The Rigidity of Graphs on a Flexible Torus

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Rigidity of Frameworks and Applications, Lancaster University
A movie
Goal of the work

- Study infinite periodic frameworks, with periodic deformations
- when is such a framework *rigid*, so that its parts cannot be moved with respect to one another?

Infinite graphs present challenges:
- row rank $\neq$ column rank
- Owen and Power, 2009, operator theory methods

We use a finite graph that captures the periodic structure of our infinite graph: *gain graphs aka voltage graphs*
Gain Graph $\langle G, m \rangle$

- directed multigraph $G = (V, E)$
- Gain assignment $m : E^+ \to \mathbb{Z}^n$
- Gain group $\mathbb{Z}^n$

Derived Graph $G^m$

- Vertices: $V \times \mathbb{Z}^n$
- Edges: $E \times \mathbb{Z}^n$
- edges determined by gains
Periodic framework $(\langle G, m \rangle, \mathbf{p})$

- **Periodic framework** $(\langle G, m \rangle, \mathbf{p})$
- $T^n = \mathbb{R}^n / \mathbb{Z}^n = [0, 1)^n$: a *fundamental region* for a tiling of $n$-space
Different versions of the problem in two dimensions

• When is a graph on the torus rigid?

\[(y_1(t), y_2(t))\]

\[(x(t), 0)\]

What torus?

1. **Fixed torus** fixed generators \(T_0^2\)
2. **Flexible torus**
   - allow one direction to scale. \(T_1^2\)
   - allow both generators to scale, but fix angle between them. \(T_2^2\)
   - allow full motion. \(T_3^2\)
   - other variations such as fixed area.
Periodic Infinitesimal Rigidity on $\mathcal{T}_0^2$

- Infinitesimal periodic motion: $u : V \rightarrow \mathbb{R}^2$ s.t. 
  $$(u_i - u_j) \cdot (p_i - (p_j + m_e)) = 0 \text{ for all } e = (i, j, m_e) \in E(G, m)$$

- a framework $\mathcal{F}$ is called infinitesimally periodic rigid if the only infinitesimal motions of the framework are infinitesimal isometries.

- The space of periodic infinitesimal motions of $(\langle G, m \rangle, p)$ is the kernel of the $|E| \times 2|V|$ periodic rigidity matrix

- $\mathcal{F}$ is infinitesimally periodic rigid $\iff$ $\text{Rank}(R_{\langle G, m \rangle}(p)) = 2|V| - 2$.

- (periodic) isostatic: (minimally) infinitesimally rigid, $|E| = 2|V| - 2$
• “Almost all" $p : V \rightarrow T^2$ are (periodic) generic.
Gain Assignments and Rigidity

- What gain assignments $m$ fail to produce rigid frameworks for any embedding $p$?

\begin{align*}
(0, 1) & \quad (1, 0) \\
(1, 0) & \quad (0, 1)
\end{align*}

infinitesimally flexible

infinitesimally rigid
Gain Assignments and Rigidity

- What gain assignments $m$ fail to produce rigid frameworks for any embedding $p$?

**Proposition**

Let $\langle G, m \rangle$ be a gain graph with gain group $\mathbb{Z}^2$, and $|E| = 2|V| - 2$. If $\langle G, m, p \rangle$ is infinitesimally rigid for any embedding $p$, then for all subgraphs $\langle G', m' \rangle \subseteq \langle G, m \rangle$ with $|E'| = 2|V'| - 2$, there is a cycle with non-zero net gain.

- Call a gain assignment $m$ satisfying this condition *constructive*.

![Inf. rigid and inf. flexible graphs](image)
Constructive Gains Determine Rigidity on $\mathcal{T}_0^2$

**Theorem**

Let $\langle G, m \rangle$ be a gain graph satisfying

(i) $|E| = 2|V| - 2$ and $|E'| \leq 2|V'| - 2$ for all $G' \subseteq G$,

(ii) $m : E^+ \to \mathbb{Z}^2$ is constructive.

Then $(\langle G, m \rangle, p)$ is infinitesimally periodic rigid for any generic embedding $p$.

- Proof uses periodic inductive constructions on the gain graph $\langle G, m \rangle$:

  - **Vertex addition:**
  - **Edge splits:**
Example

- Example: $|V| = 2, |E| = 2$
  - constructive gain assignment $\Rightarrow$ gains are not equal

- avoid non-generic embeddings $p$

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Theorem

For a multigraph \( G = (V, E) \), the following are equivalent:

(i) \( G \) is the union of 2 edge-disjoint spanning trees

(ii) \( G \) satisfies \( |E| = 2|V| - 2 \) and every (induced) subgraph \( G' \subset G \) satisfies \( |E'| \leq 2|V'| - 2 \) (Tutte, Nash-Williams, 1961)

(iii) for some gain assignment \( m \) and some embedding \( p \), the framework \( \mathcal{F} = (\langle G, m \rangle, p) \) is periodic isostatic on \( T_0^2 \) (Whiteley, 1988)

(iv) \( G \) can be constructed from a single vertex by a sequence of periodic vertex 2-additions and edge 2-splits

(v) for all constructive gain assignments \( m \) and for all generic embeddings \( p \), \( \mathcal{F} = (\langle G, m \rangle, p) \) is periodic isostatic on \( T_0^2 \)
The $2|V| - 2$ pebble game algorithm can be modified to check for infinitesimal rigidity on the fixed torus.

