RIGIDITY OF FRAMEWORKS SUPPORTED ON SURFACES

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Abstract. A theorem of Laman gives a combinatorial characterisation of the graphs that admit a realisation as a minimally rigid generic bar-joint framework in $\mathbb{R}^2$. A more general theory is developed for frameworks in $\mathbb{R}^3$ whose vertices are constrained to move on a two-dimensional smooth submanifold $\mathcal{M}$. Furthermore, when $\mathcal{M}$ is a union of concentric spheres, or a union of parallel planes or a union of concentric cylinders, necessary and sufficient combinatorial conditions are obtained for the minimal rigidity of generic frameworks.

1. Introduction

A bar-joint framework realisation of a simple finite connected graph $G = (V, E)$ is a pair (G, p) where $p = (p_1, \dots, p_n)$ is an assignment of the vertices $v_1, \dots, v_n$ in V to framework points in $\mathbb{R}^d$. In the case of frameworks in the plane there is a celebrated characterisation of those graphs G whose typical frameworks are both rigid and minimally rigid. By *rigid* we mean that any edge-length-preserving motion is necessarily a rigid motion. That is, a continuous edge-length-preserving path $p(t), t \in [0, 1]$, with $p(0) = p$, is necessarily induced by a continuous path of isometries of $\mathbb{R}^d$. The function $p(t)$ is known as a *continuous flex* of the framework (G, p) and *minimal rigidity* means that the framework is rigid with the removal of any framework edge resulting in a nonrigid framework.

In the following theorem the term *generic* means that the framework coordinates of (G, p), of which there are $2|V|$ in number, are algebraically independent over $\mathbb{Q}$. This is one way of formalising the notion of a "typical "framework for G.

Theorem 1.1. *A finite connected simple graph $G = (V, E)$ admits a minimally rigid generic realisation (G, p) in $\mathbb{R}^2$ if and only if*

(i) $2|V| - |E| = 3$ *and*

(ii) $2|V'| - |E'| \geq 3$, *for every subgraph $G' = (V', E')$ with $|E'| > 1$.*

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Moreover every generic realisation (G, p) *of such a graph is minimally rigid.*

There is a well-known notion of *infinitesimal rigidity*, which coincides with rigidity in the case of generic frameworks. See Gluck [5] and Asimow and Roth [1] for example. However frameworks may be infinitesimally flexible while being (continuously) rigid so this is a stronger notion. The theorem above is due to Laman [10] in its infinitesimally rigid formulation.

A graph satisfying (ii) above is said to be an *independent graph* for the plane, or simply an independent graph when the context is understood. The terminology here relates to the connections between the rigidity of geometric framework structures and the theory of matroids. We shall not need these connections which may be found, for example, in Graver, Servatius and Servatius [6], Jackson and Jordan [8], [7] and Whiteley [18]. When both (i) and (ii) hold then G is said to be a maximally independent graph for the plane. We also refer to these graphs as Laman graphs and as *maximally independent graphs of type* 3.

In what follows we analyse frameworks (G, p) supported on general smooth surfaces $\mathcal{M}$ embedded in $\mathbb{R}^3$. In particular in Section 3 we define continuous and infinitesimal rigidity and show that these notions are equivalent for completely regular frameworks in the sense of Definition 3.3. Also we define the ambient degrees of freedom of a framework on a surface $\mathcal{M}$ and obtain necessary counting conditions for minimally rigid completely regular realisations. The development here is in the spirit of the well-known characterisations of rigidity for free frameworks given by Asimow and Roth [1], [15], [2] where regular frameworks were identified as the appropriate topologically generic notion. The primary construct in rigidity theory is the rigidity matrix and for a framework (G, p) on $\mathcal{M}$ we form a relative rigidity matrix $R(G, p, \mathcal{M})$, with $|E| + |V|$ rows and $3|V|$ columns, which incorporates the local tangent vectors for $\mathcal{M}$ at the framework points. While we restrict attention to embedded surfaces in $\mathbb{R}^3$ there are straightforward extensions to higher dimensions, as is also the case in Asimow and Roth [1].

In Section 4, we pay particular attention to the construction of Henneberg moves between frameworks (rather than graphs) which preserve minimal rigidity. These constructions together with the graph theory of Section 2 are the main ingredients in the proof of the main result, Theorem 5.3. This shows that there is a precise version of Laman's theorem for frameworks on a circular cylinder with the class of maximally independent graphs of type 2 (see Definition 2.2) playing the appropriate role.

The approach below embraces reducible surfaces and varieties and we obtain variants of Laman's theorem for frameworks supported on parallel planes, on concentric spheres and on concentric cylinders. As a direct corollary of this for the spheres and planes cases we recover some results of Whiteley [17] on the rigidity of cone frameworks in $\mathbb{R}^3$. On the other hand from the cylinders case we deduce a novel variant for point-line frameworks in $\mathbb{R}^3$ with a single line.

The development is entirely self-contained and we begin with some pure graph theory for maximally independent graphs of type 3 and type 2. We show that with the exception of the singleton graph K_1 each maximally independent graph of type 2 is generated from K_4 by the usual Henneberg moves together with the new move of extension over a rigid subgraph in the sense of Definition 2.10. This result parallels the well-known simpler result for Laman graphs, which we also prove, to the effect that they are derivable from K_2 by Henneberg moves alone.

There are well known characterisations of maximally independent graphs of types 2 and 3 which derive from a celebrated combinatorial result of Nash-Williams for multi-graphs [12], [11]. These characterisations, in terms of spanning trees, are less suited to frameworks on surfaces where we require inductive schemes and graphs with no loops or multi-edges. Although we do not need the spanning tree viewpoint for completeness we derive the Nash-Williams characterisations (Theorems 2.12 and 2.13).

In all cases we are concerned with the usual Euclidean distance in $\mathbb{R}^3$ rather than surface geodesics or other distance measures. We note that Whiteley [18] and Saliola and Whiteley [16] examine first order rigidity for spherical spaces and various spaces where there is local flatness. (See also Connelly and Whiteley [3] for global rigidity concerns.) For the sphere there is an equivalence between the direct distance and geodesic distance viewpoints which may be exploited, however this is a special case and in general one must take account of curvature and local geometry. Thus on the flat cylinder, derived from $\mathbb{R}^2/\mathbb{Z}$ and direct distance in $\mathbb{R}^2$, a generic K_4 framework with no wrap-around edges has three infinitesimal motions, while a typical K_4 framework on the classical curved cylinder has only two.

We would like to thank Walter Whiteley for some useful discussions.

2. Graph theory.

The *Henneberg 2 move*, or Henneberg edge-split move, is an operation $G \to G'$ on simple connected graphs in which a new vertex of degree 3 is introduced by breaking an edge (v_i, v_j) into two edges $(v_i, w), (v_j, w)$ at a new vertex w and adding an edge (w, v_k) to some other vertex v_k of G. The operation maps the set of independent graphs (for the plane) to itself and also preserves maximal independence. A key step in the standard proof of Laman's theorem is to show that if the independent graph G has a minimally rigid generic framework realisation then so too does G'. In Section 4 we pursue this in wider generality by performing Henneberg moves on a minimally rigid *framework* in $\mathbb{R}^3$ where vertices (joints) are free to move on a fixed smooth surface.

We now discuss combinatorial graph theory for Laman graphs, Laman plus one graphs and maximally independent graphs of type 2.

Recall that a *Henneberg 1 move* or vertex addition move $G \to G'$ is the process of adding a degree two vertex with two new edges which are incident to any two distinct points of G.

Proposition 2.1. *Every Laman graph G arises from a sequence*

$$G_0 \to G_1 \to \cdots \to G_n = G$$

where $G_0 = K_2$, the complete graph on two vertices and where $G_k \to G_{k+1}$ is either a Henneberg 1 *move or a Henneberg* 2 *move.*

The starting point for the proof of this fact is the observation that if G is Laman with no degree 2 vertex then there are at least 6 vertices of degree 3. Indeed, if n_i is the degree of the i^{th} vertex then $\Sigma_i n_i = 2|E|$ and so

$$6 = 4|V| - 2|E| = \Sigma_i(4 - n_i).$$

On any of these vertices there is a way of performing a reverse Henneberg 2 move on G to create a Laman graph. This was established by Laman [10] and requires some care for one can easily see that there are non-Laman graphs which become Laman after a particular Henneberg 2 move. The proof is given in Lemma 2.4 and this, together with the corresponding elementary lemma for degree 2 vertices and Henneberg 1 moves, completes the proof of Proposition 2.1.

Define $f(H) = 2|V(H)| - |E(H)|$ for any graph $H = (V(H), E(H))$. This could be referred to as the *freedom number* of H (representing a sense of the total degrees of freedom when the vertices are viewed as having two degrees of freedom). We remark that the definition of a graph (V, E) entails that $|V| \geq 1$ and $|E| \geq 0$.

Definition 2.2. *(a) A simple graph G is independent of type* 3 *if $f(H) \geq 3$ for all subgraphs H containing at least one edge and is maximally independent (of type* 3*) if it is independent and $f(G) = 3$.*

(b) A simple graph G is independent of type 2 *if $f(H) \geq 2$ for all subgraphs H and is maximally independent (of type* 2*) if it is independent and $f(G) = 2$.*

The maximally independent graphs of type 1 and type 0 are similarly defined and are also relevant to frameworks on surfaces, but apart from our closing remark in Section 3 we do not consider them here.

Recall that k-connectedness means that if fewer than k vertices are removed from a graph then it remains path-connected. One can readily check that while a Laman graph is 2-connected, a maximally independent graph of type 2 is in general just 1-connected.

Lemma 2.3. *Let $r = 2$ or 3. Let G be independent of type r with subgraphs G_1 and G_2 which are maximally independent of type r. If $f(G_1 \cap G_2) \geq r$ then $G_1 \cup G_2$ is maximally independent of type r.*

Proof. We have

$$f(G_1 \cup G_2) = f(G_1) + f(G_2) - f(G_1 \cap G_2)$$

and so, from the hypotheses,

$$f(G_1 \cup G_2) \leq r + r - r = r.$$

On the other hand this quantity is greater than or equal to r, since G is independent of type r. □

Lemma 2.4. *Let G be a maximally independent graph of type 3 with a degree 3 vertex. Then there is a maximally independent graph G' with a Henneberg 2 move $G' \to G$.*

Proof. Suppose that v is a vertex of degree 3 of G with 3 distinct neighbours x, y, z. We claim that for some pair ab in $\{xy, yz, xz\}$ we have $f(H) \geq 4$ for all subgraphs H with $a, b \in V(H)$, $v \notin V(H)$. To see that this claim completes the proof note that there is a reverse Henneberg 2 move from G to $G' = G\backslash v + (a, b)$ and for every subgraph of the latter graph containing (a, b) the freedom number is no smaller than 3. The other subgraphs of G' are subgraphs of G and it follows that G' is maximally independent of type 3.

