorem 1.2. *(i) For each Toeplitz operator T_ϕ and compact operator K we have*

$$\|T_\phi + K\| \geq \|T_\phi\|.$$

(ii) The Toeplitz algebra $\mathcal{T}_{\mathbb{Z}_+}$ is equal to the set of operators $T_\phi + K$ with ϕ in $C(\mathbb{T})$ and K compact and the quotient $\mathcal{T}_{\mathbb{Z}_+}/\mathcal{K}$ is naturally isomorphic to $C(\mathbb{T})$.

On the other hand the norm closed operator algebra generated by T_z is abelian and isometrically isomorphic to the disc algebra. Indeed, it is the algebra $\{T_\phi : \phi \in A(\mathbb{D})\}$. The weak operator topology closed algebra is similarly a copy of $H^\infty(\mathbb{D})$, namely, $\{T_\phi : \phi \in H^\infty(\mathbb{D})\}$. On occasion we simply write H^∞ for this operator algebra when the context is clear. Recall that the weak operator topology is the weakest topology for which the spatial linear functionals $T \to \langle Tf, g\rangle$ are continuous.

There are a great many ways in which one can move on from the Toeplitz context above and below I discuss some aspects of the following operator algebra directions.

B: The (spatial) free semigroup context (Section 3.: The Fock space $\ell^2(\mathbb{F}_n^+)$ for the free semigroup on n generators, the freely noncommuting shifts $L_1, \ldots, L_n$ with

$$\dim(I - (L_1L_1^* + \ldots L_nL_n^*)) = 1,$$

the noncommutative disc algebra $\mathcal{A}_n$ and free semigroup algebra $\mathcal{L}_n$, and the Cuntz-Toeplitz C*-algebra on $\ell^2(\mathbb{F}_n^+)$.

C: The (spatial) graph context (Section 3.: The Fock space of a directed graph $G = (V, E)$, the freely noncommuting partial isometries $L_e, e \in E$, the tensor algebra $\mathcal{A}_G$, the free semigroupoid algebras $\mathcal{L}_G$, and the Cuntz-Krieger-Toeplitz C*-algebras $\mathcal{T}_G = C^*(\mathcal{A}_G)$.

D: The (universal) free semigroup context (Sections 2,4): The freely noncommuting isometries $S_1, \dots, S_n$ with $S_1S_1^* + \dots + S_nS_n^* = I$, the Cuntz algebras $\mathcal{O}_n = C^*(S_1, \dots, S_n)$.

E: The (universal) graph context (Section 4): The (universal) graph C*-algebra C*(G) of a countable directed graph $G = (V, E)$ with partial isometry generators S_e, for $e \in E$, and relations

$$\sum_{r(e)=x} S_eS_e^* = P_x, \quad S_e^*S_e = P_{s(e)},$$

where $\{P_x : x \in V\}$ is a family of orthogonal projections and $e = (r(e), s(e))$.

2. The Cuntz Algebras, intuitively.

The Cuntz algebra $\mathcal{O}_n$ is a certain C*-algebra generated by n isometries, $S_1, \dots, S_n$ say, satisfying $S_1S_1^* + \dots + S_nS_n^* = 1$. That is, their range projections $S_iS_i^*$ are orthogonal and sum to the identity operator. In fact I could have dropped the word ”certain” because of the following remarkable uniqueness property.

Theorem 2.1. *If $n \geq 2$ and $s_1, \dots, s_n$ is any family of n isometries in a unital C*-algebra with $s_1s_1^* + \dots + s_ns_n^* = I$, then $C^*(\{s_1, \dots, s_n\})$ is naturally isometrically isomorphic to $\mathcal{O}_n$.*

Our main aim is to obtain tools and results that will help in understanding norm closed subalgebras of the Cuntz algebras. In this connection we are prepared to consider operator algebras generated by semigroups of words in the generators and to contemplate quite general subalgebras. Perhaps it is fair to say that a C*-algebraist is largely happy with the state of knowledge of the Cuntz algebras. He/she is more interested in looking for generalisations and wider classes to classify and understand (such as Graph C*-algebras, or C*-algebras allied to dynamical systems). Our view here is quite different - we are intending to linger with $\mathcal{O}_n$ and look inside it with a view to understanding classes of nonselfadjoint operator algebras. Our orientation and motivation comes partly from the existing theory of *limit algebras* which are found as nonselfadjoint subalgebras of approximately finite C*-algebras. We shall focus on the Cuntz algebras for clarity but the methods we discuss do extend to more general graph C*-algebras.

2.1. **Cuntz algebra basics.** One direct way to define $\mathcal{O}_n$ is to look into the next section, take the freely noncommuting shifts $L_1, \ldots, L_n$ on the Fock space for the free semigroup on two generators, take the generated C*-algebra and divide out by the ideal of compact operators. (This should sound familiar if $n = 1$!) In taking the quotient we lose the Hilbert space and gain equality in place of the one dimensional defect $\dim(I - L_1L_1^* + \ldots L_nL_n^*) = 1$.

The uniqueness allows us to consider two other models for $\mathcal{O}_n$ which will in fact be our viewpoint. I call these models the *interval picture* and the *Cantorised interval picture*. The first uses isometric operators $S_1, \ldots, S_n$ on $L^2[0,1]$ whose ranges are the orthogonal subspaces $L^2[\frac{i-1}{n}, \frac{i}{n}]$. For definiteness we define $S_if(x) = \sqrt{n}f(\frac{i-1+x}{n})$.

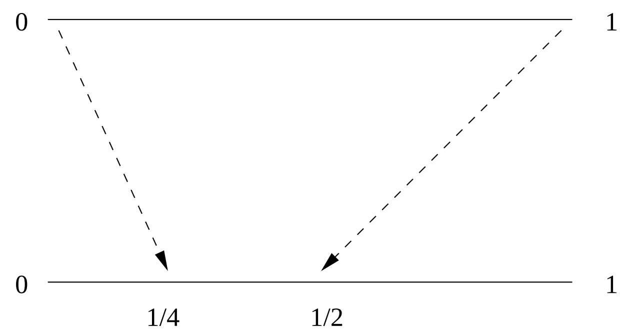

FIGURE 1. Interval picture for the operator S_1S_2 in $\mathcal{O}_2$.

Notice that a product $S_{i_1}S_{i_2}\ldots S_{i_k}$ has range $L^2[E_k]$ where E_k is an interval with $|E_k| = n^{-k}$. Write S_μ for this product where μ is the word $i_1i_2\ldots i_k$. These k-fold products have distinct ranges and so if the lengths of μ and ν are $|\mu| = |\nu| = k$ then $S_\mu^*S_\nu$ is the zero operator when μ and ν differ. (Exercise: Prove this algebraically.) It follows in general that if μ and ν have differing lengths and $S_\mu^*S_\nu$ is non zero then either $S_\mu^*S_\nu = S_{\mu'}^*$ where $\mu = \nu\mu'$, or $S_\mu^*S_\nu = S_{\nu'}$ where $\nu = \mu\nu'$.

On the other hand products with stars on the right are always non-zero. Indeed, from the interval picture we see that for $|\mu| = |\nu| = k$ the operator $S_\mu S_\nu^*$ acts as an isometry from $L^2[E_\nu]$ to $L^2[E_\mu]$ for certain intervals E_μ, E_ν with lengths $|n|^{-k}$. Moreover, the set of operators

$$\{S_\mu S_\nu^* : |\mu| = |\nu| = k\}$$

satisfies the relations of a matrix unit system. The span of this set, $\mathcal{F}_k^n$ say, is thus a copy of the matrix algebra M_{n^k}, and we have the matrix algebra tower

$$\mathcal{F}_1^n \subseteq \mathcal{F}_2^n \subseteq \mathcal{F}_3^n \ldots$$

We see then that the generators of $\mathcal{O}_n$ provide a distinguished subalgebra $\mathcal{F}_\infty^n = \cup_{k=1}^\infty \mathcal{F}_k^n$ which we refer to as a *matricial star algebra* of type n^∞. Write $\mathcal{F}^n$ for the closure of this subalgebra in $\mathcal{O}_n$.

The uniqueness proposition gives rise immediately to an important family of automorphisms of $\mathcal{O}_n$, the so called gauge automorphisms γ_z, for $z \in \mathbb{T}$, which satisfy $\gamma_z(S_i) =$

$zS_i, 1 \leq i \leq n$. (Alternatively, these automorphisms are inherited from easily defined unitarily implemented automorphisms of the Cuntz-Toeplitz C*-algebra.)

Proposition 2.2. *(i) Each operator a in the star algebra generated by $S_1, \ldots, S_n$ has a representation*

$$a = \sum_{i=1}^{N} (S_1^*)^i a_{-i} + a_0 + \sum_{i=1}^{N} a_i S_1^i$$

where $a_i \in \mathcal{F}_\infty^n$ for each i. This representation is unique if for each $i \geq 0$ we have $a_i = a_i P_i$ and $a_{-i} = P_i a_{-i}$ where P_i is the final projection of S_1^i.

(ii) The linear maps E_i, defined by $E_i(a) = a_i$, extend to continuous, contractive, linear maps from $\mathcal{O}_n$ to $\mathcal{F}^n$.

(iii) The generalised Cesaro sums

$$\Sigma_k(a) = \sum_{k=1}^{N} (1 - \frac{|k|}{N})(S_1^*)^k E_{-k}(a) + \sum_{k=0}^{N} (1 - \frac{|k|}{N}) E_k(a) s_1^k$$

converge to a as $N \to \infty$.