That is, we can check any gain graph $\langle G, m \rangle$ for rigidity as a periodic framework.
Corollary: The cylinder

- finite (not periodic) infinitesimal rigidity of frameworks on a flat cylinder (geodesic distance) has the same characterization as $T_0^2$, when we allow the gain group to be just $\mathbb{Z}$.

- periodic cylinder frameworks with fixed fundamental region have the same characterization with the gain group $\mathbb{Z}^2$.
The Flexible Torus

infinitesimally rigid on torus with fixed generators:

\[ |E| = 2|V| - 2 \]

want to more accurately model frameworks like the Kagome lattice:

\[ |E| = 2|V| \]

• Start by allowing one of the generators of the torus to scale: \( T_x^2 \)
• Rigidity matrix: add one column for the flexible generator \( x(t) \)
• We require \( |E| = 2|V| - 1 \) edges.
A full characterization for generic rigidity on $\mathcal{T}_x^2$.

Flexible torus with one degree of freedom: $\mathcal{T}_x^2$ is fixed in the $y$-direction, variable in the $x$-direction.

**Theorem**

Let $\langle G, m \rangle$ be a gain graph with $|E| = 2|V| - 1$. Then $\langle G, m \rangle$ is generically isostatic on $\mathcal{T}_x^2$ if, and only if it satisfies:

1. Any pair of spanning trees is generically infinitesimally rigid as a graph on the fixed torus $\mathcal{T}_0^2$.

2. $G$ admits a decomposition into two edge disjoint connected, spanning subgraphs: a tree and a map-graph, in which the cycle part of the map-graph is $x$-constructive.

map-graphs:
T-gains

- **T-gains**: tool to make cycles in a graph more accessible.
  - preserves rank of rigidity matrix

\[
\langle G, m \rangle \quad \langle G, m_T \rangle
\]

\[
\begin{align*}
(1, 0) & \quad (0, 1) \\
(0, 0) & \quad (0, -1)
\end{align*}
\]

- graph isomorphism between \(G^m\) and \(G^m_T\)
- not an affine transformation
- same rigidity properties
Corollary: Periodic frameworks on the cylinder

- The results on $\mathcal{T}_x^2$ also characterize periodic frameworks with flexible fundamental region on a cylinder. *(fixed circumference)*

- OR: frameworks (not periodic) on a cylinder of flexible circumference

LaTeX code:
```
\begin{itemize}
    \item The results on $\mathcal{T}_x^2$ also characterize periodic frameworks with flexible fundamental region on a cylinder. *(fixed circumference)*
    
    \begin{center}
        \includegraphics[width=0.5\textwidth]{cylinder.png}
    \end{center}

    flexible period

    \item OR: frameworks (not periodic) on a cylinder of flexible circumference
    
    \begin{center}
        \includegraphics[width=0.5\textwidth]{cylinder2.png}
    \end{center}

    flexible circumference
\end{itemize}
```
The (More) Flexible Torus

infinitesimally rigid on torus with variable $x$-generator:

\[ |E| = 2|V| - 1 \]

- Allowing both of the generators of the torus to scale: $T_2^2$
- Rigidity matrix has a column for both flexible generators $x(t)$ and $y(t)$
- We require $|E| = 2|V|$ edges.
The Flexible Torus $\mathcal{T}_2^2$: Examples

$|E| = 2|V|$

$\langle G, m \rangle$

$G^m$ inf. rigid

$\langle G, m \rangle$

$G^m$ inf. flexible

inf. motion
Necessary conditions for generic rigidity on $\mathcal{T}_2^2$

- $|E| = 2|V|$ and $|E'| \leq 2|V'|$ for all subgraphs $G' \subset G$.

$\Leftrightarrow$ The edges have a decomposition into two spanning map-graphs.

Bases of the bicircular matroid on the edges of $G$

- Two connected spanning map-graphs: one x-constructive, one y-constructive.
Further questions

Ongoing projects:
- Algorithms for checking conditions for flexible torus
- Higher dimensions
- Bar-body frameworks: have a rigidity matrix, some necessary conditions, some sufficient conditions

Still more questions:
- Symmetry adapted periodic results.
- Discrete scaling of fundamental region...
- Global rigidity on the torus, tensegrity frameworks on the torus
- Non-periodic frameworks?
- Extension to other metrics
Scaling the fundamental region

Under what conditions will discrete scaling of the fundamental region of a periodic framework maintain its *generic* rigidity properties?

![Diagram showing inf. rigid, inf. flexible, and inf. rigid regions with arrows and coordinates](image)
Lemma

Let \( \langle G, m \rangle \) be a gain graph with \( |E| = n|V| - n \), and gain group \( \mathbb{Z}^n \). If \( \langle G, m \rangle, p \) is infinitesimally rigid for any embedding \( p \), then every subgraph \( G' \subseteq G \) with \( |E'| = n|V'| - n \) has a local gain group isomorphic to either \( \mathbb{Z}^{(n-1)} \) or \( \mathbb{Z}^n \).