To prove the claim suppose, by way of contradiction, that H_{xy}, H_{yz}, H_{xz} are subgraphs of $K = G\backslash v$ containing, respectively, the pairs $\{x, y\}$, $\{y, z\}$, $\{x, z\}$ and that the freedom number of each is 3. There are a number of cases but each leads to the conclusion that

$$f(H_{xy} \cup H_{yz} \cup H_{xz}) = 3,$$

and now adding back v and its three edges leads to the contradiction

$$f(H_{xy} \cup H_{yz} \cup H_{xz} + (x, v) + (y, v) + (z, v)) = 3 + 2 - 3 = 2.$$

To see that $f(H_{xy} \cup H_{yz} \cup H_{xz}) = 3$ suppose first that $f(H_{xy} \cap H_{yz}) \geq 3$. Then by Lemma 2.3 $f(H_{xy} \cup H_{yz}) = 3$. Now apply the lemma to the pair $H_{xy} \cup H_{yz}, H_{xz}$. Their intersection contains at least two vertices and so $f(H_{xy} \cup H_{yz} \cup H_{xz}) = 3$.

It remains to consider the case where all three pairwise intersections are singletons (necessarily x, y, z). In this case note that if H is the union $H_{xy} \cup H_{yz} \cup H_{xz}$ then

$$|E(H)| = |E(H_{xy})| + |E(H_{yz})| + |E(H_{xz})|,$$
$$|V(H)| = |V(H_{xy})| + |V(H_{yz})| + |V(H_{xz})| - 3$$

and so

$$f(H) = f(H_{xy}) + f(H_{yz}) + f(H_{xz}) - 6 = 9 - 6 = 3$$

as desired. □

We now discuss a particular class of maximally independent graphs of type 2.

Definition 2.5. *A graph $G = (V, E)$ is a Laman plus one graph if it is connected and simple, with no degree* 1 *vertices and is such that the graph $G\backslash e = (V, E\backslash e)$ is a Laman graph for some edge e.*

Note that if G is constructed as two copies of K_4 joined at a common vertex, or joined by two connecting edges, then G is maximally independent of type 2 but is not a Laman plus one graph.

In terminology to follow the next proposition asserts that the family of Laman plus one graphs is the smallest Henneberg closed family containing K_4.

Proposition 2.6. *Every Laman plus one graph is obtained from K_4 by a sequence of Henneberg* 1 *and* 2 *moves.*

Proof. Let G be a Laman plus one graph which does not have a predecessor by a Henneberg move which is Laman plus one. Suppose moreover that G_* is such a graph with the smallest number of vertices. Note that G_* has no degree 2 vertex. It will be enough to show that $G_* = K_4$.

For some edge $e = (u, v)$ we have $G_* = H + e$ with H a Laman graph and for H we have the degree counting equation $6 = \sum_i (4 - n_i)$. Suppose first that the degrees of u and v in H are two and that H has exactly two degree 3 vertices, w and z say. Each of these have reverse Henneberg 2 moves $H \to H_-$ to a Laman graph. We show that either G_* is K_4 or else at least one of these moves does not reintroduce the edge e. If the latter holds then adding e to the Laman graph results in a graph which invalidates the minimality of G_* and so the proof is complete in this case. However, if e arises from reverse Henneberg 2 moves on w and on z then these vertices must each be adjacent to u and v. Let x (resp. y) be the other vertex adjacent to w (resp. z). (See Figure 1.) If $x = z$ and $w = y$ then G_* is K_4. We may assume then that $x \neq z$. Now we obtain a contradiction since any reverse Henneberg moves $H \to H_-$ on w which results in a graph H_- with added edge e leads to a graph which is not 2-connected.

On the other hand if there is one degree 2 vertex v in H (again, necessarily a vertex of the removed edge e) then there are at least four degree 3 vertices at least one of which is not adjacent to v. Thus a reverse Henneberg move is possible which does not reintroduce e and this case contradicts minimality.

Finally, if there are no degree 2 vertices in H then there are at least 6 degree 3 vertices in H. We show that there is a reverse Henneberg 2 move on H that does not add the edge e and so again obtain a contradiction. Note that G_* has at least four degree 3 vertices. Suppose first that there are exactly four. Deleting e adds at most two degree

3 vertices, so this must happen and u, v have degree 4 in G. Thus u cannot be adjacent in H to all four of the degree 3 vertices that are not equal to v. Thus we may apply the reverse Henneberg move to the nonadjacent degree 3 vertex and obtain a contradiction.

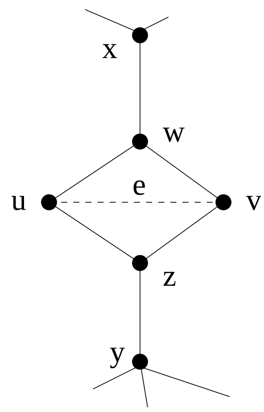

FIGURE 1. Part of $H = G_* \backslash e$.

Similarly, if G_* has exactly 5 degree 3 vertices then, since H has six such, one of u or v, say u, has degree 4 and H has 5 other degree 3 vertices. Not all of these others can be adjacent to u and so again we may perform a reverse Henneberg move that does not reintroduce the edge e. □

We note also the following alternative proof based on the following useful proposition of independent interest.

Proposition 2.7. *If G is maximally independent of type 2 then either*
(i) G has a vertex of degree 2, or
(ii) each vertex of degree 3 lies in a K_4 subgraph, or
(iii) there is a maximally independent graph G' of type 2 and a Henneberg 2 move $G' \to G$.

We may apply this in particular to a Laman plus one graph G : if (i) does not hold then there are at least 4 vertices of degree 3. This occurs when $G = K_4$ but otherwise since G is LAman plus one it follows that there is at least one degree 3 vertex which is not in a K_4 subgraph. Thus (iii) holds and this completes the alternative proof.

Proof. Suppose that (i) and (ii) do not hold. Then there is a degree 3 vertex v that is not in a K_4 subgraph. Then at least one of the neighbour pairs $(x, y), (y, z), (z, x)$ is not an edge. Call such a pair a ghost edge. We claim there is a reverse Henneberg 2 move onto one of the ghost edges which creates a maximally independent type 2 graph.

The claim follows if there is a ghost edge pair ab such that every subgraph H containing a and b but not v has $f(H) \geq 3$.

Suppose first that there are two ghost edges $(x, y), (y, z)$ and suppose by way of contradiction there are graphs H_{xy}, H_{yz} with freedom number 2. Since they share a common vertex their intersection has freedom number 2 or more and so by Lemma 2.3 their union has freedom 2. Now add v and its three edges to this union and one obtains a subgraph with freedom equal to 1, contradicting independence.

Similarly, suppose that there is only one ghost edge, (x, y) say, and there is a subgraph H with freedom number 2 containing x and y but not v. If H contains z then adding v and its edges leads to a contradiction of independence as above. If H does not contain z then adding v and z and the 5 edges $(v, x), (v, y), (v, z), (z, x), (z, y)$ provides a subgraph with freedom number 1, contradicting independence. □

Remark 2.8. A simple connected graph is said to be a *generically rigid graph* for the plane if it is rigid as a framework in $\mathbb{R}^2$ in some vertex-generic realisation. In view of Laman's theorem this means that G is a Laman graph plus some number of extra edges. More strongly, a graph G is *redundantly rigid* if it is rigid and remains so on removal of any edge. Redundant rigidity is plainly stronger than being Laman plus one and is intimately tied up with the topic of global (unique realisation) rigidity. We remark that the globally rigid graphs are K_2, K_3 and those that are derivable from K_4 by Henneberg 2 moves plus edge additions. This rather deeper result is discussed in Jackson and Jordan [8], [7].

The following simple lemma is the key to bridge the gap between Laman plus one graphs and maximally independent graphs of type 2.

Lemma 2.9. *Let G be a maximally independent graph of type 2 with at least one edge. Then one of the following holds.*

(i) G has a proper subgraph that is maximally independent of type 2.

(ii) G is a Laman plus one graph.

Proof. We may assume that G has no vertices v of degree two, for in this case (i) holds for the subgraph $G \backslash v$. Let e be any edge of G and suppose that $H = G \backslash e$, which has no degree one vertices, is not a Laman graph. Then, since all subgraphs H' of H satisfy $f(H') \geq 2$ (being subgraphs also of G) there is a subgraph H' of H with $f(H') = 2$. Thus H' is a proper maximally independent subgraph of G of type 2 and (i) holds. □

Note that, as with the K_4 examples above, two maximally independent graphs of type 2 may be joined at a common vertex, or may be joined by two disjoint edges to create a new maximally independent graph of type 2. Thus the class of maximally independent graphs of type 2 is closed under these two joining operations. Using these two moves with K_4 one obtains large graphs which are maximally independent of type 2 which have no inverse Henneberg move to a maximally independent graph of type 2.

The following contraction move, which is a companion to the last lemma, will be used in the proof of Theorem 2.11.

Let G be independent of type 2 and let H be a proper subgraph with $f(H) = 2$. Write G/H for the multigraph in which H is contracted to a single vertex. This is the graph for which the vertex set is $(V(G)\backslash V(H)) \cup \{v_*\}$ and the edge set is $(E(G)\backslash E(H)) \cup E_*$ where E_* consists of the edges (v, v_*) associated with edges (v, w) with v outside H and w in H. If G is independent of type 2 then so is G/H if it happens to be a simple graph.

To see that G/H is independent let $K \subseteq G/H$ and let $\hat{K} \subset G$ be the subgraph for which

$$V(\hat{K}) = (V(K)\backslash\{v^*\}) \cup V(H), \quad E(\hat{K}) = \pi_e^{-1}(E(K)) \cup E(H)$$

where $\pi_e : E(G) \to E(G/H)$ is the natural map defined on $E(G)\backslash E(H)$. Since $\pi_e^{-1} : E(K) \to E(G)$ is one-to-one it follows that

$$\begin{aligned} 2 \leq f(\hat{K}) &= 2(|V(K)| - 1) + 2|V(H)| - (|E(K)| + |E(H)|) \\ &= f(K) - 2 + f(H) = f(K) \end{aligned}$$

as desired.

Note that G/H is simple in the case that H is maximal with respect to inclusion. Indeed, note first that, by maximality H is vertex induced and so G/H has no loop edges. Also if w is a vertex outside H then it is incident to at most one vertex in H, for otherwise adding w and two of these edges contradicts maximality.

We now identify a natural set of moves through which we may derive from K_4 all the maximally independent graphs of type 2 with at least one edge.

Definition 2.10. *A family $\mathcal{C}$ of finite connected simple graphs is said to be Henneberg closed if it is closed under Henneberg moves and to be Henneberg complete if*

(i) $\mathcal{C}$ is Henneberg closed and

(ii) $\mathcal{C}$ is closed under subgraph extensions in the sense that, if H is a vertex induced subgraph of G and if H and G/H are in $\mathcal{C}$ then so too is G.

Theorem 2.11. *Let $\mathcal{C}$ be a Henneberg complete family of maximally independent graphs of type 2. If $\mathcal{C}$ contains K_4 then $\mathcal{C}$ contains every maximally independent graph of type 2 except, possibly, the singleton graph K_1.*

Proof. Suppose that there is a graph G_* which is maximally independent of type 2, which is not contained in $\mathcal{C}$ and which has at least one edge. Suppose also that G_* has a minimal number of vertices amongst such graphs. If K_4 is in $\mathcal{C}$ then by Proposition 2.6 G_* is not Laman plus one. By Lemma 2.9 G_* is either the singleton graph or has a proper vertex induced maximally independent subgraph H, with G_*/H also maximally independent of type 2. In the latter case, by minimality, H and G_*/H are in $\mathcal{C}$ and so G_* is also in $\mathcal{C}$ which is not the case. □

We now obtain the Nash-Williams characterisations mentioned in the introduction as the equivalences between (i) and (ii) in the next two theorems.