Proof. Our observations above show that the *linear span* of the operators $S_\mu S_\nu^*$ is the algebra generated by the generators. The representation in (i) (with the dilation actions carried by S_1 alone) follows from this by means of formulae such as $S_\mu = (S_\mu (S_1^*)^k) S_1^k = a S_1^k$ where $k = |\mu|$. The key to uniqueness is to make use of the "recovery formula" such as

$$a_0 = \int_0^{2\pi} \gamma_z(a) \frac{d\theta}{2\pi}$$

where the integral is a Riemann integral of a norm continuous function.

The representation in (i) can be viewed as a generalised Fourier series representation for the operator a. In fact to any operator a in $\mathcal{O}_n$ one may assign generalised Fourier coefficients a_k in $\mathcal{F}^n$ by means of the maps $E_k(.)$. The operators $a_k S_1^k$ $(k > 0)$ and $(S_1^*)^k a_{-k}$ $(k < 0)$ appear as the conventional Fourier series coefficients for the norm continuous operator valued function $f_a : z \to \gamma_z(a)$. The Cesaro polynomials $p_n(z)$ for the continuous function f_a converge uniformly in operator norm on $|z| = 1$ by classical theory. Finally, the specialisation $z = 1$ gives the desired norm convergence of the generalised Cesáro polynomials. □

Exercises. (i) Show that E_0 is faithful, that is, if $a \geq 0, a \neq 0$ then $E_0(a) \neq 0$. (ii) Show that

$$a_k = \left(\int_{\mathbb{T}} \gamma_z(a S_1^{*k}) dz \right) S_1^k, \; k > 0.$$

The Cantorised interval picture for $\mathcal{O}_n$ is a refinement of the interval presentation. The beauty of this perspective is that it provides both a new means of representing the algebra and yields a means for defining binary relation (and groupoid) invariants for subalgebras.

The essence of the idea is captured in Figure 2, shown with 2-fold branching relevant to the $n = 2$ case.

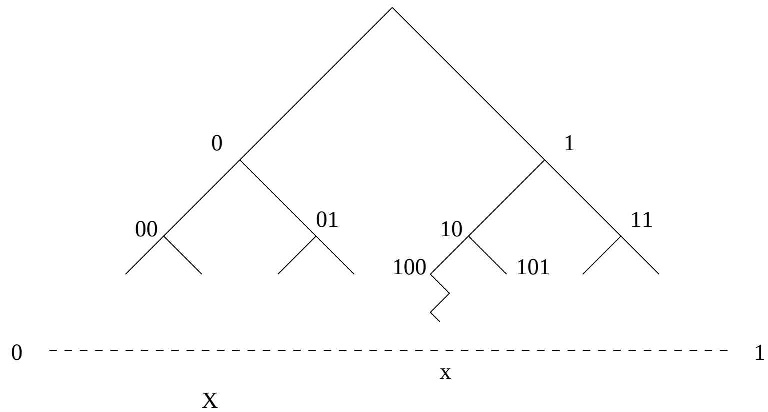

FIGURE 2. Cantorised interval picture.

Let X be the set of infinite paths on the tree, starting at vertex 0 or vertex 1. This set is identifiable with the direct product $\prod_{k=1}^{\infty}\{0,1\}$, consisting of points x which are zero-one sequence $x_1x_2\dots$. Each vertex word w in the tree, such as 1001, gives rise to an "interval" E_w of points x whose product expansion starts with w. With the product topology the direct product is a Cantor space whose topology has the set of intervals as a base of closed-open sets.

For each pair of vertex words w_1, w_2 there is a partial homeomorphism α_{w_2,w_1} from E_{w_1} to E_{w_2} defined by matching tails :

$$\text{if } x = w_1x_px_{p+1}\dots \qquad \text{then} \qquad \alpha_{w_2,w_1}(x) = w_2x_px_{p+1}\dots$$

Notice that for $|w_1| = |w_2| = k$ the partial homeomorphism has an action on the set X that bears close analogy with $S_{w_2}S_{w_1}^*$ and its interval picture, where we have relabeled the generators as S_0 and S_1. In fact we can add the natural product probability measure to X and present $\mathcal{O}_n$ on $L^2(X)$ in terms of (newly labeled) generators S_0 and S_1 induced by the partial homeomorphisms

$$\alpha_{\varnothing,0} : x \longrightarrow 0x,\ \ \alpha_{\varnothing,1} : x \longrightarrow 1x.$$

That is,

$$S_0f(x) = \sqrt{2}f(\alpha_{\varnothing,0}(x)), \quad S_1f(x) = \sqrt{2}f(\alpha_{\varnothing,1}(x)).$$

Exercise: Obtain the partial homeomorphism that is associated with the partial isometry $S_1S_0^* + S_{00}S_{11}^*$. (Here we have the 0-1 subscript labeling as opposed to the 1-2 labeling.)

2.2. **Normalising partial isometries.** We now come to an important class of partial isometries associated with $\mathcal{C}$.

Definition 2.3. A partial isometry in $\mathcal{F}^n$ (or, more generally, in $\mathcal{O}_n$) is $\mathcal{C}$-normalising if $v\mathcal{C}v^* \subseteq \mathcal{C}$ and $v^*\mathcal{C}v \subseteq \mathcal{C}$.

The obvious examples are the matrix units $S_\mu S_\nu^*$ for $|\mu| = |\nu| = k$, and the sums v of these when the initial projections are orthogonal and the final projections are orthogonal. Also we may multiply such a v by a unitary diagonal element d. The resulting $\mathcal{C}$-normalising partial isometry has support which can be indicated pictorially, as shown. The coordinates have been arranged so that the support picture represents v intuitively as a continuous matrix (although the d information is now lost). The picture should be "Cantorised" and viewed as a subset of $X \times X$.

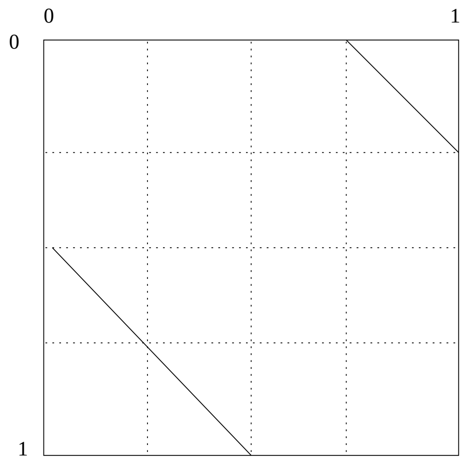

FIGURE 3. The support the partial isometry $S_2S_1^* + S_{11}S_{22}^*$.

The next theorem is an extremely useful characterisation. It shows in particular that for subalgebras of $\mathcal{F}^n$ an isometric isomorphism $\alpha : \mathcal{A}_1 \to \mathcal{A}_2$ with $\alpha(\mathcal{C}) = \mathcal{C}$ preserves the set of normalising partial isometries in the algebras.

Theorem 2.4. *Let v be an element of $\mathcal{F}^n$. Then the following assertions are equivalent:*
(i) v is a $\mathcal{C}$-normalizing partial isometry.
(ii) v is an orthogonal sum of a finite number of partial isometries of the form $dS_\mu S_\nu^*$, where $|\mu| = |\nu|$ and $d \in \mathcal{C}$.
(iii) For all projections $p, q \in \mathcal{C}$, the norm $\|qvp\|$ is equal to 0 or 1.

First we note some general "recovery facts" about $\mathcal{F}^n$ which show how operators may be approximated in an explicit manner.

For $b \in \mathcal{F}^n$ the "diagonal part" $\Delta(b) \in \mathcal{C}$ may be defined as the limit of the block diagonal operators

$$b_k := \Sigma_i e_{ii}^k b e_{ii}^k$$

where e^k_{ii} are the diagonal matrix units of $\mathcal{F}^n_k$. The map $\Delta : \mathcal{F}^n \to \mathcal{C}$ is a projection and moreover is faithful in the sense that $\Delta(b^*b) = 0$ entails $b = 0$. Likewise one can use block diagonal maps (via matrix unit projections taken from the commutants of $\mathcal{F}^n_m$ in $\mathcal{F}^n_k$, $k = m, m+1, \dots$) to define explicit maps $\Delta_m : \mathcal{F}^n \to \tilde{\mathcal{F}}^n_m$ where $\tilde{\mathcal{F}}^n_m$ is the C*-algebra $C^*(\mathcal{C}, \mathcal{F}^n_m)$ (which in fact is identifiable with $\mathcal{F}^n_m \otimes (e^m_{11}\mathcal{C}e^m_{11})$ and we have $\Delta_m(b) \to b$ as $m \to \infty$ for all $b \in F^n$. In this way (and analogously to Cesaro convergence) we can approximate a general element b in $\mathcal{F}^n$ by explicitly constructed approximants $\Delta_m(b)$ in $\tilde{\mathcal{F}}^n_m$.

Proof of Theorem 2.4: The directions $(ii) \implies (i) \implies (iii)$ are elementary so assume that v is an element which satisfies the zero one norm condition.

Choose m large so that

$$v = \Delta_m(v) + v', \ \|v'\| < 1.$$

Since v satisfies the zero one norm condition this is also true of $\Delta_m(v)$ and v'. Indeed, this holds for operators in the matrix subalgebras and by approximation holds in general. The implication (iii) implies (ii) is straightforward for elements of $\tilde{\mathcal{F}}^n_k$. Thus it remains to show that if v' has norm less than one and satisfies the zero one norm condition then $v' = 0$. This too follows from approximation.

