

# LINNIK'S PROOF OF THE WARING-HILBERT THEOREM FROM HUA'S BOOK

(with a correction)

Notes by Tim Jameson

For integers  $s \geq 1$ ,  $k \geq 2$  and  $n \geq 0$ , let  $r_s^{(k)}(n)$  denote the number of solutions  $(n_1, \dots, n_s)$  of the equation  $n_1^k + \dots + n_s^k = n$  with  $n_1, \dots, n_s \geq 0$ .

The fact that  $r_4^{(2)}(n) > 0$  for all  $n$  was probably already suspected by Diophantus (c.250). It was stated explicitly by Bachet in 1621, and later Fermat claimed to have a proof. The first accepted proof was by Lagrange in 1770, building on work of Euler.

Also in 1770, Waring wrote a letter to Euler in which he asserted that every natural number is a sum of 4 squares, 9 cubes, 19 biquadrates “and so on”. By this it is usually assumed that he meant that for each  $k$  there exists an  $s$  such that  $r_s^{(k)}(n) > 0$  for all  $n$ . We let  $g(k)$  denote the least such value of  $s$ . The problem of showing that  $g(k) < \infty$  for all  $k$  has become known as Waring's problem. It was first solved by Hilbert in 1909, by a complicated method [Hil]. Hardy and Littlewood [HL] gave a more elegant proof by the “circle method” in 1919, which was then refined and simplified by Vinogradov [Vin].

An “elementary” proof, using only number-theoretic methods, was given by Linnik in 1943 [Lin]. This method is presented in [Hua], but with one serious mistake. Here I give a version with this mistake corrected. A more general version, with  $x^k$  replaced by a polynomial, is given in [Nath].

Linnik's method depends on the notion of *Shnirelman density*, defined as follows. Let  $A$  be a set of non-negative integers (possibly including 0). For each  $n \geq 1$ , let  $A(n)$  be the number of  $a \in A$  with  $1 \leq a \leq n$ . The Shnirelman density is

$$\sigma(A) = \inf_{n \geq 1} \frac{A(n)}{n}.$$

The application to Waring's problem is via the following Lemma.

LEMMA 1. *If  $A$  contains 0 and has positive Shnirelman density, then there exists  $h$  such that every positive integer is expressible as the sum of  $h$  members of  $A$ .*

We omit the proof of this Lemma, which is quite straightforward. See [Nath, Theorem 11.4] or [Hua, Theorem 19.2.3].

So it will be enough to prove that for given  $k$ , there exists  $s$  such that  $\{n : r_s^{(k)}(n) > 0\}$  has positive Shnirelman density.

LEMMA 2. *Let*

$$q(n; h_1, h_2) = \sum_{\substack{m_1, m_2 = -M \\ h_1 m_1 + h_2 m_2 = n}}^M 1.$$

For  $(h_1, h_2) \neq (0, 0)$  we have  $q(n; h_1, h_2) = 0$  unless  $g = (h_1, h_2) | n$ , say  $h_1 = ga_1$ ,  $h_2 = ga_2$  (so  $(a_1, a_2) = 1$ ). Then

$$q(n; h_1, h_2) \leq \frac{2M}{\max(|a_1|, |a_2|)} + 1.$$

*Proof.* Say  $n = gf$ . Then the equation becomes

$$a_1 m_1 + a_2 m_2 = f.$$

On writing  $a_1 \bar{a}_1 + a_2 \bar{a}_2 = 1$ , we may write this as

$$a_1(m_1 - f\bar{a}_1) + a_2(m_2 - f\bar{a}_2) = 0.$$

Thus the general solution has the form

$$m_1 = f\bar{a}_1 + ka_2,$$

$$m_2 = f\bar{a}_2 - ka_1.$$

Wlog we may suppose  $a_1 \geq a_2 \geq 0$ . This implies  $a_1 > 0$  and

$$q(n; h_1, h_2) = \sum_{\substack{k: \\ -M - f\bar{a}_1 \leq ka_2 \leq M - f\bar{a}_1 \\ -M + f\bar{a}_2 \leq ka_1 \leq M + f\bar{a}_2}} 1.$$

By the second condition,  $k$  is constrained to lie in an interval of length  $2M/a_1$ . Such an interval contains at most  $2M/a_1 + 1$  integers, hence the statement. Note this all works fine if  $n = 0$ .  $\square$

LEMMA 3. *Let*

$$q(n) = \sum_{\substack{h_1, h_2 = -H \\ h_1, h_2 \neq 0}}^H \sum_{\substack{m_1, m_2 = -M \\ h_1 m_1 + h_2 m_2 = n}}^M 1.$$