In a similar spirit, Crapo [4] showed that maximally independent graphs of type 3 are exactly the graphs which have a 3T2 decomposition. This is a decomposition into 3 edge disjoint trees such that each vertex is in exactly 2 of them and no subgraph with at least one edge is spanned by subgraphs of two of the three trees. Spanning tree decompositions are of interest because they produce efficient polynomial time algorithms for checking generic minimal rigidity, whereas algorithms based on checking that all subgraphs satisfy the independence type are exponential in the number of vertices. See Graver, Servatius and Servatius [6] for more details on this approach.

A graph $H = (V, E)$ is said to be an edge-disjoint union of k spanning trees if there is a partition $E_1, \ldots, E_r$ of E such that the subgraphs $(V, E_1), \ldots, (V, E_r)$ are (connected) trees.

Theorem 2.12. *The following assertions are equivalent for a (simple) connected graph G.*

(i) G is maximally independent of type 3.

(ii) If G^+ is the graph (or multi-graph) obtained from G by adding an edge (including doubling an edge) then G^+ is an edge-disjoint union of two spanning trees.

(iii) G is derivable from K_2 by Henneberg moves.

Proof. That (ii) implies (i) is elementary (as given explicitly in the proof below) and that (i) implies (iii) follows from Proposition 2.1. We show by elementary induction that (iii) implies (ii).

Let $G \to G'$ be a Henneberg 1 move, adding a degree 2 vertex v, and let $(G')^+$ be obtained from G' by addition of an edge e (including doubling).

If e is added to G then we may assume $G + e$ is the union of 2 edge disjoint spanning trees. To each of the trees we may add one of the new edges.

The other case is when $e = (u, v)$ for some $u \in V(G)$, indicated in the figure below.

Suppose $G^+ = G + f, f = (g, h)$, decomposes into two edge disjoint spanning trees T_1, T_2. We now have a decomposition of G into a spanning tree T_1 and $T_2 \backslash f$ which is either (a) an edge disjoint spanning

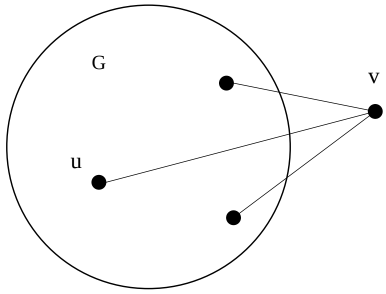

FIGURE 2. $(G')^+$, obtained from G by Henneberg 1 move plus added edge (u, v).

(disconnected) forest or (b) an edge disjoint (non-spanning) tree. In case (a) if v is adjacent to vertices in both connected components of $T_2 \backslash f$ then add both new edges (in the Henneberg move) to $T_2 \backslash f$ to form $T_2^{'}$ and add the "addition"edge to T_1 to get $T_1^{'}$. If v is adjacent to vertices in the same connected component then add one of the new edges (in the Henneberg move) to T_1 and one to $T_2 \backslash f$, then add the "addition"edge to the other component of $T_2 \backslash f$ to get $T_1^{'}$ and $T_2^{'}$.

In case (b) suppose the vertex not in $T_2 \backslash f$ is w, if w is adjacent to v then add (v, w) and some (v, x) to $T_2 \backslash f$ and add the "addition"edge to T_1 to form $T_1^{'}$ and $T_2^{'}$.

Finally if w is not adjacent to v then add the new edges one to each of T_1 and $T_2 \backslash f$ and add the "addition"edge (v, w) to $T_2 \backslash f$ to get $T_1^{'}$ and $T_2^{'}$.

By construction in each case $T_1^{'}$ and $T_2^{'}$ are edge disjoint spanning trees for $(G^{'})^+$ and a very similar elementary argument holds for the Henneberg 2 move which we leave to the reader.

Since K_2^+ is an edge-disjoint union of two spanning trees the proof is complete.

□

Theorem 2.13. *The following assertions are equivalent for a (simple) connected graph G with at least one edge.*

(i) G is maximally independent of type 2.

(ii) G is an edge-disjoint union of two spanning trees.

(iii) G is derivable from K_4 by Henneberg moves and subgraph extensions.

Proof. That (ii) implies (i) is elementary as follows. Let the two edge disjoint spanning trees be $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$. It is a simple property of trees that $|E_i| = |V| - 1$ and $|E_i'| \leq |V'| - 1$ for all subgraphs $T_i' = (V', E_i')$ of T, for $i = 1, 2$. Clearly E is the disjoint union of E_1 and E_2 and so $|E| = 2|V| - 2$ and $|E'| \leq 2|V'| - 2$.

Theorem 2.11 shows that (i) implies (iii) and we now show that (iii) implies (ii) by induction.

As in the last proof (with simplification due to the absence of edge addition) the Henneberg 1 and 2 moves preserve the spanning trees property of (ii). Suppose then that G/H and H decompose into edge disjoint spanning trees and let G be formed by the graph extension move, where $v_* \in G/H$ is replaced by H. We show that G decomposes into edge disjoint spanning trees.

Note v_* has degree $d \geq 2$. Suppose the two spanning trees for G/H are T_1 and T_2 and the two for H are H_1 and H_2. Suppose there are m edges incident to v_* in T_1 and n edges incident to v_* in T_2. Call these subsets of edges E_1 and E_2 respectively. That is, $E_i = \{(a, v_*) : a \in S_i\}$ where $S_i \subseteq V(T_i)$, $i = 1, 2$.

In the extension move these edges are replaced with edges incident to vertices in H. Call these new subsets of edges $E_1^{'}$ and $E_2^{'}$ respectively, so that

$$E_1^{'} = \{(a, w_a) : a \in S_1\}, \quad E_2^{'} = \{(a, u_a) : a \in S_2\}.$$

Then we claim that G decomposes into two edge disjoint spanning trees G_1 and G_2 where, abusing notation slightly,

$$G_i = ((T_i \backslash E_i) \cup H_i \cup E_i^{'}).$$

It is clear that every edge of G is in G_1 or G_2, that no edge is in both, that every vertex is in G_1 and in G_2, and that G_1 and G_2 are connected. It remains to show that G_1 and G_2 are trees and we need only consider G_1. Suppose that there is a cycle in G_1. Then there exists some pair of vertices $a, b \in V(G) \backslash V(H)$ incident to some edges in $E_1^{'}$ such that a and b are connected in $G \backslash H$. However this connectedness is necessarily present in $(G/H) \backslash v_*$ and so there is a cycle in G/H, a contradiction. □

Remark 2.14. The class of maximally independent *multigraphs* of type 2 has been considered by Ross [14] in the setting of periodic frameworks and has been shown to be the relevant class of graphs for a Laman type theorem for periodic isostaticity. Here the flat torus plays the role of the ambient space and finite frameworks on it, with possibly wrap-around (locally geodesic) edges model the relevant periodic frameworks. Interestingly all such graphs derive from the singleton graph by Hennenberg 1 and 2 moves together with the move of a single-vertex double-edge addition move (being a variant of the Henneberg 1 move for multigraphs) and a double-edge variant of the Henneberg 2 move (arising when, in our earlier notation, $v_k = v_i$ or v_j).

3. Frameworks on surfaces.

We now consider infinitesimal and continuous rigidity for bar-joint frameworks on general surfaces. In particular we focus on completely regular frameworks as the appropriate topologically generic notion, noting that for algebraic surfaces this includes the case of algebraically generic frameworks. It is shown that continuous rigidity and infinitesimal rigidity coincide for completely regular frameworks, a fact which will be a convenience later particularly in the consideration of frameworks on the cylinder.

We remark that the basic theory of the rigidity and flexibility of frameworks on surfaces considered here is a local one in the sense that the concepts and properties depend on the nature of $\mathcal{M}$ near the framework points $p_1, \dots, p_n$.

3.1. **Continuous rigidity.** Let $\mathcal{M} \subseteq \mathbb{R}^3$ be a surface. Formally this is a subset with the relative topology which is a two-dimensional differentiable manifold. However, of particular interest are the elementary surfaces which happen to be disjoint unions of algebraic surfaces.

A *framework on* $\mathcal{M}$ is a framework (G, p) in $\mathbb{R}^3$ with G a simple connected graph such that the framework vector $p = (p_1, \dots, p_n)$ has *framework points* p_i in $\mathcal{M}$. The framework is *separated* if its framework points are distinct.

The *edge-function* f_G of a framework (G, p) on $\mathcal{M}$ is the function

$$f_G : \mathcal{M}^{|V|} \to \mathbb{R}^{|E|}, \quad f_G(q) = (|q_i - q_j|^2)_{e=(v_i,v_j)}.$$

This is the usual edge function of the free framework in $\mathbb{R}^3$ restricted to the product manifold $\mathcal{M}^{|V|} = \mathcal{M}^n = \mathcal{M} \times \cdots \times \mathcal{M}$ consisting of all possible framework vectors for G. It depends only on $\mathcal{M}$ and the abstract graph G and for the moment, without undue confusion, we omit the dependence on $\mathcal{M}$ in the notation.

In the next definition we write (K_n, p) for the complete framework on the same set of framework vertices as (G, p).

Definition 3.1. *Let (G, p) be a framework on the surface $\mathcal{M}$ with $p = (p_1, \dots, p_n)$.*

(i) The solution space of (G, p) is the set

$$V_{\mathcal{M}}(G, p) = f_G^{-1}(f_G(p)) \subseteq \mathcal{M}^n$$

consisting of all vectors q that satisfy the distance constraint equations

$$|q_i - q_j|^2 = |p_i - p_j|^2, \text{ for all edges } e = (v_i, v_j).$$

(ii) A framework (G, p) on $\mathcal{M}$ is rigid, or, more precisely, continuously rigid, if for every continuous path $p : [0, 1] \to V_{\mathcal{M}}(G, p)$ with $p(0) = p$ there exists $\delta > 0$ such that $p([0, \delta)) \subseteq V_{\mathcal{M}}(K_n, p)$.

It is easy to see that this is equivalent to the following definition, which is simply the standard definition of continuous rigidity with $\mathbb{R}^3$ replaced by $\mathcal{M}$. A framework (G, p) on $\mathcal{M}$ is continuously rigid if it does not have a continuous flex $p(t)$ (a continuous function $p : [0, 1] \to \mathcal{M}^{|V|}$ with $p(0) = p$, $|p_i(t) - p_j(t)| = |p_i - p_j|$ for each edge) such that $p(t)$ is not congruent to p for some t.

The solution space is topologised naturally with the relative topology and, as with free frameworks, may be referred to as the realisation space of the constrained framework.

We now take into account the smoothness of $\mathcal{M}$ and the smooth parametrisations of $\mathcal{M}$ near framework points.

Let $h(x, y, z)$ be a rational polynomial with real algebraic variety $V(h)$ in $\mathbb{R}^3$. Assume that $\mathcal{M}$ is a subset of $V(h)$ which is a two-dimensional manifold, not necessarily connected, and let (G, p) be a framework on $\mathcal{M}$ with n vertices as before. We associate with the framework the following augmented equation system for the $3n$ coordinate variables of points $q = (q_1, \dots, q_n)$:

$$|q_i - q_j|^2 = |p_i - p_j|^2, \quad \text{for } (v_i, v_j) \in E,$$

$$h(q_i) = 0, \qquad \text{for } v_i \in V.$$

The solution set for these equations is the solution set $V_{\mathcal{M}}(G, p)$ which we also view as the set

$$\tilde{f}_G^{-1}(\tilde{f}_G(p))$$

where $\tilde{f}_G$ is the augmented edge function from $\mathbb{R}^{3|V|} \to \mathbb{R}^{|E|+|V|}$ given by$\tilde{f}_G(q) = (f_G(q), h(q_1), \dots, h(q_n))$, where now f_G is the usual edge function for G defined on all of $\mathbb{R}^{3n}$, rather than just on $\mathcal{M}^n$.