Exercise. Show that $\mathcal{F}^n$ has no proper closed two-sided ideals. (Hint: E_0 is faithful.)

Remark. The map $a \to \Delta(E_0(a))$ is a positive faithful contraction onto the diagonal algebras but it is *not* (as above) defined as a limit of block diagonal parts. (eg consider $a = S_1$). (However, as we note in the Notes below, it may be defined in a more subtle algebraic manner.)

2.3. **Subalgebras of $\mathcal{F}^n_\infty$.** With the two models for $\mathcal{O}_n$ above and generalised Fourier series we are almost ready to contemplate the following vague question.

What are the natural subalgebras of $\mathcal{O}_n$?

Before turning to this we should of course look first inside the C*-algebra M_n and the algebras $\mathcal{F}^n_\infty, \mathcal{F}^n$.

The most natural subalgebras of M_n are perhaps those unital subalgebras A which are spanned by a subset of the standard matrix unit system $\{e_{ij} : 1 \le i, j \le n\}$. In fact these subalgebras are precisely those that contain the diagonal algebra C spanned by $\{e_{ii}\}$. (It is this latter property that we essentially use to define infinite dimensional variants.) In this case the set $E = \{(i,j) : e_{ij} \in A\}$ can be viewed as the set of edges of a directed graph G with n vertices. It follows that $(i,i) \in E$ for all i and that $(i,k) \in E$ if $(i,j), (j,k) \in E$. Conversely, if G is the graph (V, E), with E such a reflexive and transitive relation, then

$$A(G) := \text{span}\{e_{ij} : (i,j) \in E\}$$

is a complex unital subalgebra containing C. These algebras are the building block algebras of limit algebras. It is an elementary but worthwhile exercise to show that the operator algebra $A(G)$ remembers the graph G:

Proposition 2.5. *(Recovery theorem.) If $A(G_1)$ and $A(G_2)$ are isometrically isomorphic algebras then the graphs G_1, G_2 are isomorphic.*

Sketch proof: (i) Projections must map to projections (since projections are the idempotents with norm one), (ii) minimal projections map to minimal projections, (iii) one can reduce (via unitary equivalence) to the case that diagonal projections map to diagonal projections and this gives the needed vertex map.

We can think of E both as a "support set" for the algebra (should we view a matrix (a_{ij}) as a function $ij \to a_{ij}$), and also, more usefully, as a binary relation that comes with the algebra. In these terms the proposition says that a digraph algebra has isometric isomorphism type determined by the isomorphism type of its binary relation. In Section 4 I give a generalisation of this fact for a wide class of subalgebras of $\mathcal{O}_n$. First however, let us look inside the matricial star algebra $\mathcal{F}^n_\infty$ and its closure $\mathcal{F}^n$. In fact we may as well consider a more general class of approximately finite algebras.

Definition 2.6. (a) A *unital matricial star algebra* is a complex algebra B for which there exists a spanning set

$$\{e^k_{ij} : 1 \leq i, j \leq n_k, k = 1, 2, \dots\}$$

such that
(i) for each k the set $\{e^k_{ij} : 1 \leq i, j \leq n_k\}$ is a matrix unit system for M_{n_k},
(ii) for each k, $M_{n_k} \subseteq M_{n_{k+1}}$ and moreover the inclusion map is a C*-algebra injection which maps each e^k_{ij} to a sum of matrix units from $\{e^{k+1}_{ij} : 1 \leq i, j \leq n_k\}$.

(b) A *regular matricial algebra* is a complex algebra A which is a unital subalgebra of a matricial star algebra containing the diagonal subalgebra $C = \text{span}\{e^k_{ij}\}$.

It is straightforward to see that the regular matricial algebra A in the matricial star algebra B is the union of the algebras

$$A_k = A \cap M_{n_k} = \text{span}\{e^k_{ij} : e^k_{ij} \in A\}$$

and that each A_k is a digraph algebra $A(G_k)$ relative to the given matrix unit system. Our subalgebra $\mathcal{F}^n_\infty$ is a particular unital matricial star algebra in which each inclusion map has the same multiplicity n. Further examples of such algebras can be obtained by first taking a tower of appropriate inclusions maps, $A(G_1) \to A(G_2) \to \dots$. Algebraic direct limits then provide the containing algebras B and A and the diagonal algebra C, with $C \subseteq A \subseteq B$.

Open problems. (See the Notes below.) 1. Is C unique in A up to automorphisms of A ? (If so, then $R(A)$ becomes a well-defined invariant for the algebra A.)

2. Is $R(A)$ an algebraic isomorphism invariant for a regular matricial algebra A ?

2.4. **Subalgebras of unital AF C*-algebras.** Suppose we have the purely algebraic setting $C \subseteq A \subseteq B$ given in Definition 2.4. The matricial star algebra carries a unique C*-algebra norm and taking operator norm closures gives the triple inclusion $\mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{B}$. Here $\mathcal{B}$ is a UHF C*-algebra, $\mathcal{C}$ is a particularly nice maximal abelian subalgebra (masa) in $\mathcal{B}$ and $\mathcal{A}$ is an instance of a limit algebra

$$\mathcal{A} = \varinjlim (A(G_k), \phi_k)$$

where the inclusion maps $\phi_k : A(G_k) \to A(G_{k+1})$ are particularly nice. (In the terminology of the Appendix, they are star-extendible and regular.) We give three key results for such limit algebras. The first, rather surprisingly perhaps, shows that *any* norm closed algebra $\mathcal{A}$ with $\mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{B}$ is necessarily the closure of a regular matricial algebra. This is justification for the opinion that the subalgebras of UHF C*-algebras which contain the distinguished masa are the natural generalisations of finite dimensional digraph algebras. The following "local recovery" theorem gives a key step towards understanding the limit algebras $\mathcal{A}$.

Theorem 2.7. *(Inductivity principle.) Let B be a unital matricial star algebra with subalgebra chain $\{M_{n_k}\}$ and diagonal C and let $\mathcal{B}$,$\mathcal{C}$ be their norm closures. If $\mathcal{A} \subseteq \mathcal{B}$ is a norm closed subalgebra containing $\mathcal{C}$ then $\mathcal{A}$ is the closed union of the digraph algebras $A_k = \mathcal{A} \cap M_{n_k}, k = 1, 2, \dots$.*

The next two theorems (and that above) have more general forms for subalgebras of AF C*-algebras but we state them here for subalgebras of the UHF C*-algebra $\mathcal{F}^n$. We first need to define the appropriate substitute for the graph that underlies a digraph algebra and for this the Cantorised interval picture provides what we need, both for $\mathcal{F}^n$ and for $\mathcal{O}_n$. In fact we are going to define an isometric isomorphism invariant for the algebra $\mathcal{A}$ together with its diagonal $\mathcal{C}$ which is in the category of topological binary relations. Often (always ?!) the binary relation is a complete isometric isomorphism invariant for the algebra alone.

Let X be the Cantor space $\prod_{k=1}^{\infty}\{0, 1, \dots, n-1\}$. For words μ, ν with the same length k recall that $S_\mu S_\nu^*$ is a partial isometry on $L^2(X)$ which is induced by the partial homeomorphism

$$\alpha_{\mu,\nu} : \nu x_{k+1} x_{k+2} \cdots \to \mu x_{k+1} x_{k+2} \dots .$$

Define the topological space R to be the set in $X \times X$ which is the union of the graphs of these partial homeomorphism :

$$R = \{(\alpha(x), x) : x \in \mathrm{dom}(\alpha), \alpha = \alpha_{\mu,\nu}, |\mu| = |\nu| = k, k = 1, 2, \dots\}$$

Write $E^k_{i,j}$ for the graph of the partial homeomorphism for the matrix unit $e^{(k)}_{ij}$ in $\mathcal{F}^n_k$ and one can conceive of these sets as the "support set" of the matrix units. The diagonal matrix units $e^{(k)}_{ij}$ provide closed-open sets E^k_{ii} in the diagonal $\Delta = \{(x, x) : x \in X\}$ and these give a base for the natural Cantor space topology. We topologise R by taking the sets E^k_{ij} as a base for the topology.

Exercise: Show that R is an equivalence relation. Show that the topology is not the relative product topology.

It follows from Theorem 2.7 that if $\mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{F}^n$ then $\mathcal{A}$, and the given subalgebra chain, determines a subset $R(A)$ of R, namely

$$R(A) = \cup\{E_{ij}^k : e_{ij}^k \in A_k\}.$$

With the relative topology, $R(A)$ is known as the *topological binary relation* of A. In fact the topological binary relation $R(A)$ is determined by the pair $(\mathcal{A}, \mathcal{C})$ and serves as the analogue of the graph of a digraph algebra. The following uniqueness theorem also follows from Theorem 2.7.

Theorem 2.8. *(Spectral theorem for subalgebras.) Let $\mathcal{A}_1, \mathcal{A}_2$ be norm closed subalgebras of $\mathcal{F}^n$ which contain the canonical diagonal algebra $\mathcal{C}$. If $R(\mathcal{A}_1) = R(\mathcal{A}_2)$ then $\mathcal{A}_1 = \mathcal{A}_2$.*

That $R(\mathcal{A})$ is intrinsic to the pair $\mathcal{A}, \mathcal{C}$ is also revealed by the following proposition. We identify X with the Gelfand space of $\mathcal{C}$.

