CRYSTAL FRAMEWORKS, SYMMETRY AND AFFINELY PERIODIC FLEXES

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Abstract. For an idealised bond-node crystal framework $\mathcal{C}$ in $\mathbb{R}^d$ symmetry equations are obtained for the rigidity matrices associated with various forms of infinitesimal flexibility. These are used to derive symmetry-adapted Maxwell-Calladine counting formulae for periodic self-stresses and affinely periodic infinitesimal mechanisms. The symmetry equations lead to general Fowler-Guest formulae connecting the character lists of subrepresentations of the crystallographic space and point groups associated with bonds, nodes, stresses, flexes and rigid motions. A new derivation is given for the correspondence between affinely periodic infinitesimal flex space and the nullspace of the Borcea-Streinu rigidity matrix.

1. Introduction

A finite bar-joint framework (G,p) is a graph $G=(V,E)$ together with a correspondence $v_i \to p_i$ between vertices and framework points in $\mathbb{R}^d$, the joints or nodes of (G,p). The framework edges are the line segments $[p_i,p_j]$ associated with the edges of G and these are viewed as inextensible bars. Accounts of the analysis of infinitesimal flexibility and the combinatorial rigidity of such frameworks can be found in Asimow and Roth [1], [2] and Graver, Servatius and Servatius [10].

In the presence of a spatial symmetry group $\mathcal{S}$ for (G,p), with isometric representation $\rho_{sp}:\mathcal{S}\to \mathrm{Isom}(\mathbb{R}^d)$, Fowler and Guest [8] considered certain unitary representations $\rho_n\otimes\rho_{sp}$ and ρ_e of $\mathcal{S}$ on the finite dimensional vector spaces

$$\mathcal{H}_v=\sum_{joints}\oplus\mathbb{R}^d,\quad \mathcal{H}_e=\sum_{bars}\oplus\mathbb{R}.$$

These spaces contain, respectively, the linear subspace of infinitesimal flexes of (G,p), denoted $\mathcal{H}_{\rm fl}\subseteq\mathcal{H}_v$, and the linear subspace of infinitesimal self-stresses, $\mathcal{H}_{\rm str}\subseteq\mathcal{H}_e$. Also $\mathcal{H}_{\rm fl}$ contains the space $\mathcal{H}_{\rm rig}$ of rigid motion flexes while $\mathcal{H}_{\rm mech}=\mathcal{H}_{\rm fl}\ominus\mathcal{H}_{\rm rig}$ is the space of infinitesimal mechanisms. It was shown, for $d=2,3$, that for the induced representations $\rho_{\rm mech}$ and $\rho_{\rm str}$ there is a relationship between the associated

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character lists. Specifically,

$$[\rho_{\text{mech}}] - [\rho_{\text{str}}] = [\rho_{sp}] \circ [\rho_n] - [\rho_e] - [\rho_{\text{rig}}]$$

where, for example, $[\rho_{\text{mech}}]$ is the character list

$$[\rho_{\text{mech}}] = (tr(\rho_{\text{mech}}(g_1)), \dots, tr(\rho_{\text{mech}}(g_n))),$$

for a fixed set of generators of $\mathcal{S}$, and where $[\rho_{sp}] \circ [\rho_n]$ indicates entry-wise product.

If there are no infinitesimal flexes or stresses for (G, p) the Fowler-Guest formula can be viewed as a symmetry-adapted version of Maxwell's counting rule for finite rigid frameworks. Indeed, evaluating the formula at the identity element g_1 one recovers the more general Maxwell-Calladine formula, which in three dimensions takes the form

$$m - s = 3|V| - |E| - 6,$$

where m and s are the dimensions of the spaces of infinitesimal mechanisms and infinitesimal self-stresses, respectively. Moreover it has been shown in a variety of studies that the character equation for individual symmetries can lead to useful symmetry-adapted counting conditions for rigidity and isostaticity (stress-free rigidity). See, for example, Connelly, Guest, Fowler, Schulze and Whiteley [5], Fowler and Guest [9], Owen and Power [16] and Schulze [21], [22].

The notation above is taken from Owen and Power [16] where symmetry equations were given for the rigidity matrix $R(G, p)$ and a direct proof of the character equation obtained, together with a variety of applications. In particular a variant of the character equation was given for translationally periodic infinite frameworks and for strict periodicity. In the present development, which is self-contained, we combine the symmetry equation approach with a new derivation of the rigidity matrix, identified in Borcea and Streinu [3], which is associated with affinely periodic infinitesimal flexes. This derivation is given via infinite rigidity matrices and this perspective is well-positioned for the incorporation of space group symmetries. By involving such symmetries we obtain a general symmetry-adapted affine Maxwell-Calladine equation for periodic frameworks (Theorem 3.5) and affine variants of the character list formula.

To be more precise, let $\mathcal{C}$ be a countably infinite bar-joint framework in $\mathbb{R}^d$ with discrete vertex set and with translational periodicity for a full rank subgroup $\mathbb{Z}^d$ of isometric translations. We refer to frameworks with this form of periodicity as crystal frameworks. An affinely periodic flex of $\mathcal{C}$ is one that, roughly speaking, allows periodicity relative to an affine transformation of the ambient space, including contractions, global rotations and sheering. The continuous and infinitesimal forms of this are given in Definition 2.2 and it is shown how the infinitesimal flexes of this type correspond to vectors in the null space of a finite rigidity matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$ with $|F_e|$ rows and $d|F_v| + d^2$ columns. Here

$\mathcal{M} = (F_v, F_e)$, a *motif* for $\mathcal{C}$, is a choice of a set F_v of vertices and a set F_e of edges whose translates partition the vertices and edges of $\mathcal{C}$. The "extra" d^2 columns correspond to the d^2 degrees of freedom present in an affine adjustment of the $\mathbb{Z}^d$-periodicity. When the affine adjustment matrices are restricted to a subspace $\mathcal{E}$ there is an associated rigidity matrix and linear transformation, denoted $R(\mathcal{M}, \mathcal{E})$. The case of strict periodicity corresponds to $\mathcal{E} = \{0\}$.

The symmetry-adapted Maxwell-Calladine formulae for affinely periodic flexibility are derived from the symmetry equation

$$\pi_e(g)R(\mathcal{M}, \mathbb{R}^{d^2}) = R(\mathcal{M}, \mathbb{R}^{d^2})\pi_v(g),$$

where π_e and π_v are finite-dimensional representations of the space group $\mathcal{G}(\mathcal{C})$. As we discuss in Section 3, this in turn derives from the more evident symmetry equation for the infinite rigidity matrix $R(\mathcal{C})$ by restriction to finite dimensional invariant subspaces.

The infinite dimensional vector space transformation perspective for $R(\mathcal{C})$ is also useful in the consideration of quite general infinitesimal flexes including local flexes, diminishing flexes, supercell periodic flexes and, more generally, phase-periodic flexes. In particular phase-periodic infinitesimal flexes for crystal frameworks are considered in [18] in connection with the analysis of rigid unit modes (RUMs) in material crystals. See for example [15], [17], [18], [7] and [25].

Several illustrative examples are discussed in Section 4, namely the well-known kagome framework, a framework suggested by a Roman tiling of the plane, the framework $\mathcal{C}_{\rm Hex}$ formed by a regular network of triangular hexahedra, and a related infinite hexahedron tower. In addition to such curious "mathematical" frameworks there exist a profusion of "natural" polyhedral net frameworks which are suggested by the crystalline structure of natural materials, such as quartz, perovskite, and various aluminosilicates and zeolites. A few of these frameworks, such as the cubic form sodalite framework $\mathcal{C}_{\rm SOD}$, ([7], [18]), have pure mathematical forms. See also the examples in [12], [17], and [25].

We note two recent interesting articles on periodic frameworks which also examine affinely periodic flexibility and rigidity and which provide further examples. Ross, Schulze and Whiteley [20] consider Borcea-Streinu style rigidity matrices and orbit rigidity matrices to obtain a counting predictor for infinitesimal motion in the presence of space symmetries of separable type. Using this they derive the many counting conditions in two and three dimensions, arising from the type of separable space group in combination with the type of affine distortion space $\mathcal{E}$. We indicate how the orbit matrix approach of [20] appears in our formalism.

In a more combinatorial vein Malestein and Theran [13] consider a detailed combinatorial rigidity theory for periodic frameworks which are as generic as can be modulo translational peridicity. In particular they obtain combinatorial characterisations of periodic affine infinitesimal rigidity and isostaticity for such generic periodic frameworks in $\mathbb{R}^2$.

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2. Crystal frameworks and rigidity matrices.

2.1. **Preliminaries.** For convenience suppose first that d is equal to 3. The changes needed for the extension to $d = 2, 4, 5, \dots$ are essentially notational. Let $G = (V, E)$ be a countable simple graph with $V = \{v_1, v_2, \dots\}, E \subseteq V \times V$ and let p be a sequence (p_i) where the $p_i = p(v_i)$ are corresponding points in $\mathbb{R}^3$. The pair (G, p) is said to be a *bar-joint framework* in $\mathbb{R}^3$ with framework vertices p_i (the joints or nodes) and framework edges (or bars) given by the straight line segments $[p_i, p_j]$ between p_i and p_j when (v_i, v_j) is an edge in E.

An *isometry* of $\mathbb{R}^3$ is a distance-preserving map $T : \mathbb{R}^3 \to \mathbb{R}^3$. A *full rank translation group* $\mathcal{T}$ is a set of translation isometries $\{T_k : k \in \mathbb{Z}^3\}$ with $T_{k+l} = T_k + T_l$ for all k, l, $T_k \neq I$ if $k \neq 0$, such that the three vectors

$$a_1 = T_{e_1}0, \quad a_2 = T_{e_2}0, \quad a_3 = T_{e_3}0,$$

associated with the generators $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ of $\mathbb{Z}^3$ are not coplanar. We refer to these vectors as the *period vectors* for $\mathcal{T}$ and $\mathcal{C}$. The following definitions follow the formalism of Owen and Power [17], [18].

