MATH 550: The Probability Integral Transform

Simulation and Transformation

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This forms part of the MSc in Statistics at Lancaster University

Revision of Uniform(0,1)

$U \sim Unif(0,1)$ is a (continuous) random variable uniformly distributed between 0 and 1.

The pdf is $f_U(u) = 1$ if $0 \leq u \leq 1$, and $f_U(u) = 0$ otherwise.

The cdf is

$$F_U(u) = \begin{cases} 0 & \text{if} \quad u < 0 \\ u & \text{if} \quad 0 \leq u \leq 1 \\ 1 & \text{if} \quad u > 1. \end{cases}$$

Sketch these.

A coin toss

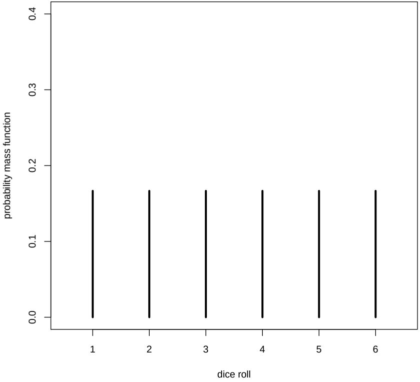
Suppose you wish to simulate the toss of a coin, but all that you are able to do is simulate uniform random numbers between 0 and 1, $Unif(0, 1)$.

You could say "if $U \leq 0.5$ then the coin is a head, and otherwise it is a tail".

Why is this reasonable?

A dice roll

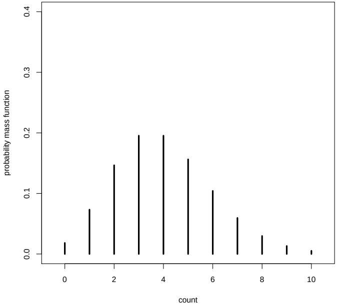
Now suppose you wish to simulate the roll of a dice D but, again, you are only able to simulate $U \sim Unif(0,1)$



Write a sensible set of conditions for D.

A Poisson(4) random variable

Next simulate from a Poisson(4) random variable P using only $U \sim Unif(0,1)$.

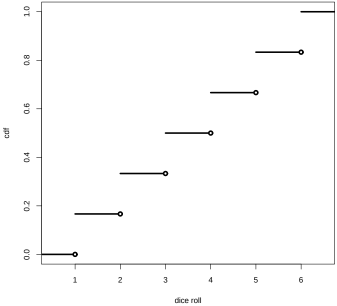


$P(P=0) \approx 0.0183$, $P(P=1) \approx 0.0733$, $P(P=2) \approx 0.1465$, $P(P=3) \approx 0.1954$ and $P(P=4) \approx 0.1954$...

Write down the decision rule for the first few values of P.

Using the CDF - dice

The CDF for a dice roll is (*Make sure you understand why!*)



In matching the probabilities we performed the following calculations

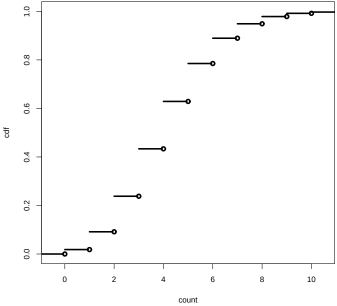
$$P(D=1) = P(D \leq 1) \quad = \quad P(U \leq 1/6)$$

$$P(D=2) = P(D \leq 2) - P(D \leq 1) \quad = \quad P(U \leq 2/6) - P(U \leq 1/6)$$

$$\cdots \qquad \cdots$$

Using the CDF - Poisson(4)

The CDF for a Po(4) r.v. is



In matching the probabilities we performed the following calculations

$$
\begin{aligned}
P(P=0) = P(P \leq 0) &= P(U \leq 0.0183) \\
P(P=1) = P(P \leq 1) - P(P \leq 0) &= P(U \leq 0.0916) - P(U \leq 0.0183) \\
P(P=2) = P(P \leq 2) - P(P \leq 1) &= P(U \leq 0.2381) - P(U \leq 0.0916)
\end{aligned}
$$

Simulating from a continuous r.v.

Above we made P and U correspond so that

$$F_P(p) = F_U(u).$$

(NB we implicitly forced the probability of non-integer P to be zero).

For a continuous random variable the probability of any given value (e.g. 0.817) is 0.

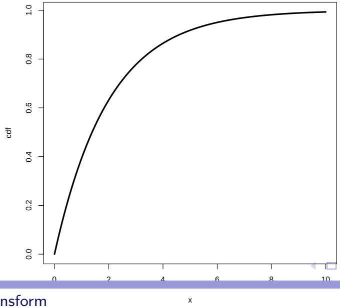
To simulate from a continuous random variable given $U \sim Unif(0,1)$ we *match the CDFs at all values*.

Simulating an Exp(2)

Suppose that you wish to simulate $Y \sim Exp(2)$ using only $U \sim Unif(0,1)$.

