Some interactions between operator theory and control engineering

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The theory of operators on Hilbert space (1906 - ) is a tool for studying functions, hence has applications in physics and engineering.

Examples of operators

Laplacian, Dirac
Multiplication, Toeplitz
Controllability, observability
Hilbert space

In this talk, a Hilbert space will be a space of functions closed under addition and scalar multiplication and having a scalar product $\langle f, g \rangle$.

Example: $L^2(0, \infty)$ consists of all functions $f$ on the positive real axis $(0, \infty)$ such that

$$\int_0^\infty |f(t)|^2 \, dt < \infty$$

with scalar product $\langle f, g \rangle = \int_0^\infty f(t) \overline{g(t)} \, dt$.

An operator $T$ is a linear transformation on a Hilbert space: $T(f + g) = Tf + Tg$, $T(\alpha f) = \alpha Tf$.

There are theories of eigenvalues, eigenvectors, diagonalization, ...
Hankel operators

These are operators on an infinite-dimensional Hilbert space with a matrix of the form

$$
\Gamma = \begin{bmatrix}
a_1 & a_2 & a_3 & \ldots \\
a_2 & a_3 & a_4 & \ldots \\
a_3 & a_4 & a_5 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

with respect to some orthonormal basis.

$\Gamma$ is a bounded operator if and only if the $a_j$ are the negative Fourier coefficients of some bounded measurable function on the circle.

The operator $H : L^2(0, \infty) \rightarrow L^2(0, \infty)$,

$$(Hf)(x) = \int_{0}^{\infty} f(t) h(x+t) \, dt,$$

for a suitable $h$, is a Hankel operator.
Pick's Theorem

Given $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$ with $\text{Re} \ z_j > 0$, $|w_j| \leq 1$, $j = 1, \ldots, n$, does there exist $S \in H^\infty$ such that $S(z_j) = w_j$ for $j = 1, 2, \ldots, n$ and $|S(s)| \leq 1$ for all $s \in \text{RHP}$?

G. Pick (1917) proved: yes if and only if

$$\left[\frac{1 - \bar{w}_i w_j}{z_i + \bar{z}_j}\right]_{i,j=1}^n \geq 0.$$

Recall: for a square matrix $M$,

$M \geq 0$ means $x^* M x \geq 0$ for every column vector $x$. 
More Pick-type questions

Describe (parametrize) all solutions $S$.

When does there exist a rational interpolating function $S$ with at most 3 poles in RHP bounded by 1 on the imaginary axis?

[Answer: if and only if Pick's matrix has at most 3 negative eigenvalues.]

The same questions for matrix-valued functions. All these questions and many more were answered by Adyan, Arov and Krein in a series of papers c. 1970 on Hankel operators.
A sample AAK theorem

Let $z_1, \ldots, z_n, w_1, \ldots, w_n$ be data for which Pick's interpolation problem is solvable. There exist functions $a, b, c, d \in \mathcal{H}^\infty$ such that the solutions of the problem are precisely the functions

$$S = \frac{ag + b}{cg + d}$$

where $\|g\|_{\mathcal{H}^\infty} \leq 1$.

$a, b, c$ and $d$ can be calculated explicitly in terms of the leading singular value and vectors of a Hankel operator constructed from the data.
Linear systems and stability

Consider a plant (physical device) modelled by a linear constant-coefficient differential eq’n

\[ p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u, \quad y(0) = y_0 \]

where

\[ u(t) \in \mathbb{R}^n \] is the input at time \( t \)
\[ y(t) \in \mathbb{R}^p \] output at time \( t \).

After Laplace transformation

\[ Y(s) = G(s)U(s) + \text{initial terms} \]

James Clerk Maxwell: the system is stable if and only if all poles of the rational function \( G(s) \) lie in the open left half plane \( \{ \text{Re} s < 0 \} \).
Stabilization

Let $G(s)$ be an unstable plant.

\[
\begin{align*}
    u & \rightarrow u - Ky \\
    \downarrow \downarrow & \downarrow \\
    G \rightarrow y \\
    \uparrow \rightarrow K \\
    Ky & \rightarrow y
\end{align*}
\]

Design a plant $K(s)$ (a "controller") such that the above feedback loop is stable.

\[
y = G(u - Ky)
\]

\[
\therefore y = (1 + GK)^{-1} Gu.
\]

Example: $G(s) = \frac{1}{s - 4}$, $K(s) = 5$

\[
(1 + GK)^{-1}G = \left(\frac{s + 1}{s - 4}\right)^{-1} \frac{1}{s - 4} = \frac{1}{s + 1}
\]

$G$ is unstable but the "closed loop" is stable.
Interpolation

Suppose $K$ stabilizes the plant $G$ — that is,

$$(1 + GK)^{-1} G \in H^\infty,$$

\[ H^\infty \overset{\text{def}}{=} \{ \text{functions bounded and analytic in } \text{RHP} \}, \]
\[ \text{RHP} \overset{\text{def}}{=} \{ s \in \mathbb{C} : \text{Re } s > 0 \}. \]

The sensitivity function of the closed loop is

$$S \overset{\text{def}}{=} (1 + GK)^{-1}.$$

$S$ satisfies

\[ S(z) = 1 \text{ for every zero } z \text{ of } G \text{ in RHP} \] \hspace{1cm} (*)
\[ S(p) = 0 \text{ for every pole } p \text{ of } G \text{ in RHP}. \]

Finding $K$ — finding $S \in H^\infty$ satisfying $(*)$.

