Observations on symmetry-breaking flexes of a kagome lattice

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Outline

A ‘perturbed’ kagome grid

Kagome lattice

Background to kinematic bifurcations

Computational method

Results for $a = 1, \ b = 1$

Results for $a = 1, \ b = 2$

Results for $a = 1, \ b = 3$

Results for $a = 2, \ b = 2$
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Results for $a = 1, b = 1$

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Results for $a = 1, b = 3$

Results for $a = 2, b = 2$
Primitive unit cell of a kagome lattice
Primitive unit cell of a ’perturbed’ kagome lattice
Perturbed kagome lattice

We will consider a perturbed kagome lattice with repetitive boundary conditions:
Perturbed kagome lattice

We will consider a perturbed kagome lattice with repetitive boundary conditions:
Deformed perturbed kagome lattice

If we follow the single flex of the perturbed lattice we pass the configuration shown:
Deformed perturbed kagome lattice

If we follow the single flex of the perturbed lattice we pass the configuration shown:
Using a supercell

We can relax the lattice constraint by using a supercell made of $a$ columns and $b$ rows of the primitive cell:

- $a = 3$
- $b = 2$

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Perturbed kagome lattice with $a = 3$, $b = 2$
Perturbed kagome lattice with $a = 3$, $b = 2$

This system still only has a single flex
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Kagome lattice

We will consider a kagome lattice with repetitive boundary conditions with various size of repetitive super-cell:
Kagome lattice

We will consider a kagome lattice with repetitive boundary conditions with various size of repetitive super-cell:
Number of independent infinitesimal mechanisms for a kagome lattice

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2 3 4</td>
</tr>
<tr>
<td>2</td>
<td>2 4 4 6</td>
</tr>
<tr>
<td>3</td>
<td>3 4 7 6</td>
</tr>
<tr>
<td>4</td>
<td>4 6 6 10</td>
</tr>
</tbody>
</table>
The Key Question

The infinitesimal mechanisms are known.

Which of them ‘extend’ to a finite flex?
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From Kumar and Pellegrino (2000)

\[
\begin{bmatrix}
0 & 10 \\
10 & 0
\end{bmatrix}
\]

whose rank is \( r = 3 \). In the configuration of Fig. 1(c), \( RC \), the equilibrium matrix is

\[
\begin{bmatrix}
2 & 664 \\
3 & 775
\end{bmatrix}
\]

whose rank is \( r = 2 \).

The number of independent inextensional mechanisms of a pin-jointed structure is given by

\[
m = j - r
\]

where \( j \) is the number of unconstrained joints of the structure. The number of independent states of self-stress, i.e. sets of bar forces that are in equilibrium with zero external loads, is

\[
s = b - r
\]

where \( b \) is the number of bars. According to Eqs. (3) and (4), the structure shown in Fig. 1 has \( m = 1 \) and \( s = 0 \) in the initial configuration, but \( m = 2 \) and \( s = 1 \) at the bifurcation point, thus showing that the bifurcation points are also special configurations that admit a state of self-stress.

Returning to the configuration shown in Fig. 1(d), let us rotate bars 2 and 3 until they overlap with bar 1. This configuration, \( RB \), is also a bifurcation point where, again, two different motions are possible, \( m = 2 \).

We can either rotate bars 1 and 2 about joint 2, which now coincides with the left-hand support, or rotate bars 2 and 3 about joint 1, which coincides with the right-hand support. Once one of these two options has been chosen, the structure starts following another single-parameter path.

A topological map of all existing kinematic paths for this structure is shown in Fig. 2. This figure shows that there are three bifurcation points (\( RB \) and \( RC \) have been discussed above, while \( RA \) corresponds to a configuration symmetric to \( RC \)) linked by three kinematic paths. The upper and lower parts of each path are labelled \( Pi \) and \( Pi_0 \). At a bifurcation point, the structure can either continue moving on a path with the same number, or it can switch to a path with a different number.
The kind of behaviour that is illustrated by the above example occurs in a variety of structures that admit one or more finite-amplitude motions, but in most cases obtaining a complete map like that shown in Fig. 2 is by no means straightforward. A three-dimensional example is the ring structure shown in Fig. 3. It is well-known (e.g. Tarnai, 1980) that in the configuration shown in Fig. 3(a), this structure has a finite mechanism where the joints of the upper hexagon move alternately inwards and outwards so that, although...
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Results for $a = 1$, $b = 3$

Results for $a = 2$, $b = 2$
Algorithm

1. Find initial augmented compatibility (rigidity) matrix $\tilde{C} (\tilde{p})$ at symmetric configuration $\tilde{p} = \begin{bmatrix} p & t \end{bmatrix}$.