Definition 2.1. A crystal framework $\mathcal{C} = (F_v, F_e, \mathcal{T})$ in $\mathbb{R}^3$, with full rank translation group $\mathcal{T}$ and finite motif (F_v, F_e), is a countable bar-joint framework (G, p) in $\mathbb{R}^3$ with a periodic labelling

$$V = \{v_{\kappa,k} : \kappa \in \{1, \dots, t\}, k \in \mathbb{Z}^3\}, \quad p_{\kappa,k} = p(v_{\kappa,k}),$$

such that

(i) $F_v = \{p_{\kappa,0} : \kappa \in \{1, \dots, t\}\}$ and F_e is a finite set of framework edges,

(ii) for each κ and k the point $p_{\kappa,k}$ is the translate $T_k p_{\kappa,0}$,

(iii) the set $\mathcal{C}_v$ of framework vertices is the disjoint union of the sets $T_k(F_v), k \in \mathbb{Z}^3$,

(iv) the set $\mathcal{C}_e$ of framework edges is the disjoint union of the sets $T_k(F_e), k \in \mathbb{Z}^3$.

The finiteness of the motif and the full rank of $\mathcal{T}$ ensure that the periodic set $\mathcal{C}_v$ is a discrete set in $\mathbb{R}^d$ in the usual sense.

One might also view the motif as a choice of representatives for the translation equivalence classes of the vertices and the edges of $\mathcal{C}$. It is natural to take F_v as the vertices of $\mathcal{C}$ that lie in a distinguished *unit cell*, such as $[0, 1) \times [0, 1) \times [0, 1)$ in the case that $\mathcal{T}$ is the standard cubic translation group for $\mathbb{Z}^3 \subseteq \mathbb{R}^3$. In this case one could take F_e to consist of the edges lying in the unit cell together with some number of cell-spanning edges $e = [p_{\kappa,0}, p_{\tau,\delta}]$ where $\delta = \delta(e) = (\delta_1, \delta_2, \delta_3)$ is nonzero. This can be a convenient assumption and with it we may refer to $\delta(e)$ as the *exponent* of the edge. The case of general motifs is discussed at the end of this section.

While we focus on discrete full rank periodic bar-joint frameworks, which appear in many mathematical models in applications, we remark that there are, of course, other interesting forms of infinite frameworks with infinite spatial symmetry groups. In particular there is the class of *cylinder frameworks* in $\mathbb{R}^d$ which are translationally periodic for a subgroup $\{T_k : k \in \mathbb{Z}^r\}$ of a full rank translation group $\mathcal{T}$. See, for example, the two-dimensional strip frameworks of [17] and the hexahedral tower in Section 5. On the other hand one can also consider infinite frameworks constrained to infinite surfaces and employ restricted framework rigidity theory [14].

2.2. **Affinely periodic infinitesimal flexes.** We now give definitions of affinely periodic infinitesimal displacement velocities and flexes starting with a matrix-data description and a connection with continuous flexes. This notation takes the form (u, A) where A is an arbitrary $d \times d$ matrix and $u = (u_\kappa)_{\kappa \in F_v}$ is a real vector in $\mathbb{R}^{d|F_v|}$ composed of the infinitesimal flex vectors $u_\kappa = u_{\kappa,0}$ that are assigned to the framework vertices $p_{\kappa,0}$. The matrix A, an *affine velocity matrix*, is viewed both as a $d \times d$ real matrix and as a vector in $\mathbb{R}^{d^2}$ in which the columns of A are written in order. Thus, any pair (v, B) represents affinely periodic infinitesimal *displacement* velocities for the framework points, and some of these infinitesimal displacements will qualify as affinely periodic infinitesimal *flexes* in the sense given below. In Theorem 2.7 we obtain a three-fold characterisation of such flexes.

By an *affine flow* we mean a differentiable function $t \to A_t$ from $[0, \delta]$ to the set of nonsingular linear transformations of $\mathbb{R}^3$ such that $A_0 = I$ and we write A for the derivative at $t = 0$. (See also Owen and Power [17].) In particular for any matrix A the function $t \to I + tA$ is an affine flow, for suitably small δ.

Definition 2.2. Let $\mathcal{C}$ be a crystal framework.

(i) A flow-periodic flex of $\mathcal{C}$ for an affine flow $t \to A_t$ is a coordinate-wise differentiable path $p(t) = (p_{\kappa,k}(t))$, for $t \in [0, t_1]$ for some $t_1 > 0$, such that

(a) the flow-periodicity condition holds;

$$p_{\kappa,k}(t) = A_t T_k A_t^{-1} p_{\kappa,0}(t), \quad \text{for all} \quad \kappa, k, t,$$

(b) for each edge $e = [p_{\kappa,k}, p_{\tau,k+\delta(e)}]$ the distance function

$$t \to d_e(t) = |p_{\kappa,k}(t) - p_{\tau,k+\delta(e)}(t)|$$

is constant.

(ii) An affinely periodic infinitesimal flex of $\mathcal{C}$ is a pair (u, A), with $u \in \mathbb{R}^{3|F_v|}$ and A a $d \times d$ matrix such that for all motif edges e

$$\langle p_{\kappa,0} - p_{\tau,\delta(e)}, u_\kappa - u_\tau + A(p_{\tau,0} - p_{\tau,\delta(e)})\rangle = 0.$$

The notion of a flow-periodic flex, with finite vertex motions, is quite intuitive and easily illustrated. One can imagine for example a periodic zig-zag bar framework (perhaps featuring as a subframework of a full rank framework) which flexes periodically by concertina-like contraction and expansion. Affinely periodic infinitesimal flexes are perhaps less intuitive. However we have the following proposition.

Proposition 2.3. *Let $p(t)$ be a flow-periodic flex of $\mathcal{C}$ for the flow $t \to A_t$ whose derivative at $t = 0$ is A. Then $(p'_{\kappa,0}(0)), A)$ is an affinely periodic infinitesimal flex of $\mathcal{C}$.*

Proof. Differentiating the flow-periodicity condition

$$A_t^{-1} p_{\tau,\delta}(t) = T_{\delta(e)} A_t^{-1} p_{\tau,0}(t)$$

gives

$$-Ap_{\tau,\delta}(0) + p'_{\tau,\delta}(0) = -Ap_{\tau,0}(0) + p'_{\tau,0}(0)$$

and so

$$p'_{\tau,\delta}(0) = p'_{\tau,0}(0) + A(p_{\tau,\delta}(0) - p_{\tau,0}(0)).$$

Differentiating and evaluating at zero the distance constancy equations

$$t \to \langle p_{\kappa,0}(t) - p_{\tau,\delta(e)}(t), p_{\kappa,0}(t) - p_{\tau,\delta(e)}(t)\rangle$$

gives

$$0 = \langle p_{\kappa,0}(0) - p_{\tau,\delta(e)}(0), p'_{\kappa,0}(0) - p'_{\tau,\delta}(0)\rangle$$

and hence

$$0 = \langle p_{\kappa,0}(0) - p_{\tau,\delta(e)}(0), p'_{\kappa,0}(0) - p'_{\tau,0}(0) + A(p_{\tau,0}(0) - p_{\tau,\delta}(0))\rangle$$

as required. □

We remark that with a similar proof one can show the following related equivalence. The pair (u, A) is an affinely periodic infinitesimal flex of $\mathcal{C}$ if and only if for the flow $A_t = I + tA$ the distance deviation

$$|p_{\kappa,0} - p_{\tau,\delta(e)}| - |(p_{\kappa,0} + tu_\kappa) - A_t T_{\delta(e)} A_t^{-1}(p_{\tau,\delta(e)} + tu_\tau)|$$

of each edge is of order t^2 as $t \to 0$. This property justifies, to some extent, the terminology that (u, A) is an infinitesimal flex for $\mathcal{C}$. Note that an infinitesimal rotation of $\mathcal{C}$ (defined naturally, or by means of Proposition 2.3) provides an affine infinitesimal flex.

2.3. **Rigidity matrices.** We first define a matrix associated with the motif $\mathcal{M} = (F_v, F_e)$ which is essentially the rigidity matrix identified by Borcea and Streinu [3]. We write it as $R(\mathcal{M}, \mathbb{R}^{d^2})$ and also view it as a linear transformation from $\mathbb{R}^{d|F_v|} \oplus \mathbb{R}^{d^2}$ to $\mathbb{R}^{|F_e|}$. In the case $d = 3$ the summand space $\mathbb{R}^{d^2}$ is a threefold direct sum $\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$. For $\mathcal{C}$ with cubic lattice structure with period vectors

$$a_1 = (1, 0, 0), a_2 = (0, 1, 0), a_3 = (0, 0, 1),$$

an *affine velocity matrix* $A = (a_{ij})$ provides a vector in $\mathbb{R}^{3^2}$ in which the columns of A are written in order. It is shown in Theorem 2.7 that the composite vector (u, A) lies in the kernel of the rigidity matrix if and only if (u, A) is an affinely periodic infinitesimal flex in the sense of Definition 2.2 (ii), and if and only if the related infinite vector $\tilde{u} = (u_\kappa - AZk)_{\kappa,k}$ is an infinitesimal flex of $\mathcal{C}$ in the usual sense.

Definition 2.4. Let $\mathcal{C} = (F_v, F_e, \mathcal{T})$ be a crystal framework in $\mathbb{R}^d$ with isometric translation group $\mathcal{T}$ and motif $\mathcal{M} = (F_v, F_e)$. Then the affinely periodic rigidity matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$ is the $|F_e| \times (d|F_v| + d^2)$ real matrix whose rows, labelled by the edges $e = [p_{\kappa,0}, p_{\tau,\delta(e)}]$ of F_e, have the form

$$[0 \cdots 0 \ v_e \ 0 \cdots 0 \ -v_e \ 0 \ \cdots \ 0 \ \delta_1 v_e \cdots \delta_d v_e],$$

where $v_e = p_{\kappa,0} - p_{\tau,\delta(e)}$ is the edge vector for e, distributed in the d columns for κ, where $-v_e$ appears in the columns for τ, and where $\delta(e) = (\delta_1, \ldots, \delta_d)$ is the exponent of e. If e is a reflexive edge in the sense that $\kappa = \tau$ then the entries in the d columns for κ are zero.

The transformation $R(\mathcal{M}, \mathbb{R}^{d^2})$ has the block form $[R(\mathcal{M}) \ X(\mathcal{M})]$ where $R(\mathcal{M})$ is the (strictly) periodic rigidity matrix, or motif rigidity matrix. We also define the linear transformations $R(\mathcal{M}, \mathcal{E})$ for linear spaces $\mathcal{E}$ of affine distortion matrices, by restricting the domain;

$$R(\mathcal{M}, \mathcal{E}) : \mathbb{R}^{3|F_v|} \oplus \mathcal{E} \to \mathbb{R}^{|F_e|}.$$

For a general countably infinite bar-joint framework (G, p) one may define a rigidity matrix $R(G, p)$ in the same way as for finite bar-joint frameworks. For the crystal frameworks $\mathcal{C}$ it takes the following form.