More generally let $\mathcal{M}$ be a surface in $\mathbb{R}^3$ for which there are smooth functions $h_1, \dots, h_n$ which determine $\mathcal{M}$ near $p_1, \dots, p_n$, respectively. Then we define the *augmented edge function* by

$$\tilde{f}_G(q) := (f_G(q), h_1(q_1), \dots, h_n(q_n)), \quad q \in \mathbb{R}^{3|V|}.$$

Suppose for the moment that (G, p) is a free framework in $\mathbb{R}^d$. Write $B(p, \delta)$ for the product $B(p_1, \delta) \times \cdots \times B(p_{|V|}, \delta)$ of the open balls $B(p_i, \delta)$ of radius δ centred at the framework points. Then (G, p) in $\mathbb{R}^d$ is *regular* if the point p in the domain of the edge function $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ is one where the derivative function $D\tilde{f}_G(\cdot)$ achieves its maximal rank. This is to say that p is a *regular point* for this function on $\mathbb{R}^{3|V|}$. The regular points form a dense open set in $\mathbb{R}^{3|V|}$, since the nonregular (singular) points are determined by a finite number of polynomial equations. By standard multivariable analysis a regular point p in $V(G, p)$ has a neighbourhood

$$V(G, p)^\delta = B(p, \delta) \cap V(G, p),$$

which is diffeomorphic to a Euclidean ball in $\mathbb{R}^k \subseteq \mathbb{R}^{3|V|}$ for some k. We take the dimension k as the definition of the ("free") dimension $\dim(G, p)$ of the framework. It follows that all points q close enough to p are regular and $\dim(G, q) = \dim(G, p)$.

These facts extend naturally to frameworks on surfaces.

Definition 3.2. *Let (G, p) be a framework on a smooth surface $\mathcal{M}$ with local coordinate functions $h_1, \dots, h_n$.*

(i) Then (G,p) is regular if p is a regular point for the augmented edge function $\tilde{f}_G$ in the sense that the rank of the derivative matrix is constant in a neighbourhood of p in $\mathbb{R}^{3|V|}$.

(ii) If (G,p) on $\mathcal{M}$ is regular then its dimension is the dimension of the kernel of the derivative of the augmented edge function evaluated at p;

$$\dim_{\mathcal{M}}(G,p) := \dim\ker D\tilde{f}_G(p).$$

For simple examples of irregular frameworks on the sphere one may take a framework which has an antipodal edge, or a triangle whose vertices lie on a great circle.

The local nature of $\mathcal{M}$ near a regular point $p = (p_1,\ldots,p_n)$ for the complete graph K_n determines what we refer to as the number of *ambient degrees of freedom at p*. We define this formally as $d(\mathcal{M},p) = \dim_{\mathcal{M}}(K_n,p)$. Thus $d(\mathcal{M},p) = 3,2,1$ or 0.

The path-wise definition of continuous rigidity of (G,p) on $\mathcal{M}$ given above is in fact equivalent to the following set-wise formula: for some $\delta > 0$ the inclusion

$$V_{\mathcal{M}}(K_n,p)^{\delta} \subseteq V_{\mathcal{M}}(G,p)^{\delta}$$

is an equality. This equivalence for a general framework p is a little subtle in that it follows from the local path-wise connectedness of real algebraic varieties. (That is, each point has a neighbourhood which is path-wise connected.) However for a regular framework $V_{\mathcal{M}}(G,p)^{\delta}$ is an elementary manifold, diffeomorphic to a Euclidean ball, with submanifold $V_{\mathcal{M}}(K_n,p)^{\delta}$ and the equivalence is evident. It follows, somewhat tautologically, that if (G,p) is a regular framework, then (G,p) is rigid on $\mathcal{M}$ if and only if $\dim_{\mathcal{M}}(G,p) = d(\mathcal{M},p)$.

As in the case of free frameworks the regular framework vectors for a graph form a dense open set in $\mathcal{M}^{|V|}$. However, the most amenable constrained frameworks are those that are completely regular in the sense of the next definition.

Definition 3.3. *A framework (G,p) on a smooth surface $\mathcal{M}$ is completely regular if (H,p) is regular on $\mathcal{M}$ for each subgraph H.*

For an example of a regular framework which is not completely regular consider the following. Let $\mathcal{M}$ consist of two parallel planes distance 1 apart and for the complete graph K_6 let (K_6,p) be a separated framework with three non-colinear framework points in each plane. Such continuously rigid frameworks are regular. However if there are points p_i, p_j on separate planes at a minimal distance of 1 apart then (K_6,p) is not completely regular.

One might view the completely regular frameworks as those that are "topologically generic" and in examples one can readily identify a dense open set of completely regular frameworks.

The next proposition establishes a necessary "Maxwell count" condition. Here p' is the restriction of p to $V(G')$.

Proposition 3.4. *Let (G,p) be a completely regular minimally rigid framework on a smooth two-dimensional manifold $\mathcal{M}$. Then*

$$2|V| - |E| = \dim(\mathcal{M}, p)$$

and for each subgraph G' with $|E(G')| > 0$,

$$2|V(G')| - |E(G')| \geq \dim(\mathcal{M}, p').$$

Proof. Let G_1 be a spanning tree with edges $e_1, \dots, e_m$ and let $G_k \subseteq G_{k+1}$ be subgraphs with $|E(G_{k+1})| = |E(G_k)| + 1$, for $1 \leq k \leq r-1$, where $m + r = |E(G)|$. Since (G_1, p) is regular we have $\dim_{\mathcal{M}}(G_1, p) = 2|V| - |E(G_1)| = |V| + 1$. By complete regularity the dimensions $\dim_M(G_k, p)$ are defined and for each k

$$\dim_{\mathcal{M}}(G_k, p) \geq \dim_{\mathcal{M}}(G_{k+1}, p).$$

Suppose that (G,p) is minimally rigid on $\mathcal{M}$. From continuous rigidity we have $\dim_{\mathcal{M}}(G,p) = d(\mathcal{M}, p)$ and by minimal rigidity the inequalities are strict. To see this note that the elementary manifolds $V_{\mathcal{M}}(G_k, p)^\delta$ are determined by multiple intersections. For example if $e_{k+1} = (v_i, v_j)$ then, for all small enough $\delta > 0$,

$$V_{\mathcal{M}}(G_{k+1}, p)^\delta = V_{\mathcal{M}}(G_k, p)^\delta \cap V(|q_i - q_j| = |p_i - p_j|).$$

Thus if there is an equality at the k^{th} step then removal of e_{k+1} does not affect the subsequent inequalities and we arrive at the rigidity of $(G \backslash e_{k+1}, p)$, contrary to minimal rigidity.

By the strict inequalities and noting that $r = |E(G)| - |E(G_1)| = |E(G)| - (|V| - 1))$ we see that

$$d(\mathcal{M}, p) = \dim_{\mathcal{M}}(G, p) = |V| + 1 - r = 2|V| - |E|$$

as desired. □

Remark 3.5. Recall that a generic point p_1 for a connected surface $\mathcal{M}$ defined by an irreducible rational polynomial equation $h(x,y,z) = 0$ is one such that every rational polynomial g vanishing at p_1 necessarily vanishes on $\mathcal{M}$. One may similarly define a generic framework (G,p) on $\mathcal{M}$ as one for which every rational polynomial g in $3n$ variables which vanishes on the framework vector $(p_1, \dots, p_n)$ necessarily vanishes on $\mathcal{M}^n$. Since the set of generic framework vectors is a dense set, generic framework vectors can be found amongst the open set of completely regular framework vectors.

3.2. **Infinitesimal rigidity.** Fix a smooth surface $\mathcal{M}$ in $\mathbb{R}^3$.

Definition 3.6. *Let (G,p) be a framework $\mathcal{M}$ in $\mathbb{R}^3$ and let $h_k(x,y,z) = 0$ be the local equation for the surface $\mathcal{M}$ in a neighbourhood of the framework point p_k, for $1 \leq k \leq |V|$. The rigidity matrix, or relative* *rigidity matrix, of* (G,p) *on* $\mathcal{M}$ *is the* $|E|+|V|$ *by* $3|V|$ *matrix defined in terms of the derivative of the augmented edge-function* $\tilde{f}_G$ *as*

$$R(G,p,\mathcal{M}) = 1/2(D\tilde{f}_G)(p).$$

The factor of $\frac{1}{2}$ is introduced for consistency with existing usage for the rigidity matrices of free frameworks. For example the usual three-dimensional rigidity matrix $R(G,p)$ for (G,p) viewed as a free framework appears as the submatrix of $R(G,p,\mathcal{M})$ given by the first $|E|$ rows. In block operator matrix terms we have

$$R(G,p,\mathcal{M}) = \begin{bmatrix} R(G,p) \\ \frac{1}{2}Dh(p) \end{bmatrix}$$

where, with $|V| = n$, the mapping $h : \mathbb{R}^{3|V|} \to \mathbb{R}^n$ is

$$h = (h_1(x_1,y_1,z_1),\dots,h_n(x_n,y_n,z_n)).$$

Note that the kernel of the matrix $(Dh)(p)$ is determined by the remaining $|V|$ rows and is the subspace of vectors $u = (u_1,\dots,u_n)$ where u_k is tangent to $\mathcal{M}$ at p_k. Thus the kernel of the relative rigidity matrix is the subspace of $\ker R(G,p)$ (the space of free infinitesimal flexes) corresponding to tangency to $\mathcal{M}$. Vectors in this kernel are referred to as *infinitesimal flexes* for (G,p) on $\mathcal{M}$. The subspace of *rigid motion flexes* is defined to be $\ker R(K_n,p,\mathcal{M})$. When (K_n,p) is regular this space has dimension $d(\mathcal{M},p)$.

Definition 3.7. *Let* (G,p) *be a regular framework with* n *framework vertices on the smooth surface* $\mathcal{M}$ *and suppose that* (K_n,p) *is regular. Then* (G,p) *is infinitesimally rigid if*

$$\dim\ker R(G,p,\mathcal{M}) = \dim\ker R(K_n,p,\mathcal{M}) = d(\mathcal{M},p).$$

The following theorem is useful when contemplating Henneberg moves on frameworks and the preservation of rigidity which we turn to in the next section.

