Operator Algebras with Unitary Commutation Relations

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Semigroup Toeplitz algebras from the left regular representation:

S : a semigroup with cancelation

$\mathcal{H}_S$: Hilbert space with o.n.b $\{\xi_s : s \in S\}$.

$\mathcal{A}_S$ and $\mathcal{L}_S$ are the norm-closed and weakly (WOT) closed operator algebras generated by the left shifts:

$$L_e \xi_f = \xi_{ef}.$$

For $S = \mathbb{F}_n^+$ we have $\mathcal{A}_n, \mathcal{L}_n$, the noncommutative disc algebra and the Free semigroup algebra.

Some topics:

Invariant subspaces: reflexivity, hyper-reflexivity, Beurling theorem

Algebraic structure: ideals, semisimplicity

Isomorphisms: automorphisms, group actions, classification

Representation theory: dilation theory, models for tuples, enveloping C*-algebras.

These topics are fairly developed for $\mathcal{A}_n, \mathcal{L}_n$, partially developed for graph variants $\mathcal{A}_G, \mathcal{L}_G$

eg : Arias, Davidson, Jaeck, Katsoulis, Kribs, Muhly, Pitts, Popescu, Power, Shpigel, Solel, Voiculescu, ...

Beginnings for higher rank:

eg : Davidson, Kribs, Power, Solel, Yang...

Unitary commutation relations:

$$e_i f_j = \sum_{k,l} u_{(i,j),(k,l)} f_l e_k$$

where $u = (u_{(i,j),(k,l)})$ is a unitary in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$.

Permutation unitary case: θ is a permutation of nm pairs (i, j).

$$e_i f_j = f_{j'} e_{i'}, \quad (i', j') = \theta(i, j)$$

θ gives a discrete semigroup $\mathbb{F}_\theta^+$ of words in the e_i and the f_j. Each word is reducible, uniquely, to a product $w_1(e)w_2(f)$ of two free words.

This is the path semigroup of a single vertex rank 2 graph (Kumjian Pask).

The unitary relation algebra $\mathcal{A}_u$

$E = \mathbb{C}^n, F = \mathbb{C}^m$

$\mathcal{H}_u$ is $\mathbb{Z}_+^2$-graded Fock space

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \oplus (E^{\otimes k} \otimes F^{\otimes l}).$$

u gives an identification of $E \otimes F$ and $F \otimes E$ via

$$e_i \otimes f_j := \sum_{i'=1}^{m} \sum_{j'=1}^{n} u_{(i,j),(i',j')} f_{j'} \otimes e_{i'}$$

This extends similarly any tensor of E's and F's; each is equal to some $E^{\otimes k} \otimes F^{\otimes l}$. All these form a product system X_u. (Fowler..)

Left creation operators L_{e_i}, L_{f_j} are thus well defined. The norm closed algebra is $\mathcal{A}_u$. (The C*-envelope is the generalised Cuntz algebra $\mathcal{O}_{X_u}$. Davidson-P-Yang.)

The Gelfand (character) space.

V_u is the variety comprised of (z, w) in $\mathbb{C}^n \times \mathbb{C}^m$ with

$$z_i w_j = \sum_{k,l} u_{(i,j),(k,l)} z_k w_l.$$

Let

$$\Omega_u = V_u \cap (\overline{\mathbb{B}}_n \times \overline{\mathbb{B}}_m)$$

where $\mathbb{B}_n$ is the open unit ball of $\mathbb{C}^n$.

PROP^n (i) The linear multiplicative functionals on $\mathbb{C}[\mathbb{F}_u^+]$ are in one-to-one correspondence with points (z, w) in V_u.

(ii) $\mathcal{M}(\mathcal{A}_u)$ is homeomorphic to Ω_u.

Voiculescu automorphisms of $\mathcal{A}_n$ (and $\mathcal{O}_n, \mathcal{E}_n$)

$\alpha \in \mathbb{B}_n$, $x_0 = (1 - \|\alpha\|^2)^{-1/2}$, $X_1 \in GL(n, \mathbb{C})$.

Let $X = \begin{pmatrix} x_0 & \eta^* \\ \eta & X_1 \end{pmatrix}$ satisfy $X^* J X = J$.

Then $\theta_X(\lambda) := \frac{X_1\lambda + \eta}{x_0 + \langle \lambda, \eta \rangle}$, $\lambda \in \mathbb{B}_n$, yields an automorphism of $\mathbb{B}_n$.