The sensitivity minimization problem: find $S$ of min.

$\| S \|_{H^\infty}$
Two cultures meet

Important objectives for a closed loop control system are robustness in the face of modelling uncertainty and low sensitivity to sensor noise and external disturbances.

Circa 1980 G. Zames and other engineers developed an approach characterized by worst-case (in contrast to stochastic) criteria for precise, if abstract, formulation of objectives in terms of norms and function spaces.

E.g., choose $K$ to stabilise not only $G$ but also $G + \Delta G$ for all $\Delta G \in H^\infty$ of norm $< \varepsilon$.

J. W. Helton made the connection to the work of AAK.
$H^\infty$ control


The space $H^\infty$ of bounded analytic functions in RHP and the $H^\infty$ norm

$$\|f\|_{H^\infty} = \sup_{\text{Re } s > 0} \|f(s)\|, \quad \|A\| = \lambda_{\max}(A^*A)^{1/2},$$

play a central role. $\|\cdot\|_{H^\infty}$ is physically significant, and is mathematically tractable, by the methods of operator theory.

$\|\cdot\|_{H^\infty}$ measures the sup of the ratios of the energies of output to input signals.

Cambridge U. Engineering Dept. - K. Glover
Benefits for operator theory

Many interesting and significant new problems.

New insights and perspectives on familiar objects.

State space realizations.

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

If \( G \in H_{p \times n}^\infty \) and \( \| G \|_{H^\infty} \leq 1 \) then there is a Hilbert space \( X \) and a contractive operator

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} : \begin{bmatrix} X \\ C^n \end{bmatrix} \rightarrow \begin{bmatrix} X \\ C^p \end{bmatrix}
\]

such that

\[
G(s) = D + C(sI - A)^{-1}B \quad \forall s \in \mathbb{R}^H.
\]
Linear matrix inequalities

Example. Given real $n \times n$ matrices $a, b, c$ with $c = c^T$, find a positive definite matrix $x$ such that

$$b(ax + xa^T + c)b^T < 0. \quad (LMI)$$

Theorem (1994)

For a plant $G(s) = D + C(sI - A)^{-1}B$ there exists a stabilizing controller with sensitivity function $S$ such that $\|S\|_{H\infty} \leq 1$ if and only if there exists a positive definite matrix $P$ such that $P$ and $P^{-1}$ are solutions of two LMIs with coefficients formed from $A, B, C, D$. 
Robust stabilization

Construct a controller that stabilizes a given plant $G$ and all nearby plants (and achieves...).

One way to model uncertainty: the true plant is

\[ (1 + G\Delta)^{-1}G \]

where the "uncertainty" or "modelling error" $\Delta$ is assumed only to be small in some $L^\infty$-type norm.

Find the largest possible $\varepsilon > 0$ such that there is a controller that stabilizes the above plant for all $\Delta$ with $\|\Delta\| < \varepsilon$ — a *Pick*-type interpolation problem.
Structured uncertainty

It can happen that we know something about the structure of the modelling error $\Delta$ - e.g. that the off-diagonal entries are exactly zero.

An extreme example: $\Delta(s) \in \mathbb{C}I$ for all $s$.

This leads to the spectral Nevanlinna-Pick problem:

given $z_1, \ldots, z_n \in \text{RHP}$ and $W_1, \ldots, W_n \in \mathbb{C}^{k \times k}$,

find if possible an analytic $F: \text{RHP} \to \mathbb{C}^{k \times k}$ such that

$$F(z_j) = W_j \quad \text{for } j = 1, \ldots, n \text{ and all eigenvalues of } F(s) \text{ lie in the disc } \{ z : |z| < 1 \} \forall s \in \text{RHP}.$$ 

Unsolved!

Solution in the case $n = k = 2$ (J. Agler - NYJ) by operator theory and several complex variables.
An interpolation theorem

Let \( z_1, z_2 \in \text{RHP} \)
\[ W_1, W_2 \in \mathbb{C}^{2 \times 2} \text{ be non-scalar} \]

\[ s_j = \text{tr} \ W_j, \quad p_j = \det W_j \text{ for } j = 1, 2. \]

There exists analytic \( F : \text{RHP} \to \mathbb{C}^{2 \times 2} \) such that

\[ F(z_1) = W_1, \quad F(z_2) = W_2 \text{ and all eigenvalues of } F(s) \text{ lie in } \{ z : |z| < 1 \} \text{ for all } s \in \text{RHP} \]

if and only if, for all \( \omega \) such that \( |\omega| = 1 \),

\[
\left[ \frac{2(1 - \bar{p}_i p_j) - \omega (s_j - \bar{s}_i p_j) - \bar{\omega} (\bar{s}_i - \bar{p}_i s_j)}{z_j + \bar{z}_i} \right]_{i,j=1}^2 \geq 0.
\]

What if there are 3 interpolation points?

What about \( 3 \times 3 \) matrices \( W_j \)?
Multidimensional systems

An engineering subculture that should connect to operator theory and several complex variables.

Consider a plant $G$ that is a function of 2 frequency variables $\delta, s \in \text{RHP}$. Regard $\delta$ as uncertainty.

Construct a controller $K(\delta, s)$ such that $K(\delta, \cdot)$ stabilises $G(\delta, \cdot)$ for all $\delta \in \text{RHP}$ and

$$\| (1 + G(\delta, s) K(\delta, s))^{-1} \| \leq 1$$

for all $\delta, s \in \text{RHP}$.

This problem also has an LMI solution.

Ripe for interaction with pure analysis?