Theorem 3.8. *Let* $\mathcal{M}$ *be a smooth surface in* $\mathbb{R}^3$*. A regular framework* (G,p) *on* $\mathcal{M}$ *is infinitesimally rigid if and only if it is continuously rigid.*

Proof. Let $p:[0,1] \to V_{\mathcal{M}}(G,p)$, as in Definition 3.1, be a (one-sided) continuous flex of (G,p) on $\mathcal{M}$. Since p is a regular point, if (G,p) is not rigid on $\mathcal{M}$ then the inclusion

$$V_{\mathcal{M}}(K_n,p)^\delta \subseteq V_{\mathcal{M}}(G,p)^\delta$$

is proper for all small enough $\delta > 0$. Since this is an inclusion of elementary smooth manifolds there exists a differentiable two-sided flex $p(t), t \in (-1,1)$ taking values in the difference set (for $t \in (0,\delta)$). Moreover $p(t)$ may be chosen so that $p'(0)$ is not in the tangent space of $V_{\mathcal{M}}(K_n,p)^\delta$ at p. Note that the derivative vector $p'(0) = (Dp)(0)$ in

$\mathbb{R}^{3n}$ lies in the kernel of $R(G, p, \mathcal{M})$. Indeed, if d_k denotes the squared length of the k^{th} edge of (G, p) then we have

$$\tilde{f}_G \circ p(t) = \tilde{f}_G(p(t)) = (d_1, \dots, d_{|E|}, 0, \dots, 0),$$

a constant function, and so the derivative (column matrix) $D(\tilde{f}_G \circ p)(0)$ is zero. By the chain rule and noting that $p(0) = p$ this is equal to the matrix product $(D\tilde{f}_G)(p).(Dp)(0)$. Thus the vector $v = (Dp)(0)$ is an infinitesimal flex of (G, p) on $\mathcal{M}$ which is not in $\ker(R(K_n, p, \mathcal{M}))$. Thus infinitesimal rigidity implies continuous rigidity.

On the other hand continuous rigidity implies equality, for sufficiently small δ, for the elementary manifold inclusion above, and hence equality of the tangent spaces at p. This equality corresponds to infinitesimal rigidity. □

In the next section we consider minimally continuously rigid completely regular frameworks. In view of the theorem above these coincide with the class of minimally infinitesimally rigid completely regular frameworks. As in the case of free frameworks, we also say that (G, p) is *isostatic* on $\mathcal{M}$ if it is minimally infinitesimally rigid.

Theorem 3.9. *Let $(K_{|V|}, p)$ be regular and let (G, p) be a completely regular framework on the smooth surface $\mathcal{M}$. Then (G, p) is isostatic if and only*

(i)

$$\operatorname{rank} R(G, p, \mathcal{M}) = 3|V| - d(\mathcal{M}, p),$$

and (ii)

$$2|V| - |E| = d(\mathcal{M}, p).$$

Proof. From the definition a framework is infinitesimally rigid if and only if

$$\operatorname{rank} R(G, p, \mathcal{M}) = 3|V| - \dim\ker R(G, p, \mathcal{M}) = 3|V| - d(\mathcal{M}, p).$$

If (G, p) is minimally infinitesimally rigid then by the last theorem and the hypotheses it is also minimally continuously rigid. Thus (ii) holds by Proposition 3.4. It remains to show that if (i) and (ii) hold then the framework, which is infinitesimally rigid (by (i)) is minimally infinitesimally rigid. This follows since if $E' \subsetneq E$ and $((V, E'), p)$ is rigid then $|E'| + |V|$ is greater than or equal to the row rank and so $|E| + |V| > 3|V| - d(\mathcal{M}, p)$ and $2|V| - |E| < d(\mathcal{M}, p)$. □

Remark 3.10. Note that for the circular cylinder $\mathcal{M}$ we have

$$\dim_{\mathcal{M}}(K_3, q) = \dim_M(K_2, r) = 3 \text{ and } \dim_{\mathcal{M}}(K_4, p) = 2,$$

when these frameworks are completely regular. These are continuously rigid frameworks, while if G is the double triangle graph obtained from K_3 by a Henneberg move, then a typical framework (G, p) is not continuously or infinitesimally rigid. In fact $G = K_4 \backslash e$ and we see that a full "rotation"(flex) of (G, p) on the cylinder passes through noncongruent realisations of the "unrotatable"framework (K_4, p).

Remark 3.11. Let (K_4, p) be a separated regular realisation of K_4 in $\mathbb{R}^3$. Then a specialisation of six vertex coordinates is sufficient to remove all continuous nonconstant flexes of (K_4, p). If the framework vertices are all constrained to a smooth surface $\mathcal{M}$ then a specialisation of at most three equations is needed to remove all continuous flexes. That three may be necessary can be seen when $\mathcal{M}$ is a plane, or a union of parallel planes, or when $\mathcal{M}$ is a sphere, or a union of concentric spheres. Let us define the *degrees of freedom* $d(\mathcal{M})$ of the surface $\mathcal{M}$ as the minimum number of vertex coordinate specifications necessary to remove the rigid motions of all proper completely regular realisations of K_4 on $\mathcal{M}$. Thus, for the sphere and the plane there are 3 degrees of freedom, for the infinite circular cylinder there are 2, and for many familiar surfaces with only rotational symmetry, such as cones, ellipsoids and tori, there is one degree of freedom. The degrees of freedom of $\mathcal{M}$ coincides with the minimum value of $\dim_{\mathcal{M}}(K_4, p)$ as p ranges over completely regular quadruples in $\mathcal{M}$. In light of this, and our Laman style theorem for the cylinder, a plausible conjecture is the following: *for reasonable manifolds the graphs for which every completely regular framework on* $\mathcal{M}$ *is continuously rigid are those that are maximally independent of type* $d(\mathcal{M})$*, together with a number of small exceptions.*

4. Henneberg moves on constrained frameworks.

We now work towards combinatorial (Laman type) characterisations of rigid frameworks on some elementary surfaces. The proofs follow a common scheme in which we are required to

(i) establish an inductive scheme for the generation of the graphs in the appropriate class $\mathcal{C}$ for the surface, where the scheme employs moves of Henneberg type or other moves such as graph extensions,

(ii) show that the moves for $\mathcal{C}$ have their counterparts for frameworks on $\mathcal{M}$ in which minimal rigidity is preserved.

We remark that in the case of algebraic manifolds one may define for each graph G the rigidity matroid $\mathcal{R}(G, p, \mathcal{M})$, determined by a generic framework vector, as the vector matroid induced by the rows of $R(G, p, \mathcal{M})$. Thus realising the proof scheme amounts to the determination of a matroid isomorphism between $\mathcal{R}(G, p, \mathcal{M})$ and the matroid defined by maximal independence counting in G. Further in the case of the plane, combining this with Laman's theorem shows that the vector matroids $\mathcal{R}(G, p, M)$ and $\mathcal{R}(G, p')$ (the standard 2-dimensional rigidity matroid) are isomorphic. See [6], [7]. This is perhaps surprising since these matroids are induced by matrices of different sizes. However the isomorphism can be seen by considering the $|V|$ rows in $R(G, p, \mathcal{M})$ as fixed (independent) and identifying the $|E|$ rows in $R(G, p, \mathcal{M})$ with the $|E|$ rows in $R(G, p')$. Of course it is only in the case of planes and spheres that such an identification can be made.

Let $G \to G'$ be the Henneberg 2 move at the graph level in which the edge $e = (v_1, v_2)$ is broken at a new vertex v_{n+1} and in which the new edge (v_3, v_{n+1}) is added. Let $p = (p_1, \dots, p_n)$. A *Henneberg 2 framework move* $(G, p) \to (G', q)$, with (G', q) also on $\mathcal{M}$, is one for which the edges associated with the common graph edges have the same length.

In constructions of such moves the framework points $q_1, \dots, q_n$ may usually be taken close to $p_1, \dots, p_n$. Indeed a Henneberg 2 framework move will arise from a sequence

$$(G, p) \to (G \backslash e, p) \to (G \backslash e, p(t)) \to (G', (p(t), p_{n+1}(t))) = (G', q)$$

where the middle step takes place by a small flex on $\mathcal{M}$ and the final step is determined by a location of $q_{n+1} = p_{n+1}(t)$ on $\mathcal{M}$, with t small, for the vertex v_{n+1}. We consider (G, p) also to be minimally rigid so that $(G \backslash e, p)$ has one degree of freedom, in the sense that the local solution space $V(G \backslash e, p)^\delta$ is a manifold of dimension $d(\mathcal{M}, p) + 1$.

To clarify the consideration of such Henneberg framework moves which preserve minimal (continuous) rigidity we first consider frameworks in the plane under the requirement of a simple geometric noncolinearity condition.

Proposition 4.1. *Let $\delta > 0$, let (G, p) with $p = (p_1, \dots, p_n), n \geq 2$ be a completely regular minimally rigid framework in the plane whose framework vector has noncolinear triples, and let $G \to G'$ be a Henneberg 2 move. Then there is a completely regular minimally rigid framework (G', p'), with $p' = (p'_1, \dots, p'_n, p'_{n+1})$, with noncolinear triples, and $|p_1 - p'_i| < \delta$, for $1 \leq i \leq n$.*

Proof. Consider the depleted framework $(G \backslash e, p)$ with $e = (v_1, v_2)$. By minimal rigidity and complete regularity this framework has one degree of freedom modulo ambient isometries or, more precisely, $\dim(G \backslash e, p) = 4$. Consider the 1-dimensional subset $\mathcal{N}$ of $V(G \backslash e, p)^\delta$ consisting of points q for which $q_1 = p_1$ and q_2 lies on the line though p_1 and p_2. Thus there is a continuous flex $p(t) = (p_1(t), \dots, p_n(t))$ in $\mathcal{N}$ for which $|p_1(t) - p_2(t)|$ is decreasing on some small interval $[0, \delta)$ and we may also assume that this flex is differentiable. Now note that this "normalised"flex $p(t)$ extends to a flex of the enlarged framework $((G \backslash e)^+, p^+)$ formed by introducing p_{n+1} on the line segment $[p_1, p_2]$, with the two new edges, $[p_1, p_{n+1}]$ and $[p_2, p_{n+1}]$. See Figure 3.

There are precisely two such extensions, according to the sense of motion of the hinge point p_{n+1}. It follows from the noncolinearity

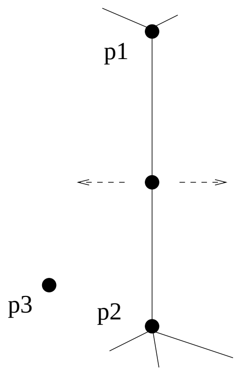

FIGURE 3. Splitting the edge $[p_1, p_2]$.

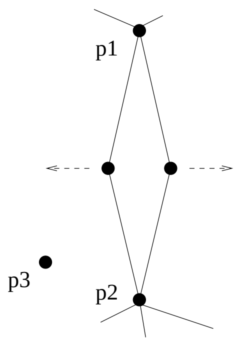

FIGURE 4. The two flexes of $((G\backslash e)^+, p^+)$.

of p_1, p_2 and p_3 that for at least one of these flexes the separation $s(t) = |p_3(t) - p_{n+1}(t)|$ is a non constant function on every interval $[0, \delta]$ for all $\delta < \delta_1$, for some δ_1. (See the next subsection for a formal proof.) Since the flex is differentiable $s(t)$ is strictly decreasing or increasing on a small interval $(0, \delta_2)$. Choose t in this interval and add the edge $[p_3(t), p_{n+1}(t)]$ to create the framework $(G', (p(t), p_{n+1}(t)))$. By construction this is continuously rigid since there is no nonconstant normalised flex (with p_1 fixed and p_2 moving on the line through p_1 and p_2). It also follows readily from the openness of the set of completely regular framework vectors that, for sufficiently small t, $(G', (p(t), p_{n+1}(t)))$ is completely regular. □

In the ensuing discussion we focus on continuous flexes and the intuitive device of hinge separation which we expect to be useful for general manifolds. However, there are alternative approaches for algebraic surfaces based on flex specialisation at generic points. We illustrate this with the following alternative proof to the generic framework variant of the proposition above. Note that Proposition 4.2 together with Proposition 2.1 provide a short proof of the interesting (sufficiency) direction of Laman's theorem.