THEOREM:

$$\Theta_X(L_\zeta) := (x_0 I + L_\eta)^{-1}(L_{X_1\zeta} + \langle \zeta, \bar{\eta} \rangle I)$$

extends to an automorphism of $\mathcal{A}_n$

Also, there is a unitary U_X with $\Theta_X(a) = U_X a U_X^*$,

$$U_X a \xi_0 = \Theta_X(a)(x_0 I + L_\eta)^{-1}\xi_0$$

KEY LEMMA. Let (z, w) lie in the **core** of $M(\mathcal{A}_u)$. Let $\alpha := \bar{z}$ and let Θ be the Voiculescu automorphism for α. Then unitary commutation relations hold:

$$\Theta(L_{e_i})L_{f_j} = \sum_{k,l} u_{(i,j),(k,l)} L_{f_l} \Theta(L_{e_k}).$$

THM. [P. and Solel, ArXiv, 2007] The following statements are equivalent for unitaries u, v in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$.

(i) There is an isometric isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$.

(ii) There is a graded isometric isomorphism from $\Psi : \mathcal{A}_u \to \mathcal{A}_v$.

(iii) The matrices u, v are **product unitary equivalent** or (in case $n = m$) the matrices $u, \tilde{v}$ are product unitary equivalent.

$n = m = 2$.

Product unitary equivalence classes:

There are 9 classes for the 24 permutation unitaries. [P : math.OA/0604630]

For general u the product equivalence class is:

$$\{(A \otimes B)u(A \otimes B)^* : A, B \in SU_2(\mathbb{C})\},$$

So the classes are parametrised by

$$U_4(\mathbb{C})/Ad(SU_2(\mathbb{C}) \times SU_2(\mathbb{C})).$$

This implies a 10-fold real parametrisation.

($U_4(\mathbb{C})$ and $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$) are real algebraic varieties of dimension 16 and 6.

The algebras $\mathcal{A}_u$ admit a 10-fold real parametrisation (with coincidences only for the classes of u, v with $u = \tilde{v}$)

Further refinement for $n = m = 2$ and u with $dim(Ker(u - I)) = 3$:

THEOREM There are two real parameters associated with u,

λ ($|\lambda| = 1$, $\lambda \neq 1$) and a $(0 \leq a \leq 1/\sqrt{2})$.

such that u and v are product unitary equivalent if and only if they have the same a, λ.

Class representatives are $u(a, \lambda)$ where, with $\mu = \lambda - 1$,

$$\begin{bmatrix} \mu a^2 + 1 & 0 & 0 & \mu a(1 - a^2)^{1/2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mu a(1 - a^2)^{1/2} & 0 & 0 & \lambda + \mu a^2 \end{bmatrix}.$$

DEFN An isomorphism $\Psi : \mathcal{A}_u \to \mathcal{A}_v$ is **bigraded** if there are unitary matrices A $(n \times n)$ and B $(m \times m)$ such that

$$\Psi(L_{e_i}) = \sum_j a_{i,j} L_{e_j} \ , \quad \Psi(L_{f_k}) = \sum_l b_{k,l} L_{f_l} .$$

If $m = n$ and Ψ is graded, with

$$\Psi(L_{e_i}) = \sum_j a_{i,j} L_{f_j} \ , \quad \Psi(L_{f_k}) = \sum_l b_{k,l} L_{e_l}$$

for $n \times m$ unitaries A and B then Ψ is a **graded exchange isomorphism**.

LEMMA If $\Psi_{A,B}$ is bigraded then $(A \otimes B)v = u(A \otimes B)$

If $m = n$ and $\tilde{\Psi}_{A,B}$ is a graded exchange isomorphism then $(A \otimes B)\tilde{v} = u(A \otimes B)$ where $\tilde{v}_{(i,j),(k,l)} = \bar{v}_{(l,k),(j,i)}$.

proof ideas for classification theorem:

Suppose $\Phi : \mathcal{A}_u \to \mathcal{A}_v$ is an isom. isom.

(i) there is an induced homeomorphism of Gelfand space: $\gamma : \Omega_\theta \to \Omega_\tau$,

(ii) with key lemma and Voiculescu automorphisms, reduce to $\gamma(0,0) = (0,0)$,

(iii) a generalised Schwarz inequality implies γ is a unitary rotation; $A \oplus B$, respecting product structure $\mathbb{C}^n \times \mathbb{C}^m$.