Definition 2.5. Let $\mathcal{C} = (F_v, F_e, \mathcal{T})$ be a crystal framework in $\mathbb{R}^d$ as in Definition 2.4. Then the infinite rigidity matrix for $\mathcal{C}$ is the real matrix $R(\mathcal{C})$ whose rows, labelled by the edges $e = [p_{\kappa,k}, p_{\tau,k+\delta(e)}]$ of $\mathcal{C}$, for $k \in \mathbb{Z}^d$, have the form

$$[\ldots 0 \cdots 0 \ v_e \ 0 \cdots 0 \ -v_e \ 0 \ .. \ 0 \ldots],$$

where $v_e = p_{\kappa,k} - p_{\tau,k+\delta(e)}$ is the edge vector for e, distributed in the d columns for κ, k and where $-v_e$ appears in the columns for $(\tau, k+\delta(e))$, and where $\delta(e) = (\delta_1, \ldots, \delta_d)$ is the exponent of e.

We view $R(\mathcal{C})$ as a linear transformation from $\mathcal{H}_v$ to $\mathcal{H}_e$ where $\mathcal{H}_v$ and $\mathcal{H}_e$ are the direct product vector spaces,

$$\mathcal{H}_v = \Pi_{\kappa,k}\mathbb{R}^3, \quad \mathcal{H}_{e,k} = \Pi_k\mathbb{R}.$$

The following theorem shows the role played by the rigidity matrices $R(\mathcal{M}, \mathbb{R}^{d^2})$ and $R(\mathcal{C})$ in locating affinely periodic infinitesimal flexes. We require the invertible transformation $Z : \mathbb{R}^d \to \mathbb{R}^d$ which maps the standard basis vectors $\gamma_1, \dots, \gamma_d$ for $\mathbb{R}^d$ to the period vectors $a_1, \dots, a_d$ of $\mathcal{C}$.

Definition 2.6. The space $\mathcal{H}_v^{\text{aff}}$ is the vector subspace of $\mathcal{H}_v$ consisting of affinely periodic displacement vectors $\tilde{v} = (\tilde{v}_{\kappa,k})$, each of which is determined by a finite vector $(v_\kappa)_{\kappa\in F_v}$ in $\mathbb{R}^{|F_v|}$ and a $d \times d$ real matrix A by the equations

$$\tilde{v}_{\kappa,k} = v_\kappa - AZk, \quad k \in \mathbb{Z}^d.$$

Note that $\mathcal{H}_v^{\text{aff}}$ is a linear subspace of $\mathcal{H}_v$. For if $\tilde{u}, \tilde{v}$ correspond to (u, A) and (v, B) respectively, then

$$\begin{aligned}\tilde{u}_{\kappa,k} + \tilde{v}_{\kappa,k} &= u_\kappa - AZk + v_\kappa - BZk \\ &= (u_\kappa + v_\kappa) - (A + B)Zk\end{aligned}$$

which defines the displacement velocity corresponding to $(u+v, A+B)$.

In Definition 2.2 (ii) an infinitesimal flex is denoted by a pair (u, A) with $u \in \mathbb{R}^{d|F_v|}$, corresponding to vertex displacement velocities in a unit cell, and a $d \times d$ matrix A, corresponding to axis displacement velocities. In the next theorem the (u, A) data corresponds to (u, AZ) data. This denotes a vector of length $d|F_v| + d^2$, with AZ determining the vector, of length d^2 given by the d vectors $AZ\gamma_1, \dots, AZ\gamma_d$.

Theorem 2.7. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ with translation group $\mathcal{T}$ and period vector matrix Z. Then the restriction of the rigidity matrix transformation $R(\mathcal{C}) : \mathcal{H}_v \to \mathcal{H}_e$ to the finite-dimensional space $\mathcal{H}_v^{\text{aff}}$ has representing matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$. Moreover, the following statements are equivalent:*

(i) (u, A) is an affinely periodic infinitesimal flex for $\mathcal{C}$, where $u \in \mathbb{R}^{d|F_v|}$ and $A \in M_d(\mathbb{R})$.

(ii) $(u, AZ) \in \ker R(\mathcal{M}, \mathbb{R}^{d^2})$.

(iii) $\tilde{u} \in \ker R(\mathcal{C})$, where $\tilde{u} \in \mathcal{H}_v$ is the vector defined by the affinely periodic extension formula

$$\tilde{u}_{\kappa,k} = u_\kappa - AZk, \quad k \in \mathbb{Z}^d.$$

Proof. Let $e = [p_{\kappa,0}, p_{\tau,\delta(e)}]$ be an edge in F_e and as before write v_e for the edge vector $p_{\kappa,0} - p_{\tau,\delta(e)}$. Note that the term $A(p_\tau - p_{\tau,\delta(e)})$ in Definition 2.2 is equal to $A(\delta_1 a_1 + \dots + \delta_d a_d) = AZ(\delta_1\gamma_1 + \dots + \delta_d\gamma_d)$.

Let $\eta_e, e \in F_e$, be the standard basis vectors for $\mathbb{R}^{|F_e|}$. The inner product in Definition 2.2 may be written

$$\begin{aligned}
&\langle p_{\kappa,0} - p_{\tau,\delta(e)}, u_\kappa - u_\tau + A(p_\tau - p_{\tau,\delta(e)})\rangle \\
&= \langle v_e, u_\kappa - u_\tau\rangle + \langle v_e, \sum_i AZ(\delta_i\gamma_i)\rangle \\
&= \langle R(\mathcal{M})u, \eta_e\rangle + \sum_i \langle \delta_i v_i, AZ\gamma_i\rangle \\
&= \langle R(\mathcal{M})u, \eta_e\rangle + \langle (X(\mathcal{M})(AZ), \eta_e\rangle \\
&= \langle R(\mathcal{M}, \mathbb{R}^{d^2})(u, AZ), \eta_e\rangle.
\end{aligned}$$

Thus the equivalence of (i) and (ii) follows.

That the statements (ii) and (iii) are equivalent follows from the following calculation. Let $w = (w_\kappa)$ be a vector in $\mathbb{R}^{d|F_v|}$, let (w, B) be a vertex-plus-axis displacement velocity vector and let $\tilde{w}$ be the associated vector in $\mathcal{H}_v^{\mathrm{aff}}$. Then the e, k coordinate of the vector $R(\mathcal{C})\tilde{w}$ in $\mathcal{H}_e$ is given by

$$\begin{aligned}
(R(\mathcal{C})\tilde{w})_{e,k} &= \langle v_e, \tilde{w}_{\kappa,k}\rangle - \langle v_e, \tilde{w}_{\tau,k+\delta(e)}\rangle \\
&= \langle v_e, \tilde{w}_{\kappa,k} - \tilde{w}_{\tau,k}\rangle + \langle v_e, BZ\delta(e)\rangle \\
&= \langle v_e, w_\kappa - w_\tau\rangle + \sum_i \langle \delta_i v_e, BZ\gamma_i\rangle \\
&= (R(\mathcal{M})w)_e + (X(\mathcal{M})(BZ))_e \\
&= (R(\mathcal{M}, \mathbb{R}^{d^2})(w, BZ))_e.
\end{aligned}$$

□

We say that a periodic bar-joint framework $\mathcal{C}$ is *affinely periodically infinitesimally rigid* if the only affinely periodic infinitesimal flexes are those in the space $\mathcal{H}_{\mathrm{rig}}$ of isometric infinitesimal motions. The following theorem was obtained by Borcea and Streinu [3] where the rigidity matrix was derived from a projective variety identification of the finite flexing space (configuration space) of the framework.

Theorem 2.8. *A periodic bar-joint framework $\mathcal{C}$ with motif $\mathcal{M}$ is affinely periodically infinitesimally rigid if and only if*

$$\operatorname{rank} R(\mathcal{M}, \mathbb{R}^{d^2}) = d|F_v| + d(d-1)/2.$$

Proof. In view of the description of affinely periodic flexes in Theorem 2.7 the definition above is equivalent to $\operatorname{rank} R(\mathcal{M}, \mathbb{R}^{d^2}) = d|F_v| + d^2 - \dim \mathcal{H}_{rig}^{\mathcal{E}}$. The space $\mathcal{H}_{\mathrm{rig}}$ has dimension $d(d+1)/2$, since a basis may be provided by d infinitesimal translations and $d(d-1)/2$ independent infinitesimal rotations and the proof is complete. □

The theorem above is generalised by the Maxwell-Calladine formula in Theorem 2.10 which incorporates the following space of self-stresses for $\mathcal{C}$ relative to $\mathcal{E}$. Symmetry adapted variants are given in Corollary 3.3 and Theorem 3.5.

Definition 2.9. An *infinitesimal self-stress* of a countable bar-joint framework (G,p), with vertices of finite degree, is a vector in $\mathcal{H}_e$ lying in the cokernel of the infinite rigidity matrix $R(G,p)$.

For a periodic framework $\mathcal{C}$ such vectors $w = (w_{e,k})$ are characterised, as in the case of finite frameworks, by a linear dependence row condition, namely

$$\sum_{e,k} w_{e,k} R(\mathcal{C})_{((e,k),(\kappa,\sigma,l))} = 0, \quad \text{for all } \kappa, \sigma, l.$$

Alternatively this can be paraphrased in terms of local conditions

$$\sum_{(\tau,l):[p_{\kappa,k},p_{\tau,l}]\in\mathcal{C}_e} w_{\tau,l}(p_{\kappa,k} - p_{\tau,l}) = 0,$$

which may be interpreted as a balance of internal stresses.

From the calculation in Theorem 2.7 it follows that the rigidity matrix $R(\mathcal{C})$ maps affinely periodic displacement velocities to periodic vectors in $\mathcal{H}_e$. Write $\mathcal{H}_e^{\text{per}}$ for the vector space of such vectors and $\mathcal{H}_{\text{str}}^{\text{aff}}$ (resp. $\mathcal{H}_{\text{str}}^{\mathcal{E}}$) for the subspace of periodic self-stresses that correspond to vectors in the cokernel of $R(\mathcal{M}, \mathbb{R}^{d^2})$ (resp. $R(\mathcal{M}, \mathcal{E})$), and let $s_{\mathcal{E}}$ denote the vector space dimension of $\mathcal{H}_{\text{str}}^{\mathcal{E}}$, Also let $f_{\mathcal{E}}$ denote the dimension of the space $\mathcal{H}_{\text{rig}}^{\mathcal{E}}$ of infinitesimal affinely periodic rigid motions with data (u, A) with affine distortion impetus $A \in \mathcal{E}$, and let $m_{\mathcal{E}}$ denote the dimension of the space of infinitesimal mechanisms of the same form. Thus $m_{\mathcal{E}} = \dim \mathcal{H}_{\text{mech}}^{\mathcal{E}}$ where $\mathcal{H}_{\text{fl}}^{\mathcal{E}} = \mathcal{H}_{\text{mech}}^{\mathcal{E}} \oplus \mathcal{H}_{\text{rig}}^{\mathcal{E}}$.

For $\mathcal{E} = \mathbb{R}^{d^2}$ the following theorem is Proposition 3.11 of [3].

Theorem 2.10. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ with given translational periodicity and let $\mathcal{E} \subset M_d(\mathbb{R})$ be a linear space of admissible affine distortion matrices. Then*

$$m_{\mathcal{E}} - s_{\mathcal{E}} = d|F_v| + \dim \mathcal{E} - |F_e| - f_{\mathcal{E}}$$

where $m_{\mathcal{E}}$ is the dimension of the space of internal (non rigid motion) $\mathcal{E}$-affinely periodic infinitesimal flexes and where $s_{\mathcal{E}}$ is the dimension of the space of periodic self-stresses for $\mathcal{C}$ and $\mathcal{E}$.

Proof. By Theorem 2.7 we may identify $\mathcal{H}_v^{\text{aff}}$ with $\mathbb{R}^{d|F_v|} \oplus \mathbb{R}^{d^2}$ and we may identify $\mathcal{H}_v^{\mathcal{E}}$ with $\mathbb{R}^{d|F_v|} \oplus \mathbb{R}^{d^2}$, the domain space of $R(\mathcal{M}, \mathcal{E})$. We have

$$\mathbb{R}^{d|F_v|} \oplus \mathbb{R}^{d^2} = (\mathcal{H}_v^{\mathcal{E}} \ominus \mathcal{H}_{\text{mech}}^{\mathcal{E}}) \oplus \mathcal{H}_{\text{mech}}^{\mathcal{E}} \oplus \mathcal{H}_{\text{rig}}^{\mathcal{E}}$$

and

$$\mathbb{R}^{|F_e|} = (\mathcal{H}_e^{\text{per}} \ominus \mathcal{H}_{\text{str}}^{\mathcal{E}}) \oplus \mathcal{H}_{\text{str}}^{\mathcal{E}}.$$

With respect to this decomposition $R(\mathcal{M}, \mathcal{E})$ takes the block form

$$R(\mathcal{M}, \mathcal{E}) = \begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

with R an invertible matrix. Since R is square it follows that

$$d|F_v| + \dim \mathcal{E} - (m_{\mathcal{E}} + f_{\mathcal{E}}) = |F_e| - s_{\mathcal{E}},$$

as required. □

We remark that one may view an affinely periodic infinitesimal flex (u, A) as an infinitesimal flex of a motif-defined finite framework located in the set $\mathbb{R}^3/\mathcal{T}$ regarded as a distortable torus. The edges of such a framework are "wrap-around" edges or "geodesic edges" determined naturally by the edges of the motif. (See also Whiteley [24].) Here, in effect, the opposing (parallel) faces of the unit cell parallelepiped are identified, so that one can define (face-to-face) periodicity in the natural way as the corresponding concept for translation periodicity. One can also specify flex periodicity modulo an affine distortional impetus of the torus, given by the matrix A. The natural abstract graphs for such "flat torus" frameworks are the gain graphs (and coloured graphs) indicated in Section 4.

2.4. **The general form of $R(\mathcal{M}, \mathbb{R}^{d^2})$.** We have so far made the simplifying assumption that the motif set F_e consists of edges with at least one vertex in the set F_v. One can always choose such motifs and the notion of the exponent of such edges is natural. However, more general motifs are natural should one wish to highlight polyhedral units or symmetry and we now give the adjustments needed for the general case. See also [18].

Let $e = [p_{\kappa,k}, p_{\tau,k'}]$ be an edge of F_e with both k and k' not equal to the zero multi-index and suppose first that $\kappa \neq \tau$. Then the column entries in row e for the κ label are, as before, the entries of the edge vector $v_e = p_{\kappa,k} - p_{\tau,k'}$, and the entries in the τ columns are, as before, those of $-v_e$. The final d^2 entries are modified using the generalised edge exponent $\delta(e) = k' - k$. Note that this row is determined up to sign by the vertex order for the framework edge and this sign could be fixed by imposing an order on the edge. (See also the discussion in Section 4 of the various labelled graphs.) If e is a reflexive edge in the sense that $\kappa = \tau$ then once again the entry for the κ labelled columns are zero and the final d^2 entries similarly use the generalised exponent.

3. Symmetry equations and counting formulae.

3.1. **Finite-dimensional representations of the space group.** We first obtain symmetry equations for the affine rigidity matrices $R(\mathcal{M}, \mathbb{R}^{d^2})$ with respect to representations of the space group of $\mathcal{C}$.

Let $\mathcal{G}(\mathcal{C})$ be the abstract crystallographic group of the crystal framework $\mathcal{C} = (F_v, F_e, \mathcal{T})$. This is the space group of isometric maps of $\mathbb{R}^d$ which map $\mathcal{C}$ to itself, viewed as an abstract group. This entails the following two assertions .

(i) Each $g \in \mathcal{G}(\mathcal{C})$ acts as a permutation of the framework vertex labels,

$$(\kappa, k) \to g \cdot (\kappa, k),$$

and this permutation induces a permutation of the framework edge labels,

$$(e, k) \to g \cdot (e, k).$$

(ii) There is a representation $\rho_{sp} : g \to T_g$ of $\mathcal{G}(\mathcal{C})$ in Isom($\mathbb{R}^d$), the spatial representation, which extends the map $\mathbb{Z}^d \to \mathcal{T}$ and is such that for each framework vertex

$$p_{g\cdot(\kappa,k)} = T_g(p_{\kappa,k}).$$

Moreover $g \cdot (\kappa, k)$ has the form $(g \cdot \kappa, k')$ where $\kappa \to g \cdot \kappa$ is the vertex class action of g. Similarly $g \cdot (e, k) = (g \cdot e, k'')$ where $e \to g \cdot e$ is the edge class action of g. For the vertex label action we also define the associated "parallel" action $g \cdot (\kappa, \sigma, k) = (g \cdot \kappa, \sigma, k')$ where σ in case $d = 3$ denotes x, y or z.

Let ρ_v be the natural permutation representation of $\mathcal{G}(\mathcal{C})$ on the displacement space $\mathcal{H}_v$, as (linear) vector space transformations, given by

$$(\rho_v(g)u)_{\kappa,\sigma,k} = u_{g^{-1}\cdot(\kappa,\sigma,k)}$$

and similarly let ρ_e be the representation of $\mathcal{G}(\mathcal{C})$ on $\mathcal{H}_e$ such that

$$(\rho_e(g)w)_{e,k} = w_{g^{-1}\cdot(e,k)}.$$

These transformations are forward shifts with respect to the natural coordinate "bases" $\{\xi_{\kappa,\sigma,k}\}$ for $\mathcal{H}_v$ and $\{\eta_{e,k}\}$ for $\mathcal{H}_e$ in the sense that for $l \in \mathbb{Z}^d$

$$\rho_v(l) : \xi_{\kappa,\sigma,k} \to \xi_{\kappa,\sigma,k+l},$$
$$\rho_e(l) : \eta_{e,k} \to \eta_{e,k+l}.$$

Here $\eta_{e,k}$ denotes the element of $\mathcal{H}_e$ given, in terms of the Kronecker symbol, by

$$(\eta_{e,k})_{e',k'} = \delta_{e,e'}\delta_{k,k'},$$

and the totality of these elements gives a vector space basis for the set of finitely nonzero elements of $\mathcal{H}_v$. The basis $\{\xi_{\kappa,\sigma,k}\}$ is similarly defined. Note the multiplicity difference in that, for $d = 3$, $\rho_v(g)$ has multiplicity three, expressed in the decomposition

$$\mathcal{H}_v = \prod_{\kappa\in F_v, k\in\mathbb{Z}^d} \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R},$$

whereas $\rho_e(g)$ has multiplicity one.

Define $\tilde{\rho_v}$, a representation of the space group $\mathcal{G}(\mathcal{C})$ as *affine* maps of $\mathcal{H}_v$ as follows. If T is an isometry of $\mathbb{R}^d$ let $\tilde{T}$ be the affine map on $\mathcal{H}_v$ determined by coordinate-wise action of T and let

$$\tilde{\rho_v}(g) := \rho_v(g)\tilde{T_g} = \tilde{T_g}\rho_v(g)$$

where $g \to T_g$ is the affine isometry representation $\rho_{\rm sp}$. One may also identify $\tilde{\rho}_v(\cdot)$ as the tensor product representation $\rho_n(\cdot) \otimes \rho_{\rm sp}(\cdot)$ where $\rho_n(\cdot)$ is the multiplicity one version of $\rho_v(\cdot)$ (where the subscript "n" stands for "node"). Here the transformations $\rho_n(g)$ are linear while $\rho_{\rm sp}(g)$ may be affine.

Lemma 3.1. *Let $\mathcal{C}$ be a crystal framework with motif (F_v, F_e) and let g be an element of the crystallographic space group $\mathcal{G}(\mathcal{C})$ with isometry $\rho_{\rm sp}(g) = T_g$ with translation-linear factorisation TB, where $Tw = w - \gamma$ for some $\gamma \in \mathbb{R}^d$. If $\tilde{u}$ in $\mathcal{H}_v^{\rm aff}$ corresponds to the pair (u, A) (as in Definition 2.2 and Theorem 2.7), then $\tilde{\rho}_v(g)\tilde{u}$ is a vector $\tilde{v}$ in $\mathcal{H}_v^{\rm aff}$ corresponding to a pair (v, BAB^{-1}).*

Proof. Let $(\kappa', k') = g \cdot (\kappa, k)$. Then

$$\begin{aligned}(\tilde{\rho}(g)\tilde{u})_{\kappa',k'} &= T_g \tilde{u}_{g^{-1}\cdot(\kappa',k')} = T_g(u_\kappa - AZk) \\ &= B(u_\kappa - AZk) - \gamma = w_1 - BAZk\end{aligned}$$

with $w_1 = Bu_\kappa - \gamma = T_g u_\kappa$.