Proposition 4.2. *Let $G \to G'$ be a Henneberg 2 move and let (G, p) and (G', p') be generic frameworks on the plane with G a Laman graph. If (G, p) is isostatic on the plane (minimally infinitesimally rigid) then so too is (G', p').*

Proof. As before we let v_1, v_2, v_3 and v_h be the vertices involved in the Henneberg move for the edge (v_1, v_2). Suppose that (G', p') is not isostatic. Since G' is a Laman graph it follows that the rank of the rigidity matrix $R(G', p')$ is less than $2|V| - 3$. Since p' is generic this is the case for any specialisation of p' and in particular for $p' = (p_1, \dots, p_n, p_h)$ where $(p_1, \dots, p_n) = p$ and p_h is any point on the open line segment from p_1 to p_2; $p_h = ap_1 + (1 - a)p_2$, with $0 < a < 1$. Thus there is an infinitesimal flex $u' = (u_1, \dots, u_n, u_h)$ for (G', p') which is not a rigid motion flex. We have

$$\langle u_1 - u_h, p_1 - p_h \rangle = 0, \quad \langle u_2 - u_h, p_2 - p_h \rangle = 0,$$

and so by the colinearity of p_1, p_2, p_h,

$$\langle u_1 - u_h, p_1 - p_2 \rangle = 0, \quad \langle u_2 - u_h, p_1 - p_2 \rangle = 0.$$

Thus

$$\langle u_1 - u_2, p_1 - p_2 \rangle = 0,$$

and so the restriction to (G, p), namely $u = (u_1, \dots, u_n)$, is an infinitesimal flex of (G, p).

By the hypotheses u is an infinitesimal rigid motion of (G, p). In particular the restriction $u_r = (u_1, u_2, u_3)$ of u to the triangle p_1, p_2, p_3 is a rigid motion infinitesimal flex for some isometry $T : \mathbb{R}^2 \to \mathbb{R}^2$. But note that u_r is also a restriction of u', and the triangle is noncolinear, so it follows that u_h must be equal to $au_1 + (1 - a)u_2$. Thus u' itself is a rigid motion flex, also associated with T, contrary to assumption. □

4.1. **Hinge frameworks.** In the proof of Proposition 4.1 the key point is that the edge $[p_1, p_2]$ is replaced by two edges $[p_1, p_{n+1}]$ and $[p_{n+1}, p_2]$ which can "hinge" in *two* directions when p_1, p_2 flex towards each other. Similarly, for frameworks on surfaces we examine the placement of p_{n+1} at such special points. With two flex directions (and with a version of the non-colinearity condition for p_3 relative to p_1 and p_2) we obtain a "proper separation" of $|p_3(t) - p_{n+1}(t)|$ on all small enough intervals for at least one of these directions. This last idea is formalised rigourously, in a three-dimensional setting, in assertion (ii) of the hinge framework lemma below. While it seems evident that, roughly speaking, generically one can make a rigidifying Henneberg 2 move, it should be borne in mind that the motion $p_3(t)$ is undetermined (and can be an arbitrary algebraic curve [9]). Thus one needs some systematic method for avoiding exceptional placements of p_{n+1} in which there is no proper separation.

Let H be the cycle graph with four edges and four vertices $v_1, \dots, v_4$ in cyclic order. Let (H, q) be a framework in $\mathbb{R}^3$ with $q = (a, b, c, d)$ where $a, \dots, d$ are points in $\mathbb{R}^3$ with $|a - b| = |a - d| \neq 0$ and $|c - b| = |c - d| \neq 0$. We refer to this as a *hinge framework* and when $b = d$ as a *closed hinge framework*.

Lemma 4.3. *Let $q(t) = (a(t), b(t), c(t), d(t))$ be a continuous flex of the closed hinge framework (H, q) in $\mathbb{R}^3$, with $q(0) = q$, such that*

$$t \to |b(t) - d(t)|$$

is nonconstant on every interval $[0, \delta), \delta < 1$, and let $v(t)$ be a path in $\mathbb{R}^3$. Then one of the following holds:

(i) for some $\delta > 0$ and all $t \in [0, \delta)$

$$\langle b(t) - d(t), a(t) - v(t) \rangle = 0,$$

(ii) at least one of the functions $t \to |v(t) - b(t)|$, $t \to |v(t) - d(t)|$ is nonconstant on all intervals $[0, \delta)$ for δ less than some δ_1.

Proof. Suppose that (ii) fails and the functions are constant in some interval $[0, \delta)$. Since $b(0) = b = d = d(0)$ the functions are equal in this interval. Then, on this interval,

$$\langle v(t) - d(t), v(t) - d(t) \rangle = \langle v(t) - b(t), v(t) - b(t) \rangle$$

and so $\langle v(t), b(t) - d(t) \rangle = (|b(t)|^2 - |d(t)|^2)/2$. The same is true with $v(t)$ replaced by $a(t)$ and so (i) follows. □

Consider a fixed value of $t > 0$ and note that apart from the exceptional case when $b(t)$ and $d(t)$ coincide and $a(t), b(t), c(t)$ are colinear there is a unique plane $P(a(t), b(t), c(t))$ which passes through the midpoint of the line segment $[b(t), d(t)]$ and is normal to the vector $b(t) - d(t)$. With $r = (x, y, z)$ this is the plane with equation

$$\langle b(t) - d(t), a(t) - r \rangle = 0.$$

Because of distance preservation in the flex $q(t)$ note that the plane $P(a(t), b(t), c(t))$ passes through $a(t)$ and $c(t)$. For $t = 0$ and a, b, c not colinear we define $P(a(0), b(0), c(0))$ simply as the plane through a, b, c. In particular, if a, b and c are not colinear and $v(0)$ does not lie on $P(a, b, c)$ then (i) fails (at $t = 0$) and (ii) holds.

This lemma may be applied, with the useful conclusion (ii), whenever one is able to place p_{n+1} on a surface $\mathcal{M}$ in such a way that the added hinge framework $(H, (p_1, p_{n+1}, p_2, p_{n+1}))$ is "opened"(on $\mathcal{M}$) by decreasing separation motion $p_1(t)$ and $p_2(t)$.

4.2. **Spheres and planes.** The case of Henneberg 2 moves on frameworks on concentric spheres and parallel planes is straightforward in that it follows the format of the proof of Proposition 4.1 for the plane, making use of the hinge lemma at an appropriate point.

Lemma 4.4. *Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_{n+1}$ be concentric spheres. Let $p_1(t), p_2(t)$ be paths on the spheres $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively with $p_1 = p_1(0)$, $p_2 = p_2(0)$ such that the separation $|p_1 - p_2|$ is not a local maximum or minimum and such that $|p_1(t) - p_2(t)|$ is decreasing. Let $p_{n+1} \in \mathcal{M}_{n+1}$ be such that p_1, p_2, p_{n+1} are not colinear and the plane $P(p_1, p_2, p_{n+1})$ is orthogonal to the tangent plane to $\mathcal{M}_{n+1}$ at p_{n+1}. Then for some $\delta_1 > 0$ the closed hinge framework $H(q_1, q_2, q_3, q_4) = H(p_1, p_{n+1}, p_2, p_{n+1})$ has a flex $q(t)$ for $t \in [0, \delta_1)$ with $q_1(t) = p_1(t)$, $q_3(t) = p_2(t)$ and $|q_2(t) - q_4(t)|$ nonconstant on all intervals $[0, \delta), \delta \leq \delta_1$.*

Proof. Note that as for a single sphere the union $\mathcal{M}$ of the spheres $\mathcal{M}_i$ has three ambient degrees of freedom. That is $d(\mathcal{M}, p) = 3$ whenever p is a separated framework vector $(p_1, \dots, p_n)$ with $n > 1$. Without loss of generality the flex may be assumed to be normalised so that $p_1(t)$ is fixed on $\mathcal{M}_1$ and $p_2(t)$ moves towards p_1 along the shorter arc of a great circle on $\mathcal{M}_2$ (whose plane meets p_1). The hypothesis on p_{n+1} ensures that it lies on a corresponding great circle, and also that p_{n+1} is not extremally separated from either of these points, or coincident to them in the case that $\mathcal{M}_1 = \mathcal{M}_{n+1}$ or $\mathcal{M}_2 = \mathcal{M}_{n+1}$. The conclusion follows readily from the simple geometry of concentric spheres. □

The case of parallel planes has a verbatim statement, with concentric spheres replaced by parallel planes, and a completely similar proof.

For the Henneberg move construction we require a mild geometric requirement, being the counterpart to noncolinearity in the case of a single plane. More precisely we require that for each pair p_i, p_j the separation $|p_i - p_j|$ is not a local maximum or minimum and that the unique plane $P(p_1, p_2)$ through the pair, which is orthogonal to the planes (or spheres) of $\mathcal{M}$, meets no other framework point. We refer to such frameworks as *geometrically generic.* (In fact one can relax the no extremals condition and treat this class of semigeneric frameworks separately, although we do not do so here.)

The next Henneberg 2 framework move proposition has an analogue for the Henneberg 1 move which is entirely elementary. These framework moves together with standard Laman graph theory are all that are needed for the proof of the sufficiency direction for Theorem 5.1.

Lemma 4.5. *Let $\mathcal{M} = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_N$ be a union of parallel planes, or a union of concentric spheres, and let (G, p) be a minimally continuously rigid geometrically generic completely regular framework on $\mathcal{M}$. Let $\delta > 0$, let $s \in \{1, \dots, N\}$ and let $G \to G'$ a Henneberg 2 move. Then there is a minimally continuously rigid geometrically generic completely regular framework (G', p') on $\mathcal{M}$, with $p' = (p'_1, \dots, p'_n, p'_{n+1})$ and $|p_i - p'_i| < \delta$, for $1 \leq i \leq n$, and $p_{n+1} \in \mathcal{M}_s$.*

Proof. The proof has exactly the same form as that of Proposition 4.1; with the notation above Lemmas 4.3 and 4.4 allow for the placement of p_{n+1} so that the flex of $(G\backslash e, p)$ extends to $((G\backslash e)^+, p^+)$ with the separation function $|p_{n+1}(t) - p_3(t)|$ nonconstant on all small intervals. □

4.3. **Cylinders and surfaces.** We now examine more generally how to place p_{n+1} to create an opening hinge in the manner of Lemma 4.4. This involves the consideration of extremal points in the sense of the next definition.

Definition 4.6. *Let $\mathcal{M} \subseteq \mathbb{R}^3$ be a smooth manifold and p_1, p_2 distinct points of $\mathcal{M}$.*

(i) A point $q \in \mathcal{M}$ is extremal for the pair p_1, p_2 if there exists a point w on the straight line through p_1, p_2, not equal to p_1 or p_2, such that $|q - w| < |q' - w|$ for all points $q' \in \mathcal{M}, q \neq q'$, with $|q - q'| < \delta$, for some $\delta > 0$.