Proposition 2.9. *As a set, $R(A)$ is the set of points (x, y) in $X \times X$ for which there exists $a \in A$ and $\delta > 0$ such that*

$$\|paq\| \geq \delta$$

for all projections p, q in $\mathcal{C}$ with $x(p) = y(q) = 1$.

We can now state the following classification theorem which, loosely paraphrased, asserts that a triangular subalgebra of $\mathcal{F}^n$ remembers its topological binary relation.

Theorem 2.10. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be norm-closed subalgebras of $\mathcal{F}^n$ with $\mathcal{A}_i \cap \mathcal{A}_i^* = \mathcal{C}$ for $i = 1, 2$. (Such algebras are said to be triangular.) Then the following statements are equivalent*

(i) $\mathcal{A}_1 \cap \mathcal{F}_\infty^n$ and $\mathcal{A}_2 \cap \mathcal{F}_\infty^n$ are isometrically isomorphic normed algebras.

(ii) $\mathcal{A}_1$ and $\mathcal{A}_2$ are isometrically isomorphic operator algebras.

(iii) The topological binary relations $R(\mathcal{A}_1), R(\mathcal{A}_2)$ are isomorphic, that is, there is a homeomorphism $\alpha : M(\mathcal{C}) \to M(\mathcal{C})$ such that $\alpha \times \alpha : R(\mathcal{A}_1) \to R(\mathcal{A}_2)$ is a homeomorphism.

The key to the proofs of Theorem 2.7, Theorem 2.8, Proposition 2.9 and Theorem 2.10 is the structure of partial isometries given in Theorem 2.4. It is this which which makes the link between operator algebra entities and the underlying topological binary relation. Remarkably, perhaps, there is a close generalisation of Theorem 2.10 to gauge invariant subalgebras of $\mathcal{O}_n$. (Theorem 4.3.)

Sketch proof of Theorem 2.10. Let $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ be an isometric isomorphism. By triangularity, the set of projections p in $\mathcal{A}_i$ generate $\mathcal{C}_i$. Since the projections are the norm one idempotents it follows that $\Phi(\mathcal{C}_1) = \mathcal{C}_2$. By the zero-one characterisation in Theorem

2.4 it follows that Φ maps the $\mathcal{C}_1$-normalising partial isometries to $\mathcal{C}_2$-normalising partial isometries. Considering the support of a normalising partial isometry as a closed-open set in $M(\mathcal{C}_i) \times M(\mathcal{C}_i)$ it follows (from the finiteness of Theorem 2.4 (ii)) that Φ maps a base of closed open sets to a base of closed-open sets. Moreover Φ induces a point bijection $\alpha : R(\mathcal{A}_1) \to R(\mathcal{A}_2)$. (Take intersections of neighbourhoods.) The point bijection is a topological homeomorphism since its map on sets extends the original bijection of closed-open sets.

2.5. **Notes.** Theorem **??** is usually referred to as Coburn's theorem. For more on this, Cuntz algebras and other C*-algebras, see, for example, Davidson [3] and the references therein. $\mathcal{O}_n$ is usually defined as the universal C*-algebra generated by isometries $S_1, \dots, S_n$ with $S_1S_1^* + \dots + S_nS_n^* = I$. The direct sum of irreducible realisations of such relations gives generators for this algebra; the universal property is that to any isometric realisation $s_1, \dots, s_n$ of the relations there should exist a canonical homomorphism $\mathcal{O}_n \to C^*(\{s_1, \dots, s_n\})$ and this is readily checked for this universal direct sum. The uniqueness theorem, Theorem **??**, will follow now from the simplicity of $\mathcal{O}_n$, and this in turn follows readily from an algebraic formulation of the map $E_0 : \mathcal{O}_n \to \mathcal{F}_n$. (See [3],[1].) For if J is an ideal and $a \in J$ is nonzero, then a^*a is a positive nonzero element in J and so it follows from the algebraic formulation that $a_0 = E_0(a)$ is in J. Since $\mathcal{F}_n$ is simple the simplicity of $\mathcal{O}_n$ follows. In fact more is true in that one can find elements $x, y \in \mathcal{O}_n$ such that $xab = I$. To do this one uses the algebraic definitions of E_0 and Δ to get, in J, an operator $d = pdp = x_1ay_2$ close to a diagonal matrix unit p. Then one finds the appropriate isometry S_μ to conjugate p to an operator close to the identity.

The normalising partial isometry theorem and Theorem 2.7, Theorem 2.8, Theorem 2.10 are discussed further in [21] where one can find their natural extensions to AF C*-algebras.

The open problems are essentially the problems 7.8, 7.9 of [21]. Those problems are stated for closed algebras but because of inductivity the problems really reside in pure algebra and are stated in these terms here. If A is self-adjoint then the masa C is unique up to automorphism. More is true: for any other matrix unit system for A, with subalgebra system $\{A'_k\}$ and masa C' (as in the definition), there is an automorphism $A \to A$ which maps C to C'. (See [21].) However in this case the automorphism, which is determined by an intertwining diagram

$$A_1 \to A'_{n_1} \to A_{m_1} \to A'_{n_2} \to \dots,$$

can be constructed in such a way so that there is an intertwining diagram with regular maps in the sense that the normaliser of the diagonal algebras map into the normalisers of the diagonal of the next algebra. (See the Appendix.) Part of the difficulty of the general nonselfadjoint problem is that it is known that there are diagonal masas (of the type above) which although automorphically equivalent are not equivalent through an intertwining diagram of regular maps. See [9] for this subtlety, and see [10], [24] for further discussions.

The importance of the problem is that all kinds of putative invariants can be associated with classes of regular direct system and one would like these constructs (homology groups for example) to be invariants for the algebra rather than the pair (A, C).

3. Toeplitz algebras for countable directed graphs

We now take up a different topic and formally define the analytic Toeplitz algebras $\mathcal{A}_G$ and $\mathcal{L}_G$.

Let G be a finite or countable directed graph, with edge set $E(G)$ and vertex set $V(G)$. The *free semigroupoid* $\mathbb{F}^+(G)$ determined by G is a set with partially defined associative multiplication. The set consists of the vertices, which act as multiplicative units, and all finite directed paths in G. The partially defined product is the natural operation of concatenation of paths, with a vertex considered as a path. Thus a nonunit element of $\mathbb{F}^+(G)$ is a path (or word) $w = e_1e_2 \dots e_n$ where the initial vertex of each e_i, for $i < n$, is equal to the final vertex of e_{i+1}. Vertices may appear in a word of edges, redundantly, to indicate information. For example given a path w in $\mathbb{F}^+(G)$ we have $w = ywx$ when the initial and final vertices of w are, respectively, x and y.

Let $\mathcal{H}_G = \ell^2(\mathbb{F}^+(G))$ be the Hilbert space with orthonormal basis of vectors ξ_w indexed by elements w of $\mathbb{F}^+(G)$. For each edge $e \in E(G)$ and vertex $x \in V(G)$ define partial isometries and projections on $\mathcal{H}_G$ by the following left-sided actions on basis vectors:

$$L_e\xi_w = \begin{cases} \xi_{ew} & \text{if } ew \in \mathbb{F}^+(G) \\ 0 & \text{otherwise} \end{cases}$$

and

$$L_x\xi_w = \begin{cases} \xi_{xw} = \xi_w & \text{if } w = xw \in \mathbb{F}^+(G) \\ 0 & \text{otherwise} \end{cases}$$

We also write P_x for the projection L_x.

If G has a single vertex x then ξ_x is referred to as the vacuum vector and the operators L_w are isometries. If there are, additionally, only finitely many edges $e_1, \dots, e_n$ then each path w is a free word in these edges and the semigroupoid of paths is simply the free (unital) semigroup $\mathbb{F}_n^+$ with n generators. For $n = 2$ one can visualise the action of the two isometries L_{e_1} and L_{e_2} as downward left and right shifts of basis vectors placed at the vertices of a downward branching tree.

Definition 3.1. (i) The *free semigroupoid algebra* $\mathfrak{L}_G$ is the weak operator topology closed algebra generated by the projections L_x and the (partial) shift operators L_e;

$$\mathfrak{L}_G \quad = \quad \text{WOT–Alg}\,\{L_e, L_x : e \in E(G), x \in V(G)\}$$

(ii) The algebra $\mathcal{A}_G$ is the norm closed algebra generated by $\{L_e, L_x : e \in E(G), x \in V(G)\}$. This Toeplitz algebra is also referred to as the *tensor algebra* for G.

The algebra $\mathfrak{L}_G$ can also be thought of as being generated by the left "regular" representation $\lambda_G : \mathbb{F}_G^+ \to \mathcal{B}(\mathcal{H}_G)$, $\lambda_G(e) = L_e$, which faithfully represents the semigroupoid $\mathbb{F}_G^+$ as partial isometries. In the case of a noncomposable elements w_1 and w_2 one can check that the corresponding partial isometries have zero product.

3.1. **Examples and Matrix Function Algebras.** It should be apparent that the algebra $\mathcal{L}_G$ for the graph with a single vertex x and single loop edge $e = xex$ is unitarily equivalent to the analytic Toeplitz algebra $\mathcal{T}_{H^\infty}$; the Fock space naturally identifies with the Hardy space H^2, and L_e is then unitarily equivalent to the unilateral shift T_z. More generally, the noncommutative analytic Toeplitz algebras $\mathfrak{L}_n$, $n \geq 2$, also known as the free semigroup algebras, arise from the graphs with a single vertex and n distinct loop edges, while $\mathfrak{L}_\infty$ comes from the single vertex graph with countably many loops.