On the other hand k may be related to k'. We have

$$p_{\kappa',0} + Zk' = p_{\kappa',k'} = T_g p_{\kappa,k} = T_g(p_{\kappa,0} + Zk)$$

and so, since $T_g^{-1} = B^{-1}T^{-1}$,

$$Zk = T_g^{-1}(Zk' + p_{\kappa',0}) - p_{\kappa,0} = B^{-1}(Zk' + p_{\kappa',0} + \gamma) - p_{\kappa,0}.$$

Thus Zk has the form $B^{-1}Zk' + w_2$ where $w_2 = T_g^{-1}p_{\kappa',0} - p_{\kappa,0}$ is independent of k'. Substituting,

$$(\tilde{\rho}(g)\tilde{u})_{\kappa',k'} = w_1 - BAZk = (w_1 - BAw_2) - BAB^{-1}Zk',$$

as required, with $v = w_1 - BAw_2$. □

It follows that $\tilde{\rho}_v$ determines a finite-dimensional representation π_v of $\mathcal{G}(\mathcal{C})$ which is given by restriction to $\mathcal{H}_v^{\rm aff}$, written $\pi_v(g) = \tilde{\rho}_v(g)|_{\mathcal{H}_v^{\rm aff}}$. The proof of the lemma provides detail for the coordinatisaiton of this representation which is somewhat complicated in general but which simplifies in particular special cases as we shall see.

Let us introduce what might be referred to to as the *unit cell representation of the space group $\mathcal{G}(\mathcal{C})$*, namely the finite-dimensional representation μ_v given by

$$(\mu_v(g)u)_\kappa = T_g u_{g^{-1}\cdot\kappa}, \quad \kappa \in F_v.$$

That this is a representation follows from the observation that it is identifiable as a tensor product representation

$$\mu_v : g \to \nu_n(g) \otimes \rho_{\rm sp}(g)$$

where ν_n is the natural vertex class permutation representation on $\mathbb{R}^{|F_v|}$, so that $\mu_v = \nu_n \otimes Id_3$. This representation features as a subrepresentation of π_v as we see below.

There is a companion edge class representation for the range space $\mathcal{H}_e$. Define first the linear transformation representation ρ_e of $\mathcal{G}(\mathcal{C})$ on $\mathcal{H}_e$ given by

$$(\rho_e(g)w)_{f,k} = w_{g^{-1}\cdot(f,k)}, \qquad f \in F_e, k \in \mathbb{Z}^d.$$

This in turn provides a finite-dimensional representation π_e of $\mathcal{G}(\mathcal{C})$ on the finite-dimensional space $\mathcal{H}_e^{per}$ of *periodic* vectors, that is, the space of $\rho_e(l)$-periodic vectors for $l \in \mathbb{Z}^d$. These periodic vectors w are determined by the (scalar) values $w_{f,0}$, for $f \in F_e$. Writing $\eta_f, f \in F_e$, for the natural basis elements for $\mathbb{R}^{|F_e|}$ we have the natural edge class permutation matrix representation

$$\pi_e(g)\eta_f = \eta_{g^{-1}\cdot f}, \quad f \in F_e, \quad g \in \mathcal{G}(\mathcal{C}).$$

In the formalism preceding Theorem 2.7, we have identified $\mathcal{H}_v^{\rm aff}$ with a finite-dimensional vector space which we now denote as $\mathcal{D}$ (being the domain of $R(\mathcal{M}, \mathbb{R}^{d^2})$:

$$\mathcal{D} := \mathbb{R}^{d|F_v|} \oplus \mathbb{R}^{d^2} = (\mathbb{R}^{|F_v|} \otimes \mathbb{R}^d) \oplus (\mathbb{R}^d \oplus \cdots \oplus \mathbb{R}^d).$$

Let $D : \mathcal{H}_v^{\rm aff} \to \mathcal{D}$ be the linear identification in which the vector $D\tilde{u}$ corresponds to the matrix-data form (u, A) written as a row vector of columns. Thus

$$\pi_v(\cdot) = D\tilde{\rho}_v(\cdot)|_{\mathcal{H}_v^{\rm aff}} D^{-1}.$$

Note that any transformation in $\mathcal{L}(\mathcal{D})$, the space of all linear transformations on $\mathcal{D}$, has a natural 2×2 block-matrix representation. With respect to this we have the block form

$$\pi_v(g) = \begin{bmatrix} \mu_v(g) & \Phi_1(g) \\ 0 & \Phi_2(g) \end{bmatrix}, \quad g \in \mathcal{G}(\mathcal{C}),$$

where $\Phi_2(g)(A) = BAB^{-1}$.

To see this, return to the proof of Lemma 3.1 and note that in the correspondence

$$\tilde{\rho}_v(g) : (u, A) \to (v, BAB^{-1}),$$

if $A = 0$ then $v_{\kappa'} = T_g u_\kappa$ and so $v = \mu_v(g)u$ in this case.

To see the nature of the linear transformation Φ_1 note that from the lemma that

$$(\Phi_1(g)(A))_{\kappa'} = (-BAw_2)_{\kappa'} = -BA(T_g^{-1}p_{\kappa',0} - p_{\kappa,0}).$$

In particular if g is a separable symmetry for the periodicity, in the sense that $T_g p_{\kappa,0}$ has the form $p_{\kappa',0}$ for all κ, then Φ_2 is the zero map and a simple block diagonal form holds for $\pi_v(g)$, namely,

$$\pi_v(g)(u, A) = (\mu_v(g)u, T_g A T_g^{-1}).$$

The separable property in fact corresponds to g being a linear isometry of separable type in the sense that the action $(\kappa, k) \to g \cdot (\kappa, k)$ is a product action, with $(g \cdot (\kappa, k) = (g \cdot \kappa, g \cdot k)$

3.2. **Symmetry equations.** The following theorem is a periodic framework version of the symmetry equation given in Owen and Power [**?**].

Theorem 3.2. *Let $\mathcal{C}$ be a crystal framework in $\mathbb{R}^d$ and let $g \to T_g$ be a representation of the space group $\mathcal{G}(\mathcal{C})$ as isometries of $\mathbb{R}^d$.*

(a) For the rigidity matrix transformation $R(\mathcal{C}) : \mathcal{H}_v \to \mathcal{H}_e$,

$$\rho_e(g)R(\mathcal{C}) = R(\mathcal{C})\tilde{\rho}_v(g), \quad g \in \mathcal{G}(\mathcal{C}).$$

(b) For the affinely periodic rigidity matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$ determined by the motif $\mathcal{M} = (F_v, F_e)$,

$$\pi_e(g)R(\mathcal{M}, \mathbb{R}^{d^2}) = R(\mathcal{M}, \mathbb{R}^{d^2})\pi_v(g), \quad g \in \mathcal{G}(\mathcal{C}),$$

where π_e is the edge class representation in $\mathbb{R}^{|F_e|}$ and where π_v is the representation in $(\mathbb{R}^{|F_v|} \otimes \mathbb{R}^d) \oplus \mathbb{R}^{d^2}$ induced by $\tilde{\rho}_v$.

Proof. By Theorem 2.7 and Lemma 3.1, and the identification of the space of affinely periodic displacement vectors with $\mathcal{D}$ we see that (b) follows from (a).

To verify (a) observe first that if (G, p) is the framework for $\mathcal{C}$ with labelling as given in Definition 2.5 then the framework for $(G, g\cdot p)$ with relabelled framework vector $g \cdot p$ with $(g \cdot p)_{\kappa,k} = p_{g\cdot(\kappa,k)}$ has rigidity matrix $R(G, g \cdot p)$. Examining matrix entries shows that this matrix is equal to the row column permuted matrix $\rho_e(g)^{-1}R(G, p)\rho_v(g)$.

On the other hand the row for the edge $e = [p_{\kappa,k}, p_{\tau,l}]$ has entries in the columns for κ and for τ given by the coordinates of $T_g v_e$ and $-T_g v_e$ respectively, where $v_e = p_{\kappa,k} - p_{\tau,l}$. If $T_g = TB$ is the translation-linear factorisation, with B an orthogonal matrix, note that

$$T_g v_e = TBp_{\kappa,k} - TBp_{\tau,l} = Bv_e.$$

Here v_e is a column vector, while the κ column entries are given by the row vector transpose $(Bv_e)^t = v_e^t B^t = v_e^t B^{-1}$. Thus

$$R(G, g \cdot p) = R(G, p)(I \otimes B^{-1}).$$

Note also, that for any rigidity matrix R and any affine translation S we have $R(I \otimes S) = R$, as transformations form $\mathcal{H}_v$ to $\mathcal{H}_e$. Thus we also have

$$R(G, g \cdot p) = R(G, g \cdot p)(I \otimes T^{-1}) = R(G, p)(I \otimes B^{-1})(I \otimes T^{-1})$$

which is $R(G, p)(I \otimes T_g^{-1})$ and now (a) follows. □

As an application we obtain a general Fowler-Guest style Maxwell Calladine formulae relating the traces (characters) of symmetries g in various subrepresentations of π_v, π_e derived from flexes and stresses.

Let $\mathcal{E}$ be a fixed subspace of $d \times d$ axial velocity matrices A, let g be a symmetry for which $T_g A = AT_g$ for all $A \in \mathcal{E}$ and let H be the cyclic subgroup generated by g. It follows from Lemma 3.1 that there is a restriction representation $\pi_v^{\mathcal{E}}$ of H on the space $\mathcal{H}_h^{\mathcal{E}} := \mathbb{R}^{d|F_v|} \oplus \mathcal{E}$

and that there is an associated intertwining symmetry equation. This equation implies that the space of $\mathcal{E}$-affinely periodic infinitesimal flexes is an invariant subspace for the representation π_v of H. Thus there is an associated restriction representation of H, namely $\pi_{\rm fl}^{\mathcal{E}}$. We note the following subrepresentations of $\pi_v^{\mathcal{E}}$:

$\pi_{\rm fl}^{\mathcal{E}}$ on the invariant subspace $\mathcal{H}_{\rm fl}^{\mathcal{E}} := \ker R(\mathcal{M},\mathcal{E})$,

$\pi_{\rm rig}^{\mathcal{E}}$ on the invariant subspace $\mathcal{H}_{\rm rig}^{\mathcal{E}} := \mathcal{H}_v^{\mathcal{E}} \cap \mathcal{H}_{\rm rig}$,

$\pi_{\rm mech}^{\mathcal{E}}$ on the invariant subspace $\mathcal{H}_{\rm mech}^{\mathcal{E}} := \mathcal{H}_{\rm fl}^{\mathcal{E}} \ominus \mathcal{H}_{\rm rig}^{\mathcal{E}}$.