The pdf is $f_Y(y) = 1/2\ e^{-y/2}$ $(y \geq 0)$ and $f_Y(y) = 0$ $(y < 0)$ (*sketch it*).

The cdf is $F_Y(y) = 1 - e^{-y/2}$ $(y \geq 0)$ and $F_Y(y) = 0$ $(y < 0)$ (*check you understand this.*)



Simulating an Exp(2)

We must match the CDFs of Y and U.

$$\begin{aligned} P(Y \leq y) &= P(U \leq u) \\ F_Y(y) &= F_U(u) \\ 1 - e^{-y/2} &= u \\ y &= -2\log(1-u) \end{aligned}$$

So if for example you simulated $u = 0.843$ then your $Exp(2)$ random variable is $-2\log(1 - .843) = 3.703$.

This is (almost) exactly how `R` simulates exponential rvs.

A more complex example

The random variable X has density function

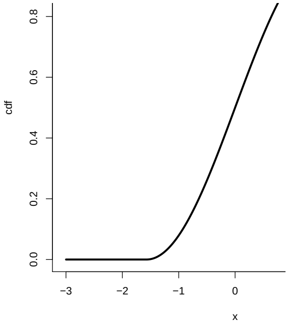
$$f_X(x) = \begin{cases} 0 & \text{if} \quad x <= -\pi/2 \\ \frac{1}{2}\cos x & \text{if} \quad -\pi/2 < x \leq \pi/2 \\ 0 & \text{if} \quad \pi/2 < x \end{cases}$$

How would you simulate from it? (Why the factor of $1/2$?).
Important: first sketch (or plot) the density function.

The cdf

The cumulative distribution function is

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x \leq -\pi/2 \\ \frac{1}{2}\sin x + \frac{1}{2} & \text{if} \quad -\pi/2 < x \leq \pi/2 \quad (\textit{why?}) \\ & \text{if} \quad \pi/2 < x \end{cases}$$



Simulation

Simulate u, then set

$$\frac{1}{2} sinx + \frac{1}{2} = u \Leftrightarrow x = \sin^{-1}(2u - 1).$$

Normal distribution

Suppose that `rnorm()` is broken, but that the following functions work.

`runif()`, `dnorm()`, `pnorm()`, `qnorm()`.

How could you use two of the above functions to simulate a $Z \sim N(0, 1)$?

NB This is NOT how `R` simulates normal random variables - there are several much faster algorithms including the Box-Muller algorithm (just beyond the scope of this course).

The Probability Integral Transform

For any continuous random variable X and $k \in [0, 1]$,

$$P\left(F_X(X) \leq k\right) = k.$$

Proof *Students!*

i.e. $A := F_X(X) \sim Unif(0, 1)$.
We can convert from any continuous distribution to a uniform random variable - this transformation has the grand title of **The Probability Integral Transform**.

The method of simulation that we have studied is called the **inverse transformation method** because we have to solve $x = F_X^{-1}(u)$.

PP-plots

A student collects the following 15 data values, $\mathbf{x}$:
234,264,214,151,321,112,255,160,235,296,238,244,226,222,94.

Do they follow a Normal distribution?

If the data are Normal then $\hat{\mu} = \overline{x} = 215.9$ and $\hat{\sigma}^2 = s^2 = 4057.8$.

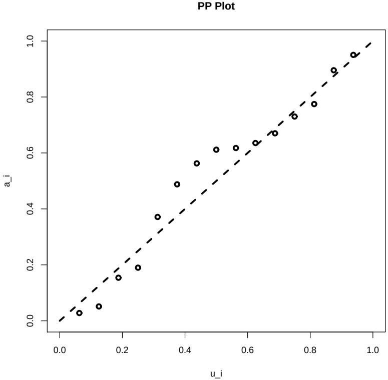
If the data are Normal then $A_i = F(X_i) \approx \Phi((X_i - \hat{\mu})/\hat{\sigma})$ should be uniformly distributed in $[0, 1]$.

Sort the a_i so that a_1 is the smallest and a_{15} is the largest.

If the a_i are uniformly distributed then this should be true:
$a_1 \approx u_1 = 1/16$, $a_2 \approx u_2 = 2/16$ ... $a_{15} \approx u_{15} = 15/16$.

Draw a picture to see why.

A plot of a_i against u_i is called a **PP Plot**.



NB(1) We can obtain confidence bands via the bootstrap.
NB(2) QQ plots (x_i vs $F^{-1}(u_i)$) are more useful for examining the tails of a distribution (read about them in your own time).

Copula Transformations

It is sometimes useful to be able to transform a random variable with one distribution into a random variable with another.

e.g. How do I turn a $N(1, 4)$ random variable Z into an Exp(2) random variable, Y?

By the PIT, $Z \sim N(1, 4) \Leftrightarrow U := \Phi((Z - 1)/2) \sim Unif(0, 1)$.

We already know how to simulate an Exp(2) from a Unif(0,1): $Y = -2 \log(1 - U)$.

The general formula for turning a continuous random variable X with cdf $F_X(x)$ into a continuous random variable Y with cdf $F_Y(y)$ is: