INFINITE FRAMEWORKS IN TWO DIMENSIONS

Stephen Power

Department of Mathematics and Statistics
Lancaster University
Lancaster LA1 4YF
UK

PART I: Miscellany

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Figure 1. A small portion of a regular infinite grid framework in the plane, $\mathbb{C}_{\mathbb{Z}^2}$. The joints (vertices/nodes/atoms/... ) are located at integer points while the bars (edges/links/bonds/... ) are between nearest neighbour joints. It is flexible. *Flexible* means that without changing bar lengths or breaking the bonds one can continuously move ("continuously flex") the entire structure (in 2D) so that at least one angle between incident edges is changing.
Figure 2. The corner-joined squares framework, $C_{sq}$, is obtained from $C_{Z2}$ by adding cross-bars (braces) to half the square as shown. This "rigidifies" those squares. Interestingly the infinite edifice has an essentially unique nontrivial flex in which half the squares rotate with alternating orientation.
Braced grids

Figure 3. Just enough braces, or cross-bars, have been added to $\mathcal{C}_{\mathbb{Z}^2}$ so that there is no nontrivial flexing.

Figure 4. With one brace removed it can now flex in an essentially unique way.
The kagome framework is the infinite bar-joint framework which is associated with the semiregular tiling of the plane by equilateral triangles and regular hexagons. Each vertex has four incident edges (as does the grid framework $C_{\mathbb{Z}^2}$) and so the average incidence is 2. Since each joint has potentially 2 degrees of freedom such frameworks are viewed as being neither over-constrained nor under-constrained. The kagome framework is a favourite standard example and has been considered from many perspectives. Like the grid framework the kagome bar-joint framework can flex continuously in infinitely many ways (although this fact is less transparent).
This framework also consists of congruent triangular rigid units with degree 4 joints throughout. The arrows indicate a local infinitesimal flex in which only 4 vertices are assigned a nonzero velocity. The pattern appears as a floor tiling in Seville Cathedral around the tomb of Christopher Columbus. (See the Images section.)
Figure 7. Roman tiling framework, with a unit cell. (The tiling pattern of this framework goes back to Roman antiquity.)
Infinite winerack

Figure 8. This winerack (cross-section), extended to a translationally periodic framework, is a 2D body-pin framework. It flexes uniquely and "explosively".
Not flexible "because infinite"¹

Figure 9. The dotted lines indicate that the construction continues infinitely to the right, with the points \(p_1, p_3, \ldots\) forming a sequence tending to the intersection of the lines through \(p_1, p_3\) and through \(p_2, p_4\). The joint at \(p_0\) has infinite degree. The whole framework is inflexible, essentially because the "flexibility room" of the small trapezia dwindles to zero. However any finite subframework which has a joint \(p_{2n}\) for some \(n \neq 0\) is flexible. There is an (essentially unique) nontrivial infinitesimal flex. By definition, this is a sequence of velocity assignments to the vertices which, to first order, do not stretch any bar. This is achieved here by applying the same horizontal velocity to the "top" joints and zero velocity to the rest.

¹This framework is bounded in the sense that it lies within a bounded region of space. It is an infinite bar-joint framework however in the sense that it has infinitely many ingredients (bars and joints).
Figure 10. This whirlpool framework is infinite with edges and joints converging to a central point. It is not continuously flexible although every finite framework is flexible. Reason: Suppose we flex the outer square at constant speed though a small change in the angles. The inner squares flex with increasing speeds, tending to infinity, and so at some stage one of the small inner squares will jam. (There is, however, an (essentially unique) infinitesimal flex.)
This infinite bar-joint framework is a whirlpool in a special symmetric position. Once again there is no continuous flex. In fact, because of the particular symmetry even the finite framework with 12 edges supported by the outer 8 vertices has no continuous flex! (As some compensation to this the space of infinitesimal flexes is infinite dimensional because each of the square subframeworks supports a local infinitesimal flex.)
How to reach infinity

Figure 12. This is a \textit{bar-pinned} framework, with straight line bars joined at the end point joints as usual and which are "clasped" or pinned together at the red joints. These lie on the negative $x$-axis in a geometric sequence. The curves are not part of the structure. They are shaped like the square root function graphs ($\pm \sqrt{|x|}$) and so define the black joint positions. If the leftmost joints move towards each other at constant speed until they meet at the $x$-axis at time $T$ then this is extendible uniquely as a flex of the entire infinite framework. It flexes towards $+\infty$ and infinity is "reached" at time $T$. 
Figure 13. Here we play a similar game, joining pairs of pinned bars (tweezers?) together. Guided by the Cantor set construction we keep doubling the tweezer components downwards. (Bars are allowed to cross over each other.) This bar-pinned infinite framework occupies a bounded 2D region. The set of "end points" is the Cantor set. The infinite framework has a (essentially) unique flex and the Cantor set of endpoints will "evolve in time", stretching or contracting in some fashion.
TO BE CONTINUED SOON .....