Also, we have a subrepresentation of $\pi_e^{\rm per}$, namely

$\pi_{\rm str}$ on the invariant subspace $\mathcal{H}_{\rm str}^{\mathcal{E}} := \operatorname{coker} R(\mathcal{M},\mathcal{E})$.

Corollary 3.3. *Let $\mathcal{C}$ be a crystal framework, let g be a space group symmetry and let $\mathcal{E} \subseteq \mathbb{R}^{d^2}$ be a space of affine velocity matrices which commute with T_g. Then*

$$tr(\pi_{\rm mech}^{\mathcal{E}}(g)) - tr(\pi_{\rm str}(g)) = tr({\pi_v}^{\mathcal{E}}(g)) - tr(\pi_e(g)) - tr(\pi_{\rm rig}^{\mathcal{E}}(g)).$$

Proof. The rigidity matrix $R(\mathcal{M},\mathcal{E})$ effects an equivalence between the subrepresentations of H on the spaces $\mathcal{H}_v^{\mathcal{E}} \ominus \mathcal{H}_{\rm fl}^{\mathcal{E}}$ and $\mathcal{H}_e \ominus \mathcal{H}_{\rm str}$. Thus the traces of the representations of g on these spaces are equal and from this the identity follows. Indeed, with respect to the decompositions

$$\mathcal{H}_v^{\mathcal{E}} = (\mathcal{H}_v^{\mathcal{E}} \ominus \mathcal{H}_{\rm fl}^{\mathcal{E}}) \oplus \mathcal{H}_{\rm mech}^{\mathcal{E}} \oplus \mathcal{H}_{\rm rig}^{\mathcal{E}}$$

$$\mathcal{H}_e^{\mathcal{E}} = (\mathcal{H}_e^{\mathcal{E}} \ominus \mathcal{H}_{\rm str}^{\mathcal{E}}) \oplus \mathcal{H}_{\rm str}^{\mathcal{E}}$$

and the rigidity matrix has block form

$$R(\mathcal{M},\mathcal{E}_\sigma) = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with R_1 square and invertible. Thus R_1 effects an equivalence between the representations on the first summands above. It follows that the traces of these representations of g are equal and so we obtain the equality

$$tr(\pi_v^{\mathcal{E}}(g)) - tr(\pi_{\rm mech}^{\mathcal{E}}(g)) - tr(\pi_{\rm rig}^{\mathcal{E}}(g)) = tr({\pi_e}^{\mathcal{E}}(g)) - tr(\pi_{\rm str}(g))$$

as required. □

We may derive from π_e and π_v various representations and maps for the point group $\mathcal{G}_{\rm pt}(\mathcal{C}) = \mathcal{G}(\mathcal{C})/\mathbb{Z}^d$ of the crystal framework $\mathcal{C}$ where $\mathbb{Z}^d$ represents the periodicity subgroup $\mathcal{T} = \{T_l : l \in \mathbb{Z}^d\}$.

Suppose first that the space group is separable in the sense that $\mathcal{G}(\mathcal{C})$ is isomorphic to $\mathbb{Z}^d \times \mathcal{G}_{\rm pt}(\mathcal{C})$. Write $\dot{g}$ for the coset element in $\mathcal{G}_{\rm pt}(\mathcal{C})$ of an element g of the space group. Then, following an appropriate shift one can recoordinatise the bar-joint framwork $\mathcal{C}$ so that for the natural inclusion map $i : \mathcal{G}_{\rm pt}(\mathcal{C}) \to \mathcal{G}(\mathcal{C})$ the map $\dot{g} \to T_{i(\dot{g})}$ is a representation by *linear* isometries. In this case we obtain, simply by restriction, the representations

$$\dot{\pi}_v : \mathcal{G}_{\mathrm{pt}}(\mathcal{C}) \to \mathcal{L}(\mathcal{D}), \quad \dot{\pi}_e : \mathcal{G}_{\mathrm{pt}}(\mathcal{C}) \to \mathcal{L}(\mathbb{R}^{|F_e|}).$$

Theorem 3.4. *For a crystal framework with separable space group $\mathcal{G}(\mathcal{C})$,*

$$\dot{\pi}_e(\dot{g})R(\mathcal{M}, \mathbb{R}^{d^2}) = R(\mathcal{M}, \mathbb{R}^{d^2})\dot{\pi}_v(\dot{g}), \quad \dot{g} \in \mathcal{G}_{\mathrm{pt}}(\mathcal{C}).$$

Moreover for a fixed element $\dot{g} \in \mathcal{G}_{\mathrm{pt}}(\mathcal{C})$ the individual symmetry equation

$$\dot{\pi}_e(\dot{g})R(\mathcal{M}, \mathcal{E}) = R(\mathcal{M}, \mathcal{E})\dot{\pi}_v(\dot{g})$$

holds where $\mathcal{E}$ is any space of matrices which is invariant under the map $A \to T_{\dot{g}}AT_{\dot{g}}^{-1}$.

Proof. The equations follow from those of Theorem 3.2. □

Note that $\dot{\pi}_e$ is quite explicit and agrees with the natural edge class permutation representation ν_e which is well-defined by

$$\nu_e(\dot{g})\eta_f = \eta_{g^{-1}\cdot f}, \quad f \in F_e, \quad g \in \dot{g} \in \mathcal{G}_{\mathrm{pt}}(\mathcal{C}).$$

Also, from our earlier remarks on the separable case $\dot{\pi}_v(\dot{g})$ has the simple block diagonal form.

3.3. **Symmetry-adapted Maxwell Calladine equations.** We now give specific symmetry adapted Maxwell-Calladine formulae which derive from the vector space of vectors *all* of which are fixed under an individual symmetry.

Suppose first that g be a space group element in $\mathcal{G}(\mathcal{C})$ which is of separable type as before. We may thus assume that T_g is an orthogonal linear transformation. In this case $\pi_v(g)$ takes the block diagonal form

$$\pi_v(g)(u, A) = (\mu_v(g)u, T_gAT_g^{-1}).$$

Let $\mathcal{H}_v^g$ be the subspace of vectors (v, A) fixed by the linear transformation $\pi_v(g)$. By separability, $\mathcal{H}_v^g$ splits as a direct sum which we write as $\mathcal{F}_g \oplus \mathcal{E}_g$. where $\mathcal{E}_g$ is the space of matrices commuting with T_g and $\mathcal{F}_g$ is the space of vectors v that are fixed by the linear transformation $\mu_v(g) = \nu_n(g) \otimes T_g$.

Let $\mathcal{H}_e^g$ be the subspace of vectors in $\mathbb{R}^{|F_e|}$ that are fixed by $\pi_e(g)$, so that $\mathcal{H}_e^g$ is naturally identifiable with $\mathbb{R}^{e_g}$ where e_g is the number of orbits of edges in F_e induced by g.

It follows from the symmetry equations in Theorem 3.4 that the rigidity matrix transformation $R(\mathcal{M}, \mathcal{E}_g)$ maps $\mathcal{H}_v^g$ to $\mathcal{H}_e^g$.

In the domain space write $\mathcal{H}_{\mathrm{rig}}^g$ for the space of ($\pi_v(g)$-invariant) rigid motions in $\mathcal{H}_v^g \subseteq \mathbb{R}^{d|F_v|} \oplus \mathcal{E}_g$ Similarly write $\mathcal{H}_{\mathrm{fl}}^g$ for the space of $\pi_v(g)$-invariant affinely periodic infinitesimal flexes (u, A) in $\mathcal{H}_v^g$ and write $\mathcal{H}_{\mathrm{mech}}^g$ for the orthogonal complement space $\mathcal{H}_{\mathrm{fl}}^g \ominus \mathcal{H}_{\mathrm{rig}}^g$.

In the co-domain space $\mathcal{H}_e^g$ of periodic $\pi_e(g)$-invariant vectors we simply have the subspace $\mathcal{H}_{\rm str}^g$ of $\pi_e(g)$-invariant periodic infinitesimal self-stresses.

Finally let

$$m_g = \dim \mathcal{H}_{\rm mech}^g, \quad s_g = \dim \mathcal{H}_{\rm str}^g, \quad e_g = \dim \mathcal{H}_e^g, \quad f_g = \mathcal{H}_{\rm rig}^g.$$

Noting that $\dim \mathcal{H}_v^g = \dim \mathcal{F}_g + \dim \mathcal{E}_g$ the following symmetry adapted Maxwell-Calladine formula follows from Corollary 3.3.

Theorem 3.5. *Let $\mathcal{C}$ be a crystal framework and let g be a separable element of the point group $\mathcal{G}_{\rm pt}(\mathcal{C})$. Then*

$$m_g - s_g = \dim \mathcal{F}_g + \dim \mathcal{E}_g - e_g - f_g.$$

The right hand side of this equation is readily computable as follows:

(i) $\dim \mathcal{F}_g$ is the dimension of the space of vectors in $\mathbb{R}^{|F_v|} \otimes \mathbb{R}^d$ which are fixed by the linear transformation $\nu_n(g) \otimes T_g$.

(ii) $\dim \mathcal{E}_g$ is the dimension of the commutant of T_g, that is of the linear space of $d \times d$ matrices A with $AT_g = T_g A$.

(iii) e_g is the number of orbits in the motif set F_e under the action of g.

(iv) f_g is the dimension of the subspace of infinitesimal rigid motions in $\mathcal{F}_g$.