(i) If G is a finite graph with no directed cycles, then the Fock space $\mathcal{H}_G$ is finite-dimensional and so too is $\mathfrak{L}_G$. As an example, consider the graph with three vertices and two edges, labelled x_1, x_2, x_3, e, f where $e = x_2ex_1$, $f = x_3fx_1$. Then the Fock space is spanned by the vectors $\{\xi_{x_1}, \xi_{x_2}, \xi_{x_3}, \xi_e, \xi_f\}$ and with this basis the general operator $X = \alpha L_{x_1} + \beta L_{x_2} + \gamma L_{x_3} + \lambda L_e + \mu L_f$ in $\mathfrak{L}_G$ is represented by the matrix

$$X \simeq \begin{bmatrix} \alpha & & & & \\ & \beta & & & \\ & & \gamma & & \\ \lambda & & & \beta & \\ \mu & & & & \gamma \end{bmatrix}.$$

Here, $\mathfrak{L}_G$ is isometrically isomorphic to (but not unitarily equivalent to) the so-called digraph algebra (see later) in $\mathcal{M}_3(\mathbb{C})$ consisting of the matrices

$$\begin{bmatrix} \alpha & 0 & 0 \\ \lambda & \beta & 0 \\ \mu & 0 & \gamma \end{bmatrix}.$$

(ii) Consider the graph G with two vertices x, y, a loop edge $e = xex$ and the edge $f = yex$ directed from vertex x to vertex y. The tree graph for Fock space takes the form

The semigroupoid algebra $\mathfrak{L}_G$ is generated by $\{L_e, L_f, P_x, P_y\}$. If we make the natural identifications $\mathcal{H}_G = P_x\mathcal{H}_G \oplus P_y\mathcal{H}_G \simeq H^2 \oplus H^2$, then

$$L_e \simeq \begin{bmatrix} T_z & 0 \\ 0 & 0 \end{bmatrix}, \quad L_f \simeq \begin{bmatrix} 0 & 0 \\ T_z & 0 \end{bmatrix}, \quad P_x \simeq \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad P_y \simeq \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Thus, $\mathfrak{L}_G$ is seen to be unitarily equivalent to a matrix Toeplitz algebra, which we can also view as (isometrically and weak star - weak star isomorphic to) the matrix function algebra

$$\begin{bmatrix} H^\infty & 0 \\ H_0^\infty & \mathbb{C}I \end{bmatrix}$$

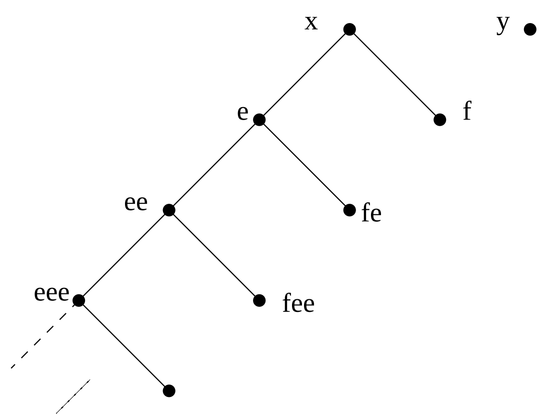

FIGURE 4. Fock space graph.

where H^∞_0 is the subalgebra of H^∞ formed by functions h with $h(0) = 0$.

Exercise. Add to G a "returning" directed edge $g = xgy$ to obtain a graph G'. Show that $\mathfrak{L}_{G'}$ contains a "copy" of $\mathcal{L}_2$ by virtue of the fact that it contains isometries with mutually orthogonal ranges.

(iii) Let $n \geq 1$ and consider the *cycle graph* C_n which has n vertices $x_1, \dots, x_n$ and n edges $e_n = x_1 e_n x_n$ and $e_k = x_{k+1} e_k x_k$ for $k = 0, \dots, n-1$. Identify $L_{x_i}\mathcal{H}_G$ with H^2 for each i in the natural way (respecting word length). Then $\mathcal{H}_G = L_{x_1}\mathcal{H}_G \oplus \dots \oplus L_{x_n}\mathcal{H}_G \simeq \mathbb{C}^n \otimes H^2$ and the operator $\alpha_1 L_{e_1} + \dots + \alpha_n L_{e_n}$ is identified with the operator matrix

$$\begin{bmatrix} 0 & & & & \alpha_n T_z \\ \alpha_1 T_z & 0 & & & \\ & \alpha_2 T_z & 0 & & \\ & & \ddots & \ddots & \\ & & & \alpha_{n-1} T_z & 0 \end{bmatrix}.$$

Write H^∞_n for the subalgebra of H^∞ arising from functions of the form $h(z^n)$ with h in H^∞. It follows that the algebra $\mathfrak{L}_{C_n}$ is isomorphic to the matrix function algebra

$$\begin{bmatrix} H^\infty_n & z^{n-1}H^\infty_n & \dots & zH^\infty_n \\ zH^\infty_n & H^\infty_n & & \vdots \\ \vdots & & \ddots & \\ z^{n-1}H^\infty_n & \dots & & H^\infty_n \end{bmatrix}.$$

3.2. **Fourier Series.** There is a companion algebra to $\mathfrak{L}_G$ coming from "right shifts". Consider the right regular representation $\rho_G : \mathbb{F}^+(G) \to \mathcal{B}(\mathcal{H}_G)$ determined by a directed graph G. This yields partial isometries $\rho_G(w) \equiv R_{w'}$ for $w \in \mathbb{F}^+(G)$ acting on $\mathcal{H}_G$ defined by the equations $R_{w'}\xi_v = \xi_{vw}$, where w' is the word w in reverse order. The corresponding algebra is

$$\mathfrak{R}_G = \text{WOT–Alg}\,\{R_e, R_x : e \in E(G), x \in V(G)\}$$
$$= \text{WOT–Alg}\,\{\rho_G(w) : w \in \mathbb{F}^+(G)\}.$$

Given edges $e, f \in E(G)$, observe that $L_e R_f \xi_w = \xi_{ewf} = R_f L_e \xi_w$, for all $w \in \mathbb{F}^+(G)$, so that $L_e R_f = R_f L_e$, and similarly for the vertex projections. In fact, we have

Proposition 3.2. *The commutant $\mathcal{L}'_G$ of $\mathcal{L}_G$ is equal to $\mathcal{R}_G$.*

The proof of this proposition makes use of an important tool, namely the Fourier series expansions of elements of $\mathfrak{L}_G$. Recall that P_x is the projection for the subspace spanned by basis vectors ξ_w with $w = xw$. Write Q_x for the projection onto the subspace spanned by basis vectors ξ_w with $w = wx$. (The projections Q_x correspond to the "components" of the Fock space when basis elements are linked by the natural tree structure.) The proposition below takes a simple form when there is a single vertex, ie. $\mathcal{L}_G = \mathcal{L}_n$.

Proposition 3.3. *Let $A \in \mathfrak{L}_G$, $x \in V(G)$, and let $a_w \in \mathbb{C}$ be the coefficients for which $A\xi_x = Q_x A \xi_x = \sum_{w=wx} a_w \xi_w$. Then the Cesaro sums associated with the formal sum $\sum_{w \in \mathbb{F}^+(G), w=wx} a_w L_w$, given by*

$$\Sigma_k(Q_x A) = \sum_{w=wx;|w|\le k} \Big(1 - \frac{|w|}{k}\Big) a_w L_w$$

converge in the strong operator topology to $Q_x A$.

3.3. $\mathfrak{L}_G$ **remembers the graph** G**.** Let us first observe why $\mathcal{L}_n$ and $\mathcal{L}_m$ are not isomorphic when $n \neq m$. This can be seen from the following theorems the first of which introduces another important tool, namely the eigenvectors for $\mathcal{L}_G^*$. By an eigenvector for $\mathcal{L}_n^*$ we really mean a vector ν which is a joint eigenvector for the n-tuple of generators $L_{e_1}^*, \dots, L_{e_n}^*$, so that

$$L_{e_i}^* \nu = \alpha_i \nu, \ \ , 1 \le i \le n,$$

for some complex numbers $\alpha_i \in \mathbb{C}$. The notation below is that $w(\lambda)$ is the complex number obtained on substituting λ_i for e_i in the word w.

Theorem 3.4. *The eigenvectors for $\mathcal{L}_n^*$ are complex multiples of the unit vectors*

$$\nu_\lambda = (1 - \|\lambda\|^2)^{1/2} \sum_{w \in \mathbb{F}_n^+} \overline{w(\lambda)} \xi_w,$$

for $\lambda = (\lambda_1, \dots, \lambda_n)$ in the open unit ball $\mathbb{B}_n \subseteq \mathbb{C}^n$. Furthermore $L_{e_i}^* \nu_i = \overline{\lambda_i} \nu_\lambda$, for each i.

Note that for λ in the open unit ball

$$\| \sum_1^n \lambda_i L_{e_i} \|^2 = \sum |\lambda_i|^2 = \|\lambda\|^2 < 1$$

so that $I - \sum_1^n \overline{\lambda}_i L_{e_i}$ is invertible, with inverse

$$\left(I - \sum_e \overline{\lambda_i} L_{e_i}\right)^{-1} = \sum_{k\geq 0}\left(\sum_e \overline{\lambda_i} L_{e_i}\right)^k = \sum_w \overline{w(\lambda)} L_w.$$

In particular $\sigma_w w(\lambda)$ is convergent, and a similar shows the norm to be $(1-\|\lambda\|^2)^{-1/2}$. Eigenvectors are important since they are allied to *characters*, ie., multiplicative linear functionals $\phi : \mathcal{L}_n \to \mathbb{C}$, $\phi : \mathcal{A}_n \to \mathbb{C}$. Indeed, the map $\phi_\nu : \mathcal{A}_n \to \mathbb{C}$ defined by

$$\phi_\nu(A) = \langle A\nu_\lambda, \nu_\lambda\rangle$$

satisfies

$$\phi_\nu(p(L_{e_1}, \dots, L_{e_n})) = \langle \nu_\lambda, \overline{p}(L_{e_1}^*, \dots, L_{e_n}^*)\nu_\lambda\rangle = \langle \nu_\lambda, \overline{p}(\overline{\lambda}_1, \dots, \overline{\lambda}_n)\nu_\lambda\rangle = p(\lambda_1, \dots, \lambda_n).$$

It follows that the vector functional actually defines a character. One can go on to show that the character space of $\mathcal{A}_n$ is homeomorphic to closed unit ball.

The dimension of the character space serves as a classifying invariant for the algebras $\mathcal{A}_n$ and $\mathcal{L}_n$. More generally one has the following theorem, and an analogous result for the weakly closed free semigroup algebras.

Theorem 3.5. *Let G, G' be directed graphs. Then the following assertions are equivalent.*
(i) G and G' are isomorphic graphs.
(ii) $\mathcal{A}_G$ and $\mathcal{A}_{G'}$ are unitarily equivalent.
(iii) $\mathcal{A}_G$ and $\mathcal{A}_{G'}$ are isometrically isomorphic.

3.4. **Notes.** There are a number of approaches to the classification theorem above. In Kribs and Power [14], following the free semigroup algebra analysis of Davidson and Pitts [5], [6], wandering vectors are analysed to obtain a Beurling type theorem for invariant subspaces of the algebra. Using this one obtains unitarily implemented automorphisms of $\mathcal{L}_G$ that act transitively on the set of eigenvectors. Now the eigenvectors are parametrised by the union of unit balls for each vertex with loop edges. In particular isomorphisms can be normalised to the special case where vacuum vectors map to vacuum vectors. Consequently the ideal $\mathcal{A}_G^0$ generated $\{L_e : e \in E(G)\}$ is preserved. Theorem **??** is straightforward in this case. See also Solel [27] and Katsoulis and Kribs [13]. Other topics that can be found in these papers, and others, are the determination of unitary automorphisms, the structure of partial isometries, the reflexivity and hyper-reflexivity of the algebras $\mathcal{L}_G$, and determination of the Jacobson radical and semisimplicity.

The Hilbert space $\mathcal{H}_n$ is readily identifiable with the Fock space

$$\mathcal{H}_n = \mathbb{C} \oplus \sum_{k\in\mathbb{Z}_+} \oplus(\mathbb{C}^n)^{\otimes k}$$

formed by the direct sum of multiple tensor products of $\mathbb{C}^n$. With this formulation the operators L_e are conveniently specified by the shift property

$$L_e(\xi_1 \otimes \cdots \otimes \xi_k) = \xi_e \otimes \xi_1 \otimes \cdots \otimes \xi_k$$

where $\xi_1 \otimes \cdots \otimes \xi_k$ is an elementary tensor in the k-fold tensor product summand. In general the generating operators L_e are partial isometries acting on a natural generalized Fock space Hilbert space, in which not all tensors are admissible. Although we have not needed the tensor formalism it is a fundamental construction allowing for further generalisations, most notably the tensor algebras of correspondences. See, for example Muhly and Solel [17].

The following result from [15] gives a graph theoretic condition corresponding, roughly speaking, to the separation of the algebras $\mathcal{L}_G$ into two classes, those which are "matrix function like" and those that are "free semigroup like". The following notion parallels somewhat the requirement that a C*-algebra contain $\mathcal{O}_2$, or that a discrete group contain a free group.

Definition 3.6. A WOT-closed algebra $\mathfrak{A}$ is *partly free* if there is an inclusion map $\mathfrak{L}_2 \hookrightarrow \mathfrak{A}$ which is the restriction of an injection between the generated von Neumann algebras. If the map can be chosen to be unital, then $\mathfrak{A}$ is said to be *unitally partly free.*

A directed graph G is said to have the *double-cycle property* if there are distinct minimal cycles $w = xwx$, $w' = xw'x$ over some vertex x in G.

Theorem 3.7. *The following assertions are equivalent for a countable directed graph G with a finite number of vertices.*
(i) G has the double-cycle property.
(ii) $\mathfrak{L}_G$ is partly free.

4. Subalgebras of $\mathcal{O}_n$

Returning to the themes of Section 2, we are now ready to look inside $\mathcal{O}_n$. Our context is that of a norm closed subalgebra $\mathcal{A} \subseteq \mathcal{O}_n$ which contains the canonical diagonal subalgebra $\mathcal{C}$ associated with the given generators of $\mathcal{O}_n$. Let us first note that there are a number of ways such algebras arise.

(i) Generator constraints: (a) If S is a semigroup of operators of the form $S_\mu S_\nu^*$ which contains all the projections $S_\mu S_\mu^*$ then the norm closed linear span is a subalgebra of $\mathcal{O}_n$ which contains the canonical diagonal subalgebra. Note that this algebra is left invariant by the gauge automorphisms of $\mathcal{O}_n$.
(b) Let $\mathcal{A}_1$ be the norm closed algebra generated by the diagonal algebra $\mathcal{C}$ and the single operator S_1. Let $\mathcal{A}_2$ be the (nonunital) subalgebra which is the ideal in $\mathcal{A}_1$ generated by $1 - S_1$. The abelain algebra $\mathcal{A}_1/com(\mathcal{A}_1)$ can be naturally identified with the disc algebra $A(\mathbb{D})$, while $\mathcal{A}_2/com(\mathcal{A}_2)$ identifies with the ideal of functions $h(z)$ with $h(0) = h(1)$. The gauge automorphisms of $\mathcal{O}_n$ rotate the ideals of $A(\mathbb{D})$ and so the algebra $\mathcal{A}_2$ is not gauge invariant.

(ii) Fourier series constraints : Let $A \subseteq \mathcal{F}^n$ be a triangular subalgebra with $A \cap A^* = \mathcal{C}$. Then

$$\mathcal{A} = \{a \in \mathcal{O}_n : E_0(a) \in A, E_k(a) = 0, k < 0\}$$

is a triangular subalgebra of $\mathcal{O}_n$. Once again, $\mathcal{A}$ is gauge invariant.

(iii) Extrinsic constraints: Let $\mathcal{N} \subseteq \mathcal{C}$ be a totally ordered family of projections. For example, $\mathcal{N}$ could consist of the projections corresponding to the intervals $[0, k/2^n]$ in the interval picture of $\mathcal{O}_n$. To such a nest of projections one can assign the nest subalgebra

$$\mathcal{A} = \mathcal{O}_n \cap Alg\mathcal{N} = \{a \in \mathcal{O}_n : (1-p)ap = 0, \ \textit{for all } p \in \mathcal{N}\}.$$

Once again, masa normalising partial isometries give a key tool for recovering the underlying "coordinates" from the algebra structure. Each partial isometry $S_\mu S_\omega^*$ is $\mathcal{C}$-normalising, as are finite sums of these when they have orthogonal ranges and orthogonal domains. Also we may multiply these sums by unitary elements of $\mathcal{C}$ to obtain further examples. These turn out to be all the normalising partial isometries and they may be characterised in intrinsic terms as in the following Theorem. In terms of the interval picture, or the Cantor interval picture one can, once again, indicate pictorially the support of such a partial isometry, as shown.

FIGURE TO DO

Theorem 4.1. *Let v be a contraction in $\mathcal{O}_n$. Then the following assertions are equivalent:*
(i) v is a $\mathcal{C}$-normalizing partial isometry.
(ii) v is an orthogonal sum of a finite number of partial isometries of the form $dS_\mu S_\nu^*$, where

$d \in \mathcal{C}$.

(iii) For all projections $p, q \in \mathcal{C}$, the norm $\|qvp\|$ is equal to 0 or 1.

Proof. The implications from (ii) to (i) to (iii) are routine and left to the reader. Let v be a contraction with the zero one norm condition. We claim first that $E_0(v)$ is a $\mathcal{C}$-normalizing partial isometry in $\mathcal{F}^n$. The intuitive reason for this follows on contemplating the (Cantor space) support $supp(v)$ of v which is defined as the set of points (y, x) in $X \times X$ such that for some $\delta > 0$ $\|qvp\| = 1$ for all projections p, q in $\mathcal{C}$ with $y(q) = x(p) = 1$. Considering continuous matrices we can see that the support $supp(w)$ of a finite sum w of products of the generators and their adjoints will be the union of $supp(E_0(w))$ and $supp(w - E_0(w))$, the former set consisting of the diagonal segments parallel to the main diagonal. Because of the essential disjointness of these sets it follows from (iii) that $E_0(w)$ satisfies the zero-one condition and so by Theorem 2.10 is a normalising partial isometry.

