

## Evaluation of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by a double integral

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Let  $\sum_{n=1}^{\infty} \frac{1}{n^2} = S$ . Numerous proofs are known of the well-known identity  $S = \pi^2/6$ . Robin Chapman's website [1] contains a very useful collation of 14 such proofs. Here we give a short one based on a certain double integral. The first two proofs in [1] give alternative double integral proofs from [2, 3] and another can be seen in [4]. Note first that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{3}{4}S.$$

We consider the double integral  $I = \int_0^1 J(x) dx$ , where

$$J(x) = \int_0^{\pi/2} \frac{\sin \theta}{(1 - x^2 \sin^2 \theta)^{1/2}} d\theta.$$

First we evaluate  $J(x)$ . Let  $x^* = (1 - x^2)^{1/2}$ , so that  $x^2 + x^{*2} = 1$ . Then  $1 - x^2 \sin^2 \theta = x^{*2} + x^2 \cos^2 \theta$ . Substitute  $t = x \cos \theta$  to obtain

$$J(x) = \frac{1}{x} \int_0^x \frac{1}{(x^{*2} + t^2)^{1/2}} dt = \frac{1}{x} \sinh^{-1} \frac{x}{x^*}.$$

Now  $\sinh^{-1} u = \log[u + (1 + u^2)^{1/2}]$  and  $1 + x^2/x^{*2} = 1/x^{*2}$ , so

$$J(x) = \frac{1}{x} \log \frac{1+x}{x^*}.$$

Also,

$$\frac{1+x}{x^*} = \frac{1+x}{(1-x^2)^{1/2}} = \frac{(1+x)^{1/2}}{(1-x)^{1/2}}.$$

By the well-known power series for  $\log(1+x)$ , we deduce that

$$J(x) = \frac{1}{2x} \log \frac{1+x}{1-x} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n+1}.$$

Termwise integration now gives

$$I = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}S.$$

Now reverse the double integral. The substitution  $u = x \sin \theta$  gives

$$\int_0^1 \frac{\sin \theta}{(1 - x^2 \sin^2 \theta)^{1/2}} dx = \int_0^{\sin \theta} \frac{1}{(1 - u^2)^{1/2}} du = \sin^{-1}(\sin \theta) = \theta,$$

so that

$$I = \int_0^{\pi/2} \theta \, d\theta = \frac{\pi^2}{8},$$

hence  $S = \pi^2/6$ .

### References

1. R. J. Chapman, Evaluating  $\zeta(2)$ ,  
<http://empslocal.ex.ac.uk/people/staff/rjchapma/rjc.html>
2. T. M. Apostol, A proof that Euler missed: evaluating  $\zeta(2)$  the easy way, *Math. Intelligencer* **5** (1983), 59–60.
3. F. Beukers, J. A. C. Kolk and E. Calabi, Sums of generalised harmonic series and volumes, *Nieuw Archief voor Wiskunde* **4** (1993), 217–224.
4. N. Lord, Yet another proof that  $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ , *Math. Gazette* **86** (2002), 477–479.