In the nonseparable case the above applies except for the splitting of the space $\mathcal{H}_v^g$ of $\pi(g)$-fixed vectors. In this case we obtain the Maxwell-Calladine formula

$$m_g - s_g = \dim \mathcal{H}_v^g - e_g - f_g$$

and $\dim \mathcal{H}_v^g$ is computed by appeal to the full block triangular representation of $\pi_v(g)$. Although somewhat involved this representation is quite explicit as an affine map of $\mathbb{R}^{2|F_v|} \oplus \mathbb{R}^{d^2}$; the formulae of Section 3.1, give

$$\pi_v(g) = \begin{bmatrix} \mu_v(g) & \Phi_1(g) \\ 0 & \Phi_2(g) \end{bmatrix}$$

where $\Phi_2(g)(A) = BAB^{-1}$ and where

$$(\Phi_1(g)(A))_{g\cdot\kappa} = -BA(T_g^{-1} p_{g\cdot\kappa,0} - p_{\kappa,0}).$$

In particular the dimension of the space $\mathcal{H}_v^g$ may be determined computationally and this could well be of use for large motif frameworks whose only nontranslational symmetries are of nonseparable type.

For strictly periodic flexes rather than affinely periodic flexes, that is, for the case $\mathcal{E} = \{0\}$, there is a modified formula with replacement of f_g by the dimension, f_g^{per} say, of the subspace $\mathcal{H}_v^{per}$ of g-symmetric periodic displacement vectors corresponding to rigid motions. Thus for the identity symmetry g we obtain

$$m - s = d|V_f| - |V_e| - d$$

as already observed in Theorem 2.10.

We remark that the restricted transformation $R(\mathcal{M}, \mathcal{E}_g) : \mathcal{H}_v^g \to \mathcal{H}_e^g$ is a coordinate free form of the orbit rigidity matrix used by Ross, Schulze and Whiteley [20] (see also Schulze and Whiteley [23]), where they give counting inequality predictors for infinitesimal mechanisms. A general such formula is given in the next corollary. They also indicate how finite motions (rather than infinitesimal ones) follow if the framework is sufficiently generic, that is, roughly speaking, algebraically vertex-generic apart from the demands of specific symmetry and periodicity.

Corollary 3.6. *Let $\mathcal{C}$ be a crystal framework and let g be a separable element of the point group of $\mathcal{C}$. If*

$$e_g < \dim \mathcal{F}_g + \dim \mathcal{E}_g - f_g$$

then there exists a non-rigid motion infinitesimal flex which is T_g-symmetric.

Proof. Immediate from Theorem 3.5. □

Finally we remark that if (G, p) is an infinite bar-joint framework in $\mathbb{R}^3$ which is generated by the translates of a finite motif under a translation *subgroup* $\{T_k : k \in \mathbb{Z} \subseteq \mathbb{Z}^3\}$ then one can similarly obtain symmetry equations for the rigidity matrices $R(G, p)$. One such example is the hexahedron tower illustrated in the next section.

4. Some crystal frameworks.

We now examine some illustrative frameworks and note in particular that the various Maxwell-Calladine equations give readily computable information. This can be useful given the formidable size of the rigidity matrix for even moderate crystal frameworks.

A motif $\mathcal{M} = (F_v, F_e)$ for $\mathcal{C}$, with specified period vectors, is a geometrical construct that carries all the ingredients necessary for determining of the various rigidity matrices. It may be specifed in more combinatorial terms as a *labelled motif graph*, by which we mean the simple abstract graph $(V(F_e), F_e)$ (with *all* vertices from F_e) together with a vertex labelling given by the (κ, k) labelling and a distance specification $d : F_e \to \mathbb{R}_+$ given by the lengths of the edges. In our discussions, in contrast with [13], [4], [19], we have not been concerned with combinatorial characterisations, nor with realizability, and so have had no need for this abstract form. Even so we note that there are equivalences between such doubly labelled graphs and the formalism of other authors, who incorporate gain graphs (Ross, Schulze, Whiteley [20]), coloured graphs (Malestein and Theran [13]), and quotients of periodic graphs (Borcea and Streinu [3]). For example the gain graph arises from the labelled motif graph $(V(F_e), F_e)$ by merging vertices with the same κ label and assigning to the new edges a direction to one of the vertices, coming from the ancestral vertices (κ, k), (τ, k'), say κ, and assigning to this edge a label, namely $k - k' \in \mathbb{Z}^d$ for the choice of κ (and $k' - k$ for the choice of τ).

4.1. **The kagome framework $\mathcal{C}_{\text{kag}}$.** The kagome framework has been examined in a variety of engineering contexts (see [11] for example). It also features as slices in higher dimensional crystal frameworks such as the kagome net framework of β–cristobalite. Its hexagonal structure is implied by Figure 1 which indicates a motif

$$\mathcal{M} = (F_v, F_e), \quad F_v = \{p_1, p_2, p_3\}, \quad F_e = \{e_1, \dots, e_6\}.$$

Consider a scaling in which the period vectors are

$$a_1 = (1, 0), \quad a_2 = (1/2, \sqrt{3}/2),$$

so that the equilateral triangles of the framework have side length 1/2.

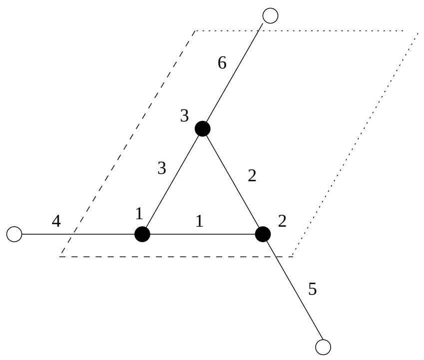

FIGURE 1. Motif for the kagome framework, $\mathcal{C}_{\text{kag}}$.

The motif rigidity matrix $R(\mathcal{M})$ is 6×6 while $R(\mathcal{M}, \mathbb{R}^{d^2})$ is 6×10. Examining the local condition for a self-stress (or by examining the co-kernel of $R(\mathcal{M})$) one soon finds a basis of three vectors in $\mathbb{R}^{|F_e|}$ for the space of periodic self-stresses. In fact it is natural to take the three vectors which assign coefficients of 1 to a colinear pair of edges of F_e and the coefficient 0 to the other four edges.

For strictly periodic vertex displacement velocities and flexes, that is, for the case $\mathcal{E} = \{0\}$, the Maxwell-Calladine equation of Theorem 2.10 takes the form

$$m - s = 2|F_v| + 0 - |F_e| - 2 = -2.$$

Since $s = 3$ we have $m = 1$ and this corresponds to the unique (up to scalar) "alternating rotation mechanism", with the motif triangle infinitesimally rotating about its centre. We may write this in matrix data form as $(u_{\mathrm{rot}}, 0)$

To compute the motif rigidity matrix $R(\mathcal{M})$ and the affine rigidity matrix

$$R(\mathcal{M}, \mathbb{R}^{d^2}) = [R(\mathcal{M}) \quad X(\mathcal{M})]$$

we note that

$$v_{e_4} = (1/2, 0), \quad v_{e_5} = (-1/2, \sqrt{3}/2), \quad v_{e_4} = (-1/2, -\sqrt{3}/2),$$

$$\delta(e_4) = (-1, 0), \quad \delta(e_5) = (1, -1), \quad \delta(e_6) = (0, 1).$$

This leads to the identification of $R(\mathcal{M}, \mathbb{R}^{d^2})$ as the matrix

$$\begin{bmatrix} -1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 0 & 0 & 0 & 0 \\ -1/4 & -\sqrt{3}/4 & 0 & 0 & 1/4 & \sqrt{3}/4 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 & -1/2 & \sqrt{3}/2 & 1/2 & -\sqrt{3}/2 \\ 1/4 & \sqrt{3}/4 & 0 & 0 & -1/4 & -\sqrt{3}/4 & 0 & 0 & -1/2 & -\sqrt{3}/2 \end{bmatrix}.$$

In the affine case, with $\mathcal{E} = M_2(\mathbb{R})$ we now have $f_{\mathcal{E}} = 3$, since infinitesimal rotation is now admissible, and so

$$m_{\mathcal{E}} - s_{\mathcal{E}} = d|F_v| + \dim \mathcal{E} - |F_e| - f_{\mathcal{E}} = 6 + 4 - 6 - 3 = 1.$$

From the affine rigidity matrix it follows readily that there are no linear dependencies among the rows and so $s_{\mathcal{E}} = 0$. (Roughly speaking, no periodic self-stresses remain on admitting all affine motions.) Thus $m_{\mathcal{E}} = 1$ indicating that affine freedom confers no additional (linearly independent) affinely periodic infinitesimal flex beyond infinitesimal rotation. One can check in particular that while there is "purely affine" flex fixing the motif vertices, with matrix data $(0, A)$), this flex is a linear combination of rigid rotation and the alternation rotation $(u_{\mathrm{rot}}, 0)$.

The kagome framework has nonseparable symmetries g such as the isometry $T_g = BT_{(1/2,0)}$ where B is the linear isometry reflecting in the x-axis and where $T_{(1/2,0)}$ is translation by $1/2$ in the x-direction. However there are evident separable symmetries whose Maxwell-Calladine type formulae offer more straightforward computations.

Consider the rotation symmetry g whose isometry T_g is rotation by $\pi/3$ about the centre of the triangle in the motif. The space $\mathcal{E}_g \subseteq M_d(\mathbb{R})$, of the symmetry-adapted Maxwell equation in Theorem 3.5, has dimension 2. Also $f_g = 1$ since now only infinitesimal rotation is admissible as a rigid motion. The space $\mathcal{F}_g$ is the space of fixed vectors for the linear transformation $\mu_v(g) = \nu_n(g) \otimes T_g$ which has the form $S \otimes T_g$ where S is a cyclic shift of order 3. It follows that $\dim \mathcal{F}_g$ is 2. Thus

$$m_g - s_g = \dim \mathcal{F}_g + \dim \mathcal{E}_g - e_g - f_g = 2 + 2 - 2 - 1 = 1.$$

Since we now know that $m_g \leq 1$ it follows that $s_g = 0$. (While there is a g-symmetric self-stress this stress is invalidated by the admission of affine motion and does not contribute to s_g.)

4.2. **A Roman tile framework.** A rather different hexagonal framework is the crystal framework $\mathcal{C}_{\text{Rom}}$ based on a Roman tiling indicated in Figure 2. The parallelogram unit cell encloses 6 framework vertices which we may take to be the motif set F_v. The 8 internal edges together with an appropriate choice of 4 cell-spanning edges provides a possible set F_e. For example we may take these edges to be the two westward edges and the two south-east edges. The 12×16 rigidity matrix $R(\mathcal{M}, \mathbb{R}^{d^2})$ determines the affinely periodic infinitesimal flexes and the associated self-stresses.