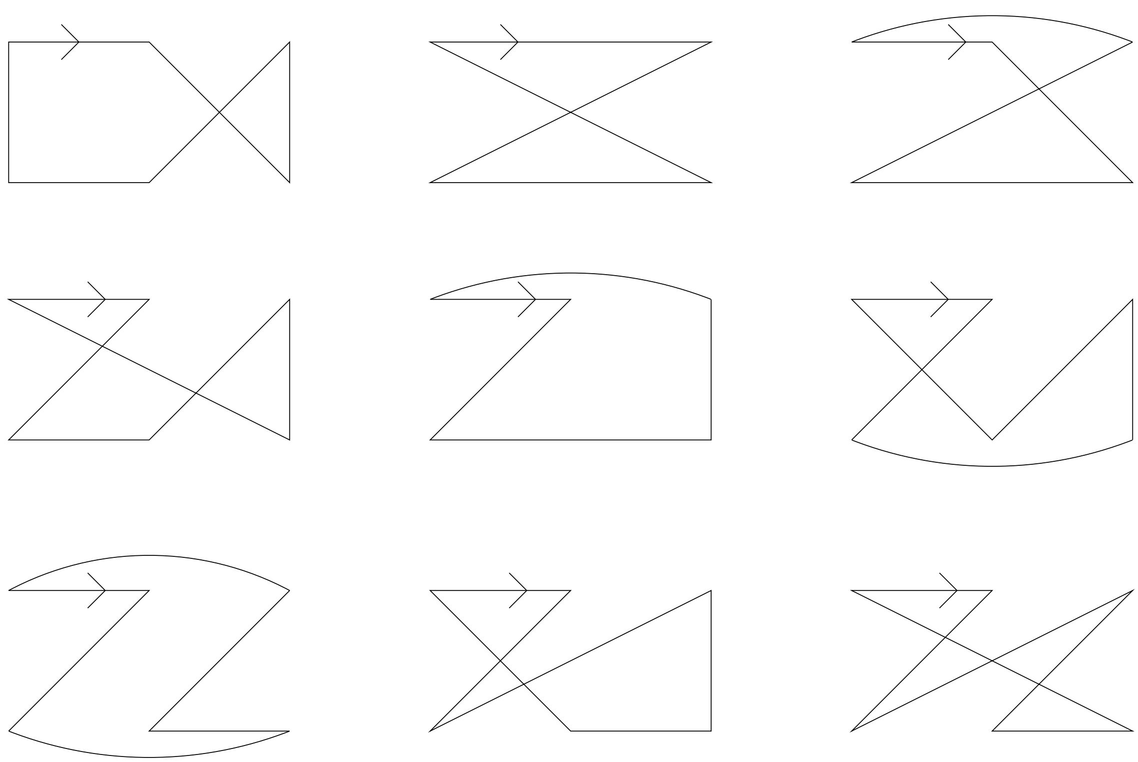
(iv) deduce Φ is bi-graded and unitary and determined by (quantisation of) $A \oplus B$.

(vi) lemma gives product unitary equivalence.

To go further, for permutations, we ask:

Is product unitary equivalence the same as *product conjugacy*, ie, conjugacy relative to $S_n \times S_m$?

Cycle diagrams for the 14 $S_2 \times S_3$ product conjugacy classes of 6 cycles.



THM [P] For $(n, m) = (2, 3)$ there are precisely 14 pairwise nonisomorphic Toeplitz algebras $\mathfrak{L}_\theta$ with (the same) minimal Gelfand space.

COR [P] The analytic Toeplitz algebra $\mathcal{L}_\theta$ for the 2-graph determined by the permutation $\theta = (124653)$ is not isometrically isomorphic to its commutant.

How many semigroups/2-graphs ?

ie

How many $S_n \times S_m$ conjugacy classes of θ:

n,m	Number of 2-graphs
2,2	9
2,3	84
3,4	3,333,212

Lemma Let C be a 2×3 matrix of rank 1 with more than one non-zero entry and not all entries equal. Suppose that $\theta \in S_6$ is a cycle order 6 and $\theta^k[C]$ has rank 1 for $k = 1, \ldots, 5$. Then one of the following holds.