For a rigourous proof we can argue as follows. If $E_0(v)$ were not a normalising partial isometry then by Theorem 2.10 we would be able to find $\delta > 0$ and projections p, q in $\mathcal{C}$ so that

$$\delta \leq \|qE_0(v)p\| \leq 1 - \delta.$$

Moreover, for each N it is possible to choose the projections in such a way so that $qwp = 0$ for any standard partial isometry w of the form $S_\mu S_\nu^*$ with $|\mu| \neq |\nu|$ and $0 \leq |\mu|, |\nu| < N$. Choose v' in the star algebra generated by the generators with $\|v - v'\| < \delta/3$. Since $v' - E_0(v')$ is a linear combination of $S_\mu S_\nu^*$ with $|\mu| \neq |\nu|$ we may choose p, q as above so that $pvq - pv'q = pvq - pE_0(v')q$. It follows that $\|pvq\|$ is not zero or one, a contradiction.

Now suppose that $m > 0$. If $|\nu| = m$ and $|\lambda| - |\mu| = m$, then the product $S_\nu^* S_\lambda S_\mu^*$ is either zero or of the form $S_{\lambda_1} S_\mu^*$ with $|\lambda_1| = |\mu|$. It follows that if $\Phi_m(v)$ is the m^{th} term in the series expansion of v ($\Phi_m(v) = a_m S_1^m$ for $m \geq 0$), then $S_\nu^* \Phi_m(v) = E_0(S_\nu^* v)$. Since v satisfies the zero one condition, so does $S_\nu^* v$ and the argument above shows that $S_\nu^* \Phi_m(v)$ is $\mathcal{C}$-normalizing and so has the desired form. This, in turn, implies that $S_\nu S_\nu^* \Phi_m(v)$ is $\mathcal{C}$-normalizing and has the required form for any word ν of length m. Consequently, $\Phi_m(v)$ has the desired form. In a similar fashion, we can show that when $m < 0$, $\Phi_m(v)$ also has the desired form (consider adjoints, for example).

Finally, if w is a partial isometry and $ww^*xw^*w \neq 0$ then $\|w + ww^*xw^*w\| > 1$. From this observation and the Cesáro convergence of generalised Fourier series, it follows that the operators $\Phi_m(v)$ are non-zero for only finitely many values of m and that v is the orthogonal sum of these operators. Thus v itself has the desired form. □

Recall the Cantor interval picture and the partial homeomorphisms $\alpha_{\mu\nu}$. The dilation factor of $\alpha_{\mu\nu}$ we define to be $k = |\nu| - |\mu|$. Previously we focused on the case $k = 0$ appropriate to $\mathcal{F}^n$. We now define the counterpart to $R(\mathcal{F}^n)$.

The *Cuntz groupoid* $R(\mathcal{O}_n)$ is, intuitively speaking, the support of the algebra in the Cantorised interval picture, with record taken of the dilation factors. More formally it is the set

$$R(\mathcal{O}_n) = \{(x, k, y) : x = \alpha_{\mu\nu}(y) \text{ for some } \alpha_{\mu\nu}\},$$

together with

(i) the totally disconnected topology with (as before) the set of graphs $E_{\mu\nu}$, for the partial homeomorphisms $\alpha_{\mu\nu}$, as a base,

(ii) the natural partially defined multiplication coming from composition of appropriate partial homeomorphisms.

It is natural now to seek to obtain for subalgebras of $\mathcal{O}_n$ results analogous to those in Section 2. It turns out that there is a complication in that "synthesis", as expressed in Theorem 2.7, may fail. However, it is precisely the *gauge invariant* closed subalgebras containing $\mathcal{C}$ that are determined by their groupoid support:

Theorem 4.2. *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{O}_n$ containing the canonical diagonal masa $\mathcal{C}$. Then $\mathcal{A}$ is generated by the partial isometries $S_\mu S_\nu^*$ belonging to $\mathcal{A}$ if and only if $\mathcal{A}$ is invariant under the gauge automorphisms γ_z for $|z| = 1$.*

For a gauge invariant algebra as above we define an associated topological semigroupoid $R(\mathcal{A})$. This is the set

$$R(\mathcal{A}) = \{(x, k, y) : x = \alpha_{\mu\nu}(y) \text{ for } \mu, \nu \text{ such that } S_\mu S_\nu^* \in \mathcal{A}\}$$

with the relative topology and partially defined multiplication. With the characterisation of normalising partial isometries given above it is now possible to prove the following analogue of Theorem 2.10, and in a similar manner to the proof of that theorem. To paraphrase, gauge invariant triangular subalgebras of $\mathcal{O}_n$ remember their semigroupoids and are classified by them.

Theorem 4.3. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be norm-closed subalgebras of $\mathcal{O}_n$ with $\mathcal{A}_i \cap \mathcal{A}_i^* = \mathcal{C}$ for $i = 1, 2$. Then the following statements are equivalent*
(i) $\mathcal{A}_1$ and $\mathcal{A}_2$ are isometrically isomorphic operator algebras.
(ii) The semigroupoids $R(\mathcal{A}_1), R(\mathcal{A}_2)$ are isomorphic, that is, there is a homeomorphism $\alpha : R(\mathcal{A}_1) \to R(\mathcal{A}_2)$ which respects the partially defined multiplication.

Proof. The direction (i) $\implies$ (ii) is similar to that of the proof of Theorem 2.10. The direction (ii) $\implies$ (ii) is more straightforward. One lifts the map α to a map on the semigroup of normalising partial isometries generated by the $S_\mu S\nu^*$ and this can be extended to an isometric algebra isomorphism.

□

Suppose now that G is a countable directed graph (V,E) with range and source maps $r,s:E\to V$. One can generalise the Cuntz relations as we indicated above in context E of the introduction. To each edge e there is a partial isometry S_e and to each vertex x a projection P_x. The initial projection of S_e is $P_{s(e)}$ while the range projection is dominated by $P_{r(e)}$. Moreover, under a given P_y the range projections sum to that projection:

$$\sum_{e:r(e)=y} S_eS_e^* = P_y$$

with weak operator topology convergence if the edge incidence is infinite. We see then that it is simply the graph that encodes partial isometry generators and relations. The graph C*-algebra $C*(G)$ is defined to be the associated universal C*-algebra. Much is known about the structure of this diverse class. See for example [25].

Once again, words in the generators and their adjoints have a reduced form $S_\alpha S_\beta^*$ where, in the graph case, $\alpha=\alpha_1\dots\alpha_n$ is a directed path in G (directed from left to right in our convention) and there are natural counterpart to methods and results in Section 2.1. Moreover, if G has no source vertices, in the sense that r is onto, then the abelian C*-algebra generated by the projections $S_\alpha S_\alpha^*$ is a masa. In this setting, with the simplifying assumption of finite incidence at every vertex (so-called row finiteness) one has exact counterparts to all the results of this section.

Notes. Theorems 4.1, 4.2 and 4.3 are taken from Hopenwasser, Peters and Power [11] where one can also find the more general variants for graph C*-algebras. The methods related to the AF classification, Theorem 2.10, has assisted in the analysis of many particular families of subalgebras of AF C*-algebras and much is known of the structure of ideals and representations, for example. On the other hand subalgebras of graph C*-algebras have not received such attention but it may be timely to do so.

In these lectures we have been led from operator algebra considerations to specific topological groupoids and semigroupoids, for $\mathcal{F}^n$ and $\mathcal{O}_n$. It is an important and natural consideration to complete the circle and construct operator algebras associated with general abstract topological groupoids. For this see Renault [26], Paterson [18] and Raeburn [27]. For further perspectives on non-self-adjoint operator algebras see the recent article of Donsig and Pitts [8] who also comment on variants of the open problems in Section 2.

5. Appendix : Digraph Algebras and Limit Algebras.

We give a brief self contained account of digraph algebras $A(G)$ and some examples of direct limit algebras.

5.1. **Digraph Algebras.** A *digraph* is a directed graph $G = (V, E)$ with no multiple directed edges, so that $E \subseteq V \times V$ and each edge e in E can be written as (x, y) with initial vertex y and final vertex x. If $V = \{1, \ldots, n\}$ then the subspace $A(G) \subseteq M_n(\mathbb{C})$ is defined by

$$A(G) := \text{span}\{e_{i,j} : (i, j) \in E\},$$

where $\{e_{ij}\}$ is the *standard matrix unit system* for $M_n(\mathbb{C})$. Suppose further that E, when regarded as a binary relation, is reflexive : $(v, v) \in E \quad$ for all $v \in V$. Then $D_n \subseteq A(G)$, where $D_n =$ span $\{e_{ii}\}$. Note that $A(G)$ is a *complex algebra* if and only if G is transitive, that is, if (i, j) and (j, k) are edges then so is (i, k). We say that $A(G)$ is a *standard digraph algebra* in this case.

Examples. (i) If K_m is the complete digraph on $\{1, \ldots, m\}$ then $A(K_m) = M_m(\mathbb{C})$. (ii) Let D_{2m} be a 2m-sided polygon with alternating directions on the edges (and loops at each vertex). Then $A(D_{2m})$ is identifiable as a rather sparse algebra of matrices. (iii) Let T_n be the subalgebra of upper triangular matrices in $M_n(\mathbb{C})$. Then T_n is a digraph algebra. (iv) Given the digraph G, we can construct $G \times K_m$, the *relative product* by replacing each vertex of G by K_m, and replacing each proper edge of G by all the n^2 edges between the new vertices. The digraph algebra $A(G \times K_m)$ is identifiable with $A(G) \otimes M_m(\mathbb{C})$.

Definition 5.1. $A \subseteq M_n(\mathbb{C})$ is a *digraph algebra* if A is a complex algebra and A contains a *maximal abelian self-adjoint algebra* (masa) D.