(ii) A point $q \in \mathcal{M}$ is critical for the pair $p_1, p_2 \in \mathcal{M}$ if for some $\delta > 0$ there are no other points q' in $\mathcal{M}$ such that $|q' - q| < \delta$ and

$$|q' - p_1| \leq |q - p_1|, \quad |q' - p_2| \leq |q - p_2|.$$

If q is extremal for a pair, as above, then the tangent plane T_q to $\mathcal{M}$ at q is normal to $q - w$ and moreover the surface $\mathcal{M}$ near q lies in the halfspace of T_q that is opposite the halfspace for w, and strictly so, apart from q itself. Also, for small δ the curve

$$\mathcal{M} \cap S(p_1, |p_1 - q|) \cap B(q, \delta)$$

on the surface $S(p_1, |p_1 - q|)$ of the closed ball $B(p_2, |p_2 - q|)$ is tangential at q to this ball and lies outside this ball, apart from the point q. (Below we give conditions to show how, with p_1 fixed, the two parts of this curve are the paths taken by the hinge frameworks points that separate from the extremal point $p_h = q$.)

We remark that if w lies between p_1 and p_2 (see Figures 5, 6) then q is a critical point for the pair. To see this note that from the colinearity of p_1, w, p_2 the ball $B(w, |q - w|)$ in $\mathbb{R}^3$ contains the intersection of the closed balls $B(p_1, |q - p_1|), B(p_2, |q - p_2|)$. In particular any point near q which is as close to both p_1 and p_2 as q lies in this intersection and so lies in $B(w, |q - w|)$, and so is at least as close to w as q is, contrary to the definition of q.

Finally, note that if the tangent plane at p_1 is not normal to $p_2 - p_1$ then, by taking w close to p_1 there are extremal points arbitrarily close to p_1. We shall make use of this fact for the Henneberg 2 move on the cylinder.

Figure 4 is indicative of a critical point q, where the plane of the diagram is the plane $P(p_1, p_2, q)$ through the triple, the bold curve is in the intersection of this plane with $\mathcal{M}$, and the tangent plane T_q to $\mathcal{M}$ at q is orthogonal to $P(p_1, p_2, q)$. Figure 5 is indicative of the plan view

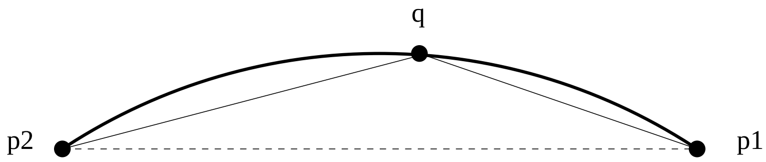

FIGURE 5. A critical point q for p_1, p_2, elevation view.

of such a point, characterised by the tangency of $B(p_1, |q - p_1|) \cap \mathcal{M}$ and $B(p_2, |q - p_2|) \cap \mathcal{M}$ at q.

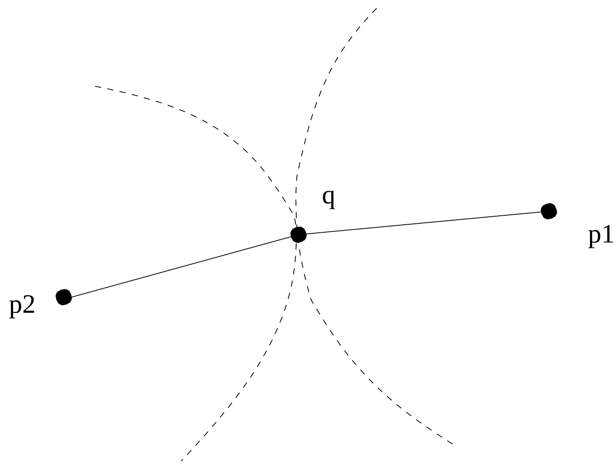

FIGURE 6. A critical point q for p_1, p_2, plan view.

Suppose now that $\mathcal{M}$ is a (circular) cylinder. If $p_1(t)$ and $p_2(t)$ are continuous paths emanating from p_1 and p_2 respectively then by rigid motion normalisation we may assume $p_1(t) = p_1$ for all t. There is now no further normalisation available for the adjustment of the motion of $p_2(t)$ or the specification of $p_2'(0)$. This makes the location of p_{n+1} more problematical in the case that the derivative of the separation $|p_1 - p_2(t)|$ vanishes at $t = 0$. However this complexity only arises (in our edge-deleted framework context) when (G, p) on $\mathcal{M}$ is infinitesimally flexible (before the edge deletion). In view of the equivalence between continuous rigidity and infinitesimal rigidity this difficulty does not occur. Explicitly, we have the following condition which expresses succinctly that the separation motion of the pair $p_1(t)$ and $p_2(t)$ is a *nontangential separation*:

$$\langle p_2'(0) - p_1'(0), p_2 - p_1 \rangle < 0.$$

It need not be the case that any extremal point q can be used for the appropriate ("hinge separating") placement of p_{n+1}. However, using the nontangentiality of separation we show in Lemma 4.8 that there are critical points for the pair p_1, p_2 close to p_2 or p_1 that can be used.

The following simple lemma is needed. In paraphrase it asserts the geometrical fact that the tangential departure $q(t)$ of the point q from the surface of the ball $B(0,|q|)$, together with an acute-to-q departure $p(s)$ from the origin allows for the solution of the distance equation $|q(t)-p(s(t))| = |q|$ for all t in some small interval, where $t \to s(t)$ is a continuous parameter change.

Lemma 4.7. *Let $q(s), s \in [0,1]$, be a path in $\mathbb{R}^3$ starting at $q = q(0) \neq 0$ such that $\langle q'(0), q\rangle = 0$ and such that for $s > 0$ the path points $q(s)$ lie outside the closed ball $B(0,|q|)$. Also, let $p(t), t \in [0,1]$, be a path starting at $p = p(0) = 0$ with*

$$\langle p'(0), q\rangle > 0.$$

Then there is a continuous parameter change $t = t(s)$, for some range $s \in [0,\delta]$, such that in this range

$$|p(t(s)) - q(s)| = |p-q|.$$

Proof. Let $f(s,t) = |p(t)-q(s)|^2 - |q|^2$. Consider first that the function

$$t \to f(t,t) = |p(t)-q(t)|^2 - |q|^2,$$

which is zero at $t = 0$. In view of the hypotheses, for some positive number c we have $\langle p(t), q(t)\rangle \geq ct$ in some small interval $[0,\delta_1]$. It follows readily that the function $t \to f(t,t)$ is strictly decreasing, and in particular $f(t,t) < 0$, for all t in some small interval $[0,\delta_2]$.

We now see that for fixed s in $[0,\delta_2]$ the function $t \to f(s,t)$ has a strictly positive value at $t = 0$ and is negative at $t = s$. By the intermediate value theorem there is a first point $t(s)$ with $f(s,t(s)) = 0$ and moreover, the function $s \to t(s)$ is continuous. □

Lemma 4.8. *Let p_1, p_2 be distinct points on a cylinder $\mathcal{M}$ such that the line segment from p_1 to p_2 does not lie in $\mathcal{M}$. Let $p_2(t)$ be a path on $\mathcal{M}$ with $p_2(0) = p_2$ such that*

$$\langle p_2'(0), p_2 - p_1\rangle < 0.$$

Then there is an extremal point p_{n+1} for the pair p_1, p_2 such that the closed hinge framework $H(q_1,q_2,q_3,q_4) = H(p_1,p_{n+1},p_2,p_{n+1})$ has a flex $q(t), t \in [0,\delta_1)$, on $\mathcal{M}$, with $q_1(t) = p_1$, $q_3(t) = p_2(t)$ and $|q_2(t) - q_4(t)|$ a nonconstant function on all intervals $[0,\delta), \delta \leq \delta_1$ for some $\delta_1 > 0$.

Proof. In view of the discussion above we may choose an extremal point q for the point pair p_1, p_2 such that

$$\langle p_2'(0), p_2 - q\rangle < 0.$$

For example, q may be chosen close to p_1, as a local closest point to a point w close to p_1. Let $q(t), t \in [-1,1]$ be a parametrisation of the curve

$$\mathcal{M} \cap B(p_1, |p_1 - q|) \cap B(q,\delta)$$

for appropriate $\delta > 0$, as above. Apply the lemma to the path pair $p_2(t), q(t), t \in [0,1]$, to create $q_2(t)$ (as $q(t^{-1}(s))$, from the parametrisation $t(s)$) and also to the path pair $p_2(t), q(-t), t \in [0,1]$ to create $q_4(t)$. □

A similar hinge construction lemma holds for frameworks supported on a union $\mathcal{M}$ of concentric cylinders $\mathcal{M}_i$. The new aspect is that the extremal point q must be chosen on a preassigned cylinder $\mathcal{M}_k$ of $\mathcal{M}$ and we must maintain the inequality

$$\langle p_2'(0), p_2 - p_1\rangle > 0.$$

when p_1 is replaced by q. Maintaining this inequality corresponds to choosing q in the halfspace of points z with $\langle p_2'(0), p_2 - z\rangle > 0$. To see this note that the line of points $w_t = p_2 + t(p_1 - p_2), t \in \mathbb{R}$, is not parallel to the common cylinder axis (by assumption) and also that the line is not orthogonal to $p_2'(0)$. Thus for all t large w_t lies in the half space. Since the cylinder $\mathcal{M}_k$ passes through the half space it follows (from simple geometry) that for large enough t the closest point q_t on $\mathcal{M}_k$ to w_t will lie in the half space, as required.

5. Combinatorial characterisations of rigid frameworks

We now obtain variants of Laman's theorem for bar-joint frameworks constrained to parallel planes, to concentric spheres and to concentric cylinders. In each case the proof scheme is the same.

Theorem 5.1. *Let $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_N$ be parallel planes or concentric spheres in $\mathbb{R}^3$ with union $\mathcal{M}$, let G be a simple connected graph and let $\pi : V \to \{1, \dots, |V|\}$. Then G admits a minimally rigid completely regular framework (G,p) on $\mathcal{M}$, with $p_k \in \mathcal{M}_{\pi(k)}$ for each k, if and only if G is a Laman graph.*

Proof. Section 3 shows that the Laman counting conditions are necessary. For sufficiency note that there are minimally rigid completely regular frameworks (K_2, p) with p_1, p_2 placed on any pair $\mathcal{M}_i, \mathcal{M}_j$. Thus the constructions of the last section together with the graph theory of Section 2 lead to the stated framework realisations if G is a Laman graph. □

We now turn to the proof of a Laman theorem for the cylinder for which we require the following matricial companion to the rigid graph extension move.

Lemma 5.2. *Let $\mathcal{M}$ be a cylinder in $\mathbb{R}^3$ and let H be a subgraph of the simple connected graph G such that $K = G/H$ is simple and suppose that G, H, K are maximally independent of type 2. Suppose that for H and K all completely regular framework realisations on $\mathcal{M}$ are isostatic. Then the same is true of G.*

Proof. Let (G,p) be completely regular for $\mathcal{M}$ and let $n = |V(G)|$. Let v_* be a fixed vertex of H. Consider the rigidity matrix $R(G,p,\mathcal{M})$ with column triples in the order of $v_1,\dots,v_{r-1},v_*,v_{r+1},\dots,v_n$ where $v_1,\dots,v_{r-1}, v_r = v_*$ are the vertices of H. Order the rows of $R(G,p,\mathcal{M})$ in the order of the edges $e_1,\dots,e_{|E(H)|}$ for H followed by the n rows of the block diagonal matrix whose diagonal entries are the vectors $h_1(p_1),\dots,h_n(p_n)$ in $\mathbb{R}^3$, followed by the remaining rows for the edges of $E(G)\backslash E(H)$. Note that the principle submatrix formed by the first $|E(H)| + r$ rows is the 2 by 1 block matrix $[R(H,p,\mathcal{M})\ 0]$.