(i) θ is product conjugate to θ_1, in which case C can be arbitrary,

(ii) θ is product conjugate to one of the (up-down alternating) permutations θ_2, θ_3, in which case C either has a zero row or the rows of C each have 3 equal entries,

(iii) θ is product conjugate to the rectangular permutation θ_4, in which case C has exactly two non-zero entries in consecutive locations for the cycle θ.

(iv) θ is product conjugate to θ_9, in which case the two rows of C are equal.

EXTRAS

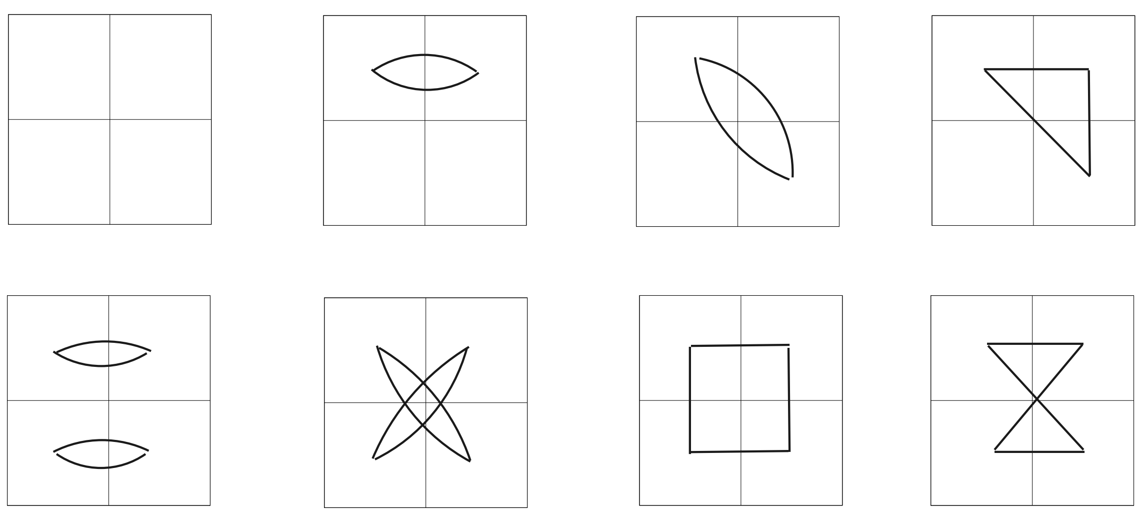


Figure 1: Cycle diagrams for $(n, m) = (2, 2)$

Two Semigroupoid algebras:

(i) Sparse matrix algebra:

$$\begin{bmatrix} \alpha & & & & \\ & \beta & & & \\ & & \gamma & & \\ \lambda & & & \beta & \\ \mu & & & & \gamma \end{bmatrix}$$

(ii) The cycle graph C_n gives a matrix function algebra :

$$\begin{bmatrix} H^\infty(z^n) & z^{n-1}H^\infty(z^n) & \dots & zH^\infty(z^n) \\ zH^\infty(z^n) & H^\infty(z^n) & & \vdots \\ \vdots & & \ddots & \\ z^{n-1}H^\infty(z^n) & \dots & & H^\infty(z^n) \end{bmatrix}.$$

THM [Kribs-P] The algebras $\mathcal{L}_G$ and $\mathcal{L}_{G'}$ are unitarily equivalent IFF $G \approx G'$.

THM [Kribs-P] $\mathcal{L}_G$ is reflexive.

THM [Jaeck-P] $\mathcal{L}_G$ is a hyper-reflexive if G is finite.

DEFN $\mathcal{A}$ is *partly free* if there is an inclusion map $\mathfrak{L}_2 \hookrightarrow \mathcal{A}$ which is the restriction of an injection between the generated von Neumann algebras.

THM The FAE:

(i) G has the aperiodic path property.

(i') G either has a "double cycle" or a proper infinite path

(ii) $\mathcal{L}_G$ is partly free.

THM : [Kribs-P] $\mathfrak{L}_\Lambda$ is semisimple IFF every edge in Λ lies on a cycle.

THM : [Kribs-P] (i) Let Λ be a higher rank graph with no multiplicity one radiating vertices. Then $\mathfrak{L}_\Lambda$ is reflexive.

(ii) If Λ has a single vertex then $\mathfrak{L}_\Lambda$ is reflexive.