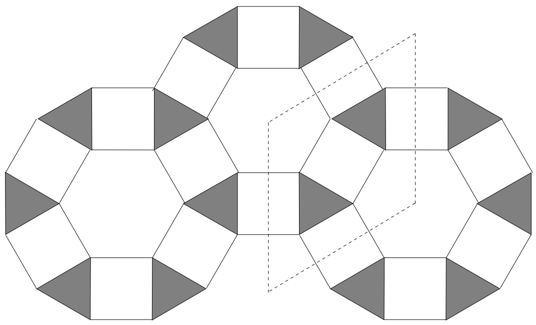

FIGURE 2. Part of the Roman tiling framework $\mathcal{C}_{\text{Rom}}$.

One can verify by examining the stress condition at the vertices in the unit cell that up to a scalar multiple there are two linearly independent nonzero periodic infinitesimal self-stresses providing a basis for the cokernel of $\mathcal{R}(\mathcal{M})$. A fully symmetric self-stress is evident. Additionally there is an independent periodic self-stress $w = (w_{e,k})$ which in the unit cell has coefficient $w_{e,0}$ equal to zero for the two triangle edges that are inward facing.

From Theorem 2.10 the dimension m_{per} of the space of strictly periodic mechanisms ($\mathcal{E} = \{0\}$) satisfies

$$m_{per} - s = d|F_v| + \dim E - |F_e| - 2 = 12 + 0 - 12 - 2 = -2$$

and it follows that (unlike the kagome framework) there are no nontrivial (strictly) periodic infinitesimal mechanisms. On the other hand the unrestricted affine variant Maxwell-Calladine equation for the identity symmetry gives

$$m_{\text{aff}} - s_{\text{aff}} = d|F_v| + d^2 - |F_e| - f_{\text{rig}} = 12 + 4 - 12 - 3 = 1,$$

with $f_{\text{rig}} = 3$ since there are 3 rigid motion flexes since infinitesimal rotation is now included. Thus there exists at least one affinely periodic mechanism.

4.3. **Rigid unit frameworks.** The planar diagram of Figure 3 shows a template for a translationally periodic body-pin framework in the plane with kite shaped rigid unit bodies. This framework is equivalent, from the point of view of rigidity, to a bar-joint framework formed by the edges of the quadrilaterals together with added internal edges and we are at liberty to add all these edges vertically to preserve the reflective and inversion symmetries.

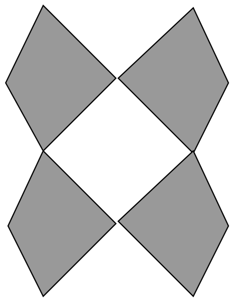

FIGURE 3. Template for $\mathcal{C}_{\text{kite}}$.

That there is an affinely periodic infinitesimal flex may be seen from the symmetry adapted Maxwell-Calladine equation for inversion symmetry g. Note that one may choose a motif set F_v which is g-symmetric. Also the vertex motif vector space $\mathcal{F}_g$ is the sum of four 2 dimensional spaces. The commutant space $\mathcal{E}_g$ for the linear inversion $T_g : (x, y) \to (-x, -y)$ is the set of all matrices and so has dimension 4. The number e_g of edge classes (including two classes for the added vertical edges) is 10, while the number of independent inversion symmetric rigid (congruence) motions f_g is 1, for rotation. It follows that

$$m_g - m_s = \dim \mathcal{F}_g + \mathcal{E}_g - e_g - f_g = 8 + 4 - 10 - 1 = 1$$

and so $m_g \geq 1$. In fact $m_g = 1$ and there are no inversion symmetric periodic self-stresses (for the given periodicity).

4.4. **The hexahedron framework $\mathcal{C}_{\text{Hex}}$.** A hexahedron in $\mathbb{R}^3$ with equilateral triangle faces and distinct vertices may be formed by joining two tetrahedra at a common face. Let $\mathcal{C}_{\text{Hex}}$ be the hexahedron crystal framework in which such hexahedral subunits are vertically aligned and horizontally connected in regular triangular fashion. Thus the polar vertices have degree 6 while the equatorial vertices have degree 12. The framework is determined by a single subunit and the period vectors, assuming unit length sides, are given by

$$a_1 = (1, 0, 0), \quad a_2 = (1/2, \sqrt{3}/2, 0) \quad a_3 = (0, 0, 2h),$$

where $h = \sqrt{2}/\sqrt{3}$.

A motif $\mathcal{M} = (F_v, F_e)$ for $\mathcal{C}_{\text{Hex}}$ may be formed from a single hexahedron unit, with $F_v = \{p_1, p_2\}$, a choice of equatorial vertex $p_1 = (0, 0, 0)$ and south polar vertex $p_2 = (1/2, \sqrt{3}/6, -h)$, and with F_e the set of nine edges $\{e_1, \ldots, e_9\}$. In particular the motif rigidity matrix $R(\mathcal{M})$ is 9×6 while the affine rigidity matrix $R(\mathcal{M}, \mathbb{R}^9)$ is 9×15. The framework is periodically infinitesimally rigid, essentially since the unit cell is occupied by the single rigid hexahedron unit. In fact one can compute the RUM spectrum in various ways (see the methods in [18]) and this reveals that $\Omega(\mathcal{C}_{\text{Hex}})$ is as trivial as it can be, being the singleton set $\{(1, 1, 1)\}$ corresponding to translational flexes. In particular there are no supercell periodic infinitesimal flexes.

Despite having up-down reflective symmetry and rotational symmetry the symmetry adapted Maxwell-Calladine formulae are inconclusive for the existence of affinely periodic infinitesimal flexes for the given periodicty.

4.5. **The hexahedron tower.** We define the hexahedron tower framework $\mathcal{G}_{\text{Hex}}$ as the one-dimensionally translationally periodic framework in $\mathbb{R}^3$ formed by stacking identical hexahedron 3-rings infinitely in both directions, as indicated in Figure 4.

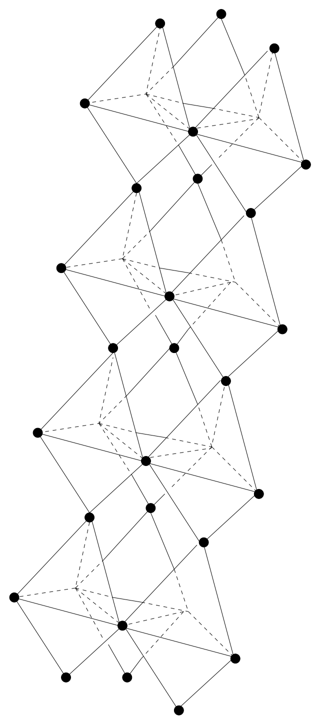

FIGURE 4. Part of the hexahedron tower.

A single hexahedron 3-ring has 3 degrees of freedom beyond the 6 isometric motion freedoms. Indeed, if the connecting triangle is fixed, removing all continuous spatial motion possibilities, then each of the three hexahedron units can flex independently. Because of the connections between the hexahedron 3-rings it follows that any finite tower also has 3 degrees of freedom in this sense.

The hexahedron tower has an evident rotationally symmetric continuous flex $p(t)$ in which the triples of degree 6 vertices alternately move a finite distance towards or away from the central axis. Viewing a single 3-ring as the unit cell this flex is flow-periodic with respect to 2-cell periodicity and with respect to affine contraction along the central axis. The associated infinitesimal flex $p'(0)$ is an affine (2-cell) periodic infinitesimal flex of contractive affine type.

Such flexes are characterised by the property that the connecting vertex triples lie on planes which are parallel, as we see below. Moreover we have the following curious dichotomy.

Proposition 4.1. *Let $p(t), t \in [0,1]$, be a continuous flex of the hexahedron tower $\mathcal{G}_{\mathrm{Hex}} = (G_{\mathrm{Hex}}, p)$ which fixes the connecting triangle of a single hexahedron 3-ring. Then either $(G_{\mathrm{Hex}}, p(t))$ is axially rotationally symmetric for all t or for some $s \in (0,1]$ the framework $(G_{\mathrm{Hex}}, p(s))$ is bounded.*

Proof. Assume first that the framework $(G_{\mathrm{Hex}}, p(s))$ has a hexahedron 3-ring whose polar triples determine planes which are not parallel. The position of one polar triple in $\mathbb{R}^3$ determines (uniquely) the position of the next triple while the subsequent triple is determined by reflection of the first triple in the plane through the second triple. From this reflection principle it follows that these planes and all subsequent planes, similarly identified, pass through a common line. Also it follows that $(G, p(s))$ is a bounded framework which circles around this line. (We are assuming here that self-intersections are admissible.) If the angle between consecutive planes happens to be a rational multiple of 2π then the vertices of the deformed tower occupy a finite number of positions with infinite multiplicity.

To complete the proof it suffices to show that if a single hexahedron 3-ring (H, p) is perturbed so that the plane of the three south pole vertices p_1', p_2', p_3' is parallel to the plane of the three north pole vertices p_4', p_5', p_6', then the 3-ring is rotationally symmetric (about the line through the centroid of the connecting triangle of equatorial vertices).

Consider the three-dimensional manifold of "flex positions" (H, p') for which $p_1' = p_1$ is fixed, p_2'lies on a fixed open line segment between p_1 and p_2, and p_3' has two degrees of freedom in a small open disc in the fixed plane through p_1, p_2, p_3. The parallelism condition corresponds to the two equations

$$\langle p_4' - p_3', (p_2 - p_1) \times (p_3 - p_1) \rangle = 0, \quad \langle p_5' - p_3', (p_2 - p_1) \times (p_3 - p_1) \rangle = 0.$$

It follows that there is a one-dimensional submanifold of positions p' with parallel triple-point planes. Since there is a one-dimensional submanifold of rotationally symmetric positions these positions provide all the positions p', as required. □

By reversing the time parameter t in the proof above, and assuming a decrease of the angle to zero at time $t = 1$, one obtains a curious continuous "unwrapping flex", from a bounded "rationally wrapped" hexahedron tower at time $t = 0$ to the unbounded tower $\mathcal{G}_{\rm Hex}$ at time $t = 1$, with bounded frameworks at each intermediate time.

More "realistically", if we regard the hexahedron units as solid rigid units, not admitting self-intersections, then it follows that the hexahedron tower is essentially uniquely deformable with axial symmetry. Similarly, since the hexahedron towers in the crystal framework $\mathcal{C}_{\rm Hex}$ cannot simultaneously flex with axial symmetry it follows readily that the solid unit framework $\mathcal{C}_{\rm Hex}$ admits no nontrivial continuous deformation.

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