Suppose, by way of contradiction, that (G,p) is not isostatic on $\mathcal{M}$. Since $2|V| - |E| = 2$ there is a vector u in the kernel of $R(G,p,\mathcal{M})$ which is not a rigid motion (infinitesimal) flex. Since completely regular points form an open set we may assume that p is generic for $\mathcal{M}$, in the sense of Remark 3.5. By adding to u some rigid motion flex we may assume that $u_r = 0$. Write $u = (u_H, u_{G\backslash H})$ where $u_H = (u_1,\dots,u_r)$. The matrix $R(G,p,\mathcal{M})$ has the block form

$$R(G,p,\mathcal{M}) = \begin{bmatrix} R(H,p,\mathcal{M}) & 0 \\ X_1 & X_2 \end{bmatrix}$$

where $X = [X_1\ X_2]$ is the matrix formed by the last $|E(G)| - |E(H)|$ rows. Since (H,p) is isostatic on $\mathcal{M}$ and $R(H,p,\mathcal{M})u_H = 0$ it follows that u_H is a rigid motion infinitesimal flex. But $u_r = 0$ and so $u_H = 0$.

Consider now the framework vector

$$p' = (p_r,\dots,p_r,p_{r+1},\dots,p_n)$$

in which the first r framework vertices are specialised to p_r and let

$$p_* = (p_r,p_{r+1},\dots,p_n)$$

be the reduced length framework vector with associated generic and completely regular framework $(G\backslash H, p_*)$. By the hypotheses this framework is infinitesimally rigid.

The matrix $X_2 = X_2(p)$ is square with nonzero vector $u_{G\backslash H}$ in the kernel and so the determinant is a polynomial in the coordinates of the p_i which vanishes identically. It follows that $\det X_2(p')$ vanishes identically and that there is a nonzero vector, $v_{G\backslash H}$ say, in the kernel of $X_2(p')$. But now we obtain the contradiction

$$R(G/H,p_*,\mathcal{M})((0,0,0),v_{G\backslash H}) = (0, X(p')((0,0,0),v_{G\backslash H})) = 0.$$

□

An alternative proof of this key lemma can be given which is based on a continuity argument and the fact that the infinitesimal flexibility of a single generic framework (G,q) on $\mathcal{M}$ ensures the infinitesimal flexibility of all generic frameworks (G,q') on $\mathcal{M}$. Consider a sequence p_N of generic framework vectors converging to the specialised framework vector p'. Arguing by contradiction one obtains a sequence of unit norm flexes $(0,u_N)$ for the generic frameworks (G,p_N) which, by the compactness of the set of unit norm displacements, provides a unit norm flex $u_* = (0, u)$ for the degenerate framework (G, p'). Plainly this infinitesimal flex gives an infinitesimal flex of $(G/H, p_*)$, contrary to the hypotheses.

Theorem 5.3. *Let $\mathcal{M}$ be a circular cylinder in $\mathbb{R}^3$ and let G be a simple connected graph. Then G admits a minimally rigid completely regular framework (G, p) on $\mathcal{M}$ if and only if G is maximally independent of type 2.*

Proof. Note that the necessity of the condition on the graph follows from Proposition 3.4.

Let $\mathcal{C}$ be the class of graphs that admit a minimally rigid completely regular framework on the cylinder. Then $\mathcal{C}$ contains K_4 since there are frameworks (K_4, p) on the cylinder that are regular and minimally rigid as free frameworks. Also $\mathcal{C}$ is closed under Henneberg 1 moves and by the results of the last section, by Henneberg 2 moves. This, together with the lemma above shows that $\mathcal{C}$ is Henneberg complete. Thus sufficiency follows from Theorem 2.11 and the fact that the singleton graph evidently has an isostatic framework. □

From the discussion in the last section we also obtain a similar combinatorial characterisation for frameworks on concentric cylinders, with statement and proof in the style of Theorem 5.1. The final ingredient of the proof is to show that the key lemma above also holds for the reducible manifold formed by a finite number of concentric cylinders. The proof is as before with the following appropriate form of generic point for $\mathcal{M}$.

Recall first that for the irreducible case, with $p = (p_1, \dots, p_n) \in \mathcal{M}^n \subset \mathbb{R}^{3n}$, the genericness of p for $\mathcal{M}$ amounts to consideration of the quotient ring $\mathbb{Q}[x_1, y_1, z_1, \dots, x_n, y_n, z_n]/I$ where I is the maximal ideal generated by the polynomials $h(x_i, y_i, z_i) = 0$ defining $\mathcal{M}$. Thus p (and (G, p)) is generic if the associated field of fractions is isomorphic to $\mathbb{Q}(p)$. (An equivalent assertion is that p is generic if the transcendence degree of the field extension $\mathbb{Q}(p) : \mathbb{Q}$ is $2n$.)

The reducible case is similar. Take a reducible surface $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ defined by a product $h_1 h_2$ of irreducible rational polynomials, where the varieties $V(h_1) \cong \mathcal{M}_1$ and $V(h_2) \cong \mathcal{M}_2$ are irreducible. Let $p = (p_1, p_2)$ with $p_1 = (p_{1,1}, \dots, p_{1,n}) \in \mathcal{M}_1^n \subset \mathbb{R}^{3(m+n)}$ and $p_2 = (p_{2,1}, \dots, p_{2,m}) \in \mathcal{M}_2^m \subset \mathbb{R}^{3(m+n)}$ (for $n, m \neq 0$) and let $p_{i,j} = (x_{i,j}, y_{i,j}, z_{i,j})$. For the corresponding indeterminates we have the tensor product decomposition of the quotient ring

$$\frac{\mathbb{Q}[x_{1,1}, y_{1,1}, z_{1,1}, \dots, x_{1,n}, y_{1,n}, z_{1,n}, x_{2,1}, y_{2,1}, z_{2,1}, \dots, x_{2,m}, y_{2,m}, z_{2,m}]}{\langle h_{1,1}, \dots, h_{1,n}, h_{2,1}, \dots, h_{2,m} \rangle} \cong$$

$$\frac{\mathbb{Q}[x_{1,1}, y_{1,1}, z_{1,1}, \dots, x_{1,n}, y_{1,n}, z_{1,n}]}{\langle h_{1,1}, \dots, h_{1,n} \rangle} \otimes \frac{\mathbb{Q}[x_{2,1}, y_{2,1}, z_{2,1}, \dots, x_{2,n}, y_{2,n}, z_{2,n}]}{\langle h_{2,1}, \dots, h_{2,n} \rangle}$$

with each factor ideal prime and hence these integral domains have fields of fractions $\mathbb{F}_1, \mathbb{F}_2$ providing the field of fractions $\mathbb{F}_1 \otimes \mathbb{F}_2$. The $(n+m)$-tuple p is said to be generic on $\mathcal{M}$ if $\mathbb{Q}(p) \cong \mathbb{F}_1 \otimes \mathbb{F}_2 \cong \mathbb{Q}(p_1) \otimes \mathbb{Q}(p_2)$. (Equivalently, $\mathbb{Q}(p)$ has transcendence degree $2(n+m)$.) The case of reducible surfaces with k irreducible components follows similarly.

Remark 5.4. We note that Whiteley [18] discusses analogous results for frameworks on the flat (geodesic) cylinder and other flat spaces. The cylinder context here concerns infinitesimal rigidity on the cyclic plane and the infinitesimal motion equations derive from equations in the plane. While this keeps some aspects of the rigidity matrix analogous to the plane there is the added feature of geodesic edges which wrap around the cylinder (for which k-frame matroids are introduced to play a role).

5.1. **Cone graphs.** We say that a graph $G = (V, E)$ is a *cone graph* if there is at least one distinguished vertex v which is adjacent to every other vertex. The following corollary and its plane variant indicated below is due to Whiteley [17]. It is well-known that the equivalence of (i) and (ii) does not hold in general (as the so called double banana graph reveals).

Corollary 5.5. *Let $G = (V, E)$ be a cone graph. Then the following statements are equivalent.*

(i) G is maximally independent for $\mathbb{R}^3$, so that $3|V| - |E| = 6$ and $3|V'| - |E'| \geq 6$ for every subgraph $G' = (V', E')$ with $|V'| > 2$.

(ii) There is a minimally rigid completely regular framework realisation (G, p) in $\mathbb{R}^3$.

Proof. One can readily see that, with $v_1 = v$ and $G_0 = G \backslash v$, the set of points q in $V_{\mathbb{R}^3}(G, p) \subseteq \mathbb{R}^{3n}$ with fixed "centre" $q_1 = p_1$ is in bijective isometric correspondence with the variety $V_{\mathcal{M}}(G_0, (p_2, \ldots, p_n))$ where $\mathcal{M}$ is the union of the p_1-centred spheres $S(p_1, |p_k - p_1|)$, for $k = 2, \ldots, n$. The stated equivalence follows from this correspondence. $\square$

There is a companion result for free bar-joint frameworks in $\mathbb{R}^3$ subject to the family of constraints that all points are a specified distance from a single plane. This follows from the parallel planes Laman theorem above. With the plane playing the role of a vertex, for the purposes of counting, the counting requirement is as above.

In a similar way we obtain from the concentric cylinders theorem the following corollary.

Let $G = (V, E)$ be a cone graph with $|V| = n + 1$ and with distinguished cone vertex v_{n+1}. Let $p = (p_1, \ldots, p_n)$ be a framework vector, as usual and let p_* be a straight line. Then the triple (G, p, p_*) is a point-line distance framework for the cone graph G. A line has 4 degrees of freedom and so the natural class of graphs for "typical" general point-line distance frameworks are those for which

$$3|V_p| + 4|V_l| - |E| = 6$$

with a corresponding inequality for subgraphs. We refer to such graphs as maximally independent point-line graphs. One can define the infinitesimal rigidity of general point-line-distance frameworks and also generic frameworks in a natural way. (See for example Owen and Power, [13].)

Corollary 5.6. *Let $G = (V, E)$ be a cone graph, viewed as a point-line graph $G = (V_p \cup V_l, E)$ with a single line corresponding to the cone vertex. Suppose also that the subgraph induced by V_p is connected with at least 4 points. Then the following statements are equivalent.*

(i) G is maximally independent point-line graph.

(ii) There is a minimally infinitesimally rigid point-line framework realisation (G, p) in $\mathbb{R}^3$.

Proof. Infinitesimal rigidity of the point-line distance framework is equivalent to the infinitesimal rigidity of the subframework of points constrained to the concentric cylinders (one for each point) determined by the point line distances. This, by the concentric cylinders theorem above is equivalent to

$$2|V_p| - |E_{pp}| = 2$$

with a similar inequality for subgraphs. For each point there is a point-line edge, and so $|V_p| = |E_{pl}|$. Thus,

$$3|V_p|+4|V_l|-|E| = (2|V_p|+|V_p|)+4-|E_{pl}|-|E_{pp}| = 2|V_p|+4-|E_{pp}| = 6,$$

with an associated inequality for subgraphs, as desired. □

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