Proposition 5.2. *If $D \subseteq M_n(\mathbb{C})$ is a masa then there is a unitary matrix $u \in M_n(\mathbb{C})$ such that $uDu^* = D_n$, the standard masa.*

Proof. D being a masa means that if D' properly contains D and D' is also a self-adjoint algebra, then D' is not abelian. Let $\{p_1, \ldots, p_t\} \subseteq D$ be a maximal set of pairwise orthogonal projections. Then rank $p_i = 1$, for all i, by maximality of D. For if not split the projection into a sum of two projections to obtain a larger abelian algebra. It follows, again by maximality, that $t = n$ and so there is a unitary u such that for all $i, up_iu^* = e_{ii}$ as required. □

Proposition 5.3. *If $D \subseteq A(G)$ is a masa then there exist a unitary $u \in A(G)$ such that $uDu^* = D_n$. Thus, maximal abelian self-adjoint subalgebras of digraph algebras are unique up to inner conjugacy (inner unitary equivalence).*

Proof. Use the previous proposition, combining unitaries in each block of the block diagonal self-adjoint subalgebra $A(G) \cap A(G)^*$. □

Since an algebra in $M_n(\mathbb{C})$ which contain the standard masa is a standard digraph algebra we have the following corollary.

Corollary 5.4. *Every digraph algebra $A \subseteq M_n(\mathbb{C})$ is inner conjugate to a standard digraph algebra. That is $uAu^* = A(G)$ for some digraph G and some unitary u in A.*

If standard digraph algebras are unitarily equivalent then by Proposition 5.2 we can assume that the unitary equivalence maps the standard masa to the standard masa and it follows readily that the graphs are isomorphic.

5.2. **Maps between digraph algebras.** To begin to understand algebras of the form $A = \bigcup_k A(G_k)$, where the building block algebras are nested, ie. $A(G_k) \subseteq A(G_{k+1})$, we must consider the nature and variety of the possible *inclusion maps* $A(G_k) \to A(G_{k+1})$. Let $(f_{ij})_{i,j=1}^m \subseteq M_n(\mathbb{C})$ be operators with the relations of an $m \times m$ *matrix unit system*, that is,

$$\left.\begin{array}{rl} f_{ij}f_{jk} = & f_{ik} \;\; \forall \;\; ijk \\ f_{ij}^* = & f_{ji} \\ f_{ij}f_{kl} = & 0 \;\text{ if }\; j \neq k \end{array}\right] (*)$$

Then the map $\phi : M_m \to M_n$ which is defined to be the *linear extenson* of the correspondences $e_{ij} \to f_{ij}$, is an injective C^*-algebra homomorphism. Conversely, it is straightforward to show that if ϕ is such a map, then $\{\phi(e_{ij})\}$ satisfy the relations (*).

Definition 5.5. Let $A(G_1) \subseteq M_m, A(G_2) \subseteq M_n$ be unital standard digraph algebras with connected graphs. Then an algebra homomorphism $\phi : A(G_1) \to A(G_2)$ is said to be *star-extendible* if ϕ is the restriction of a C*-algebra map between the finite-dimensional C*-algebras $C^*(A(G_1))$ and $C^*(G_2)$.

Example (i) The map $\phi : T_2 \to T_4$ given by

$$\phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b/\sqrt{2} & -b/\sqrt{2} \\ 0 & a & b/\sqrt{2} & b/\sqrt{2} \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$$

is a star-extendible algebra injection.

Example (ii) The map $\phi : T_2 \to T_4$ given by

$$\phi : \begin{pmatrix} a & b \\ & c \end{pmatrix} \to \begin{pmatrix} a & b & 0 & 0 \\ & c & 0 & 0 \\ & & a & 0 \\ & & & 0 \end{pmatrix}$$

is not star-extendible (and is an isometric algebra injection).

Example (iii) $\phi : T_2 \to T_4$ given by

$$\phi : \begin{pmatrix} a & b \\ & c \end{pmatrix} \to \begin{bmatrix} a & & & b \\ & a & b & \\ & & c & \\ & & & c \end{bmatrix}$$

is a star-extendible algebra injection.

Remark 1 Star-extendible injective maps between digraph algebras are necessarily *isometric*: since they are restrictions of C*-algebra maps. This in turn is an elementary fact from spectral theory. Indeed, note first that if $p \in M_m$ is a nonzero self-adjoint projection then $||p|| = 1$, by the definition of the operator norm. If ϕ is a star homomorphism then $\phi(p)$ is a projection, so $\|\phi(p)\| = 1 = \|p\|$. We claim that $\|\phi(a)\| = \|a\|$ if a is a self-adjoint operator in M_m. By the spectral theorem $a = \lambda_1 p_1 + \cdots + \lambda_m p_m$, and we can suppose $\|a\| = |\lambda_1| \geq |\lambda_k|$ for $k = 2, \ldots, m$, where $p_1, \ldots, p_m$ are pairwise orthogonal projections. Let $q_i = \phi(p_i)\ \ 1 \leq i \leq m$. Then $\phi(a) = \lambda_1 q_1 + \cdots + \lambda_m q_m$, with $q_1 \ldots q_m$ pairwise orthogonal projections. Thus (exercise) $\|\phi(a)\| = |\lambda_1| = \|a\|$. Finally if $b \in M_m$ is a general element, then

$$\|\phi(b)\|^2 = \|\phi(b)^*\phi(b)\| = \|\phi(b^*)\phi(b)\| = \|\phi(b^*b)\| = \|b^*b\| = \|b\|^2.$$

Definition 5.6. Let $A(G_1), A(G_2)$ be digraph algebras with standard matrix unit systems $\{e^k_{ij} : (ij) \in E(G_k)\}$, $k = 1, 2$, as usual.

(i) An algebra injection $\phi : A(G_1) \to A(G_2)$ is a *standard regular injection*, with respect to $\{e^1_{ij}\}$, *and* $\{e^2_{ij}\}$, if ϕ is star-extendible and maps each e^1_{ij} to a sum of matrix units in $\{e^2_{kl}\}$

(ii) An algebra injection $\psi : A(G_1) \to A(G_2)$ is a *regular (star-extendible) injection* if there exists a unitary operator u in $A(G_2)$ such that $\psi(a) = u\phi(a)u^* \quad \forall \ \ a \in A(G_1)$, where ϕ is a standard regular injection. We say that ψ is *inner equivalent* (or, less precisely, unitarily equivalent) to ϕ when such a relationship holds.

Remark. The unitary u belongs to $A_2 \cap A_2^*$. Indeed, by the spectral theorem, $u = \lambda_1 p_1 + \cdots + \lambda_r p_r$ where p_i is the spectral projection for the eigenspace for λ_i and each p_i lies in the self-adjoint subalgebra.

Exercises (i) Prove that there are uncountably many inner conjugacy classes of embedding $\phi : T_2 \to T_4$ which are star-extendible and unital.
(ii) Prove that there are only finitely many inner conjugacy classes of regular embeddings between two digraph algebras.

5.3. **Direct Limits.** We leave it to the reader to recall the definition of the direct limit algebra of a direct system. Here are some standard direct systems of triangular matrix algebras.

Standard limits. Let $n_2 = rn_1$ and let $\sigma : M_{n_1} \to M_{n_2}$ be the inclusion map such that

$$\sigma(e^{(1)}_{i,j}) = e^{(2)}_{i,j} + e^{(2)}_{i+n_1,j+n_1} + \cdots + e^{(2)}_{i+(r-1)n_1,j+(r-1)n_1}$$

or equivalently, identifying M_{n_2} with $M_r \otimes M_{n_1}$, $\sigma(a) = I \otimes a$, so that, in block matrix terms,

$$\sigma(a) = \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & & \\ & & & \\ 0 & & & a \end{bmatrix}.$$

Then $\sigma(T_{n_1}) \subseteq T_{n_2}$, and so, repeating, we may construct the regular matricial algebra $A_\sigma = \lim(T_{n_k}, \sigma)$, when $n_k | n_{k+1}$ for all k.

Refinement Limits. Let $n_2 = rn_1$ and let $\rho : M_{n_1} \to M_{n_2}$ be the inclusion map such that

$$\rho(e^{(1)}_{i,j}) = e^{(2)}_{(i-1)n_1+1,(j-1)n_1+1} + e^{(2)}_{(i-1)n_1+2,(j-1)n_1+2} + \cdots + e^{(2)}_{(i-1)n_1+r,(j-1)n_1+r}$$

or equivalently, identifying M_{n_k} with $M_{n_1} \otimes M_r$, $\rho(a) = a \otimes I$, so that, $\rho((a_{ij}))$ is the inflated matrix $(a_{ij} I_r)$. Again, $\rho(T_{n_1}) \subseteq T_{n_2}$, and for a sequence with $n_k | n_{k+1}$ for all k we can define the regular matricial algebra $A_\rho = \lim(T_{n_k}, \rho)$. This algebra is not isometrically isomorphic to the A_σ algebra for the same sequence (n_k) despite the fact that their generated C*-algebras are isomorphic.

Countable total order limits. The standard embeddings σ and the refinement embeddings ρ can also be alternated in which case one obtains a more general class of algebras, the so called alternation algebras. The three classes described correspond to a Cantor space product coordinate indexing by $\mathbb{Z}_-, \mathbb{Z}_+$, and $\mathbb{Z}$ respectively. One can generalise this further to obtain strange triangular algebras whose background Cantor product is indexed over an arbitrary countable order. Moreover this countable order is an algebra isomorphism invariant. For further detail see [23].

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