2. Write $m$ independant infinitesimal mechanisms as columns of $M$. Displace configuration by $d = M \delta$: $\tilde{q} = \tilde{p} + d$. $\delta$ is a (small) random vector.

3. Find current error in length of members $e_c$.

$$d_c = -\tilde{C}^+ e_c \quad ; \quad \tilde{q} \leftarrow \tilde{q} + d_c$$

Repeat as necessary

4. Find the converged path

$$m_f = \tilde{q} - \tilde{p}$$

Is this a new path? What is the rank of $\tilde{C} (\tilde{q})$?
Algorithm

1. Find initial augmented compatibility (rigidity) matrix $\tilde{\mathbf{C}}(\tilde{\mathbf{p}})$ at symmetric configuration $\tilde{\mathbf{p}} = \begin{bmatrix} \mathbf{p} \\ \mathbf{t} \end{bmatrix}$.

2. Write $m$ independant infinitesimal mechanisms as columns of $\mathbf{M}$. Displace configuration by $\mathbf{d} = \mathbf{M}\delta$: $\tilde{\mathbf{q}} = \tilde{\mathbf{p}} + \mathbf{d}$. $\delta$ is a (small) random vector.

3. Find current error in length of members $\mathbf{e}_c$.

$$\mathbf{d}_c = -\tilde{\mathbf{C}}^+\mathbf{e}_c; \quad \tilde{\mathbf{q}} \leftarrow \tilde{\mathbf{q}} + \mathbf{d}_c$$

Repeat as necessary

4. Find the converged path

$$\mathbf{m}_f = \tilde{\mathbf{q}} - \tilde{\mathbf{p}}$$

Is this a new path? What is the rank of $\tilde{\mathbf{C}}(\tilde{\mathbf{q}})$
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1. Find initial augmented compatibility (rigidity) matrix $\tilde{C}(\tilde{p})$ at symmetric configuration $\tilde{p} = \begin{bmatrix} p \\ t \end{bmatrix}$.

2. Write $m$ independant infinitesimal mechanisms as columns of $M$. Displace configuration by $d = M\delta$: $\tilde{q} = \tilde{p} + d$. $\delta$ is a (small) random vector.

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$$d_c = -\tilde{C}^+ e_c; \quad \tilde{q} \leftarrow \tilde{q} + d_c$$

Repeat as necessary

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$$m_f = \tilde{q} - \tilde{p}$$

Is this a new path? What is the rank of $\tilde{C}(\tilde{q})$
Algorithm

1. Find initial augmented compatibility (rigidity) matrix $\tilde{\mathbf{C}}(\tilde{\mathbf{p}})$ at symmetric configuration $\tilde{\mathbf{p}} = \begin{bmatrix} \mathbf{p} \\ \mathbf{t} \end{bmatrix}$.

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3. Find current error in length of members $e_c$.

   $$\mathbf{d}_c = -\tilde{\mathbf{C}}^+ e_c ; \quad \tilde{\mathbf{q}} \leftarrow \tilde{\mathbf{q}} + \mathbf{d}_c$$

   Repeat as necessary

4. Find the converged path

   $$\mathbf{m}_f = \tilde{\mathbf{q}} - \tilde{\mathbf{p}}$$

   Is this a new path? What is the rank of $\tilde{\mathbf{C}}(\tilde{\mathbf{q}})$
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Single path, with $m = 1$
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Single path, with $m = 1$ (identical to $a = 1$, $b = 1$ mechanism)
Single path, with $m = 1$
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Single path, with $m = 1$ (identical to $a = 1$, $b = 1$ mechanism)
Single path, with $m = 1$

There is also another identical, but offset, mechanism.
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Results for \( a = 2, \ b = 2 \)
Single path, with $m = 1$ (identical to $a = 1$, $b = 1$ mechanism)
Path (identical to $a = 1, b = 2$ mechanism); but now $m = 3$
Hyperplane of configurations with $m = 3$ — one example configuration
Hyperplane of configurations with $m = 3$ — another example configuration