Then

$$q(n) \leq \begin{cases} 20HM\sigma_{-1}(n) & (n \neq 0, H \leq M), \\ 20H^2M & (n = 0). \end{cases}$$

*Proof.* We have

$$\begin{aligned}
q(n) &= \sum_{\substack{h_1, h_2 = -H \\ h_1, h_2 \neq 0}}^H q(n; h_1, h_2) \\
&= 4 \sum_{h_1, h_2 = 1}^H q(n; h_1, h_2) \\
&\leq 4H^2 + 8M \sum_{\substack{g|n \\ g \leq H}} \sum_{\substack{1 \leq a_1 \leq H/g \\ 0 \leq a_2 \leq H/g}} \frac{1}{\max(a_1, a_2)}.
\end{aligned}$$

We could have said  $a_2 \geq 1$  here, but have allowed  $a_2 = 0$  also (since it naturally leads to no worse an estimate and) so that the following applies to the variant of this lemma required for Lemma 7b:

$$\begin{aligned}
q(n) &\leq 4H^2 + 8M \sum_{\substack{g|n \\ g \leq H}} \left( \sum_{1 \leq a_1 \leq H/g} \sum_{0 \leq a_2 < a_1} \frac{1}{a_1} + \sum_{1 \leq a_2 \leq H/g} \sum_{1 \leq a_1 \leq a_2} \frac{1}{a_2} \right) \\
&= 4H^2 + 16M \sum_{\substack{g|n \\ g \leq H}} \sum_{1 \leq a \leq H/g} 1 \\
&\leq 4H^2 + 16HM \sum_{\substack{g|n \\ g \leq H}} \frac{1}{g},
\end{aligned}$$

and the result follows. □

Although we have thrown away a lot in the case  $n = 0$  (we could give the result as  $4H^2 + 16HM \log(eH)$ ) this will have little effect in the application.

The following lemmas are of some interest in their own right.

LEMMA 4. *We have*

$$\sum_{\substack{a, b=1 \\ (a, b)=1}}^{\infty} \frac{1}{a^2 b^2} = \frac{5}{2}.$$

*Proof.* Denote the sum by  $S$ . We have

$$\sum_{d, e=1}^{\infty} \frac{1}{d^2 e^2} = \zeta(2)^2.$$

Now write  $(d, e) = g$  and  $d = ga, e = gb$ , so that  $(a, b) = 1$ . Then

$$\sum_{d, e=1}^{\infty} \frac{1}{d^2 e^2} = \sum_{\substack{g, a, b=1 \\ (a, b)=1}}^{\infty} \frac{1}{g^4 a^2 b^2} = \zeta(4)S.$$

Hence

$$S = \frac{\zeta(2)^2}{\zeta(4)} = \frac{\pi^4/36}{\pi^4/90} = \frac{5}{2}. \quad \square$$

LEMMA 5. *We have*

$$\sum_{d,e=1}^{\infty} \frac{1}{de[d,e]} = \frac{5}{2}\zeta(3).$$

*Proof.* Denote the sum by  $C$ . With  $g, a, b$  as above, we have  $[d, e] = gab$ , hence

$$C = \sum_{\substack{g,a,b=1 \\ (a,b)=1}}^{\infty} \frac{1}{g^3 a^2 b^2} = \zeta(3)S. \quad \square$$

LEMMA 6. *We have*

$$\sum_{n \leq x} \sigma_{-1}(n)^2 \leq \frac{5}{2}\zeta(3)x.$$

*Proof.* This sum is

$$\begin{aligned} \sum_{n \leq x} \sum_{d,e|n} \frac{1}{de} &= \sum_{d,e \leq x} \frac{1}{de} \sum_{\substack{n \leq x \\ n \equiv 0 \pmod{[d,e]}}} 1 \\ &\leq \sum_{d,e \leq x} \frac{x}{de[d,e]} \\ &\leq Cx, \end{aligned}$$

where  $C$  is as in Lemma 5. □

A few further estimates show that in fact

$$\sum_{n \leq x} \sigma_{-1}(n)^2 = \frac{5}{2}\zeta(3)x + O(\log^2 x).$$

LEMMA 7 (for the inductive step in Theorem 1). *For  $H \leq M$  we have*

$$\sum_{\substack{h_1, \dots, h_4 = -H \\ h_1, \dots, h_4 \neq 0}}^H \sum_{\substack{m_1, \dots, m_4 = -M \\ h_1 m_1 + h_2 m_2 = h_3 m_3 + h_4 m_4}}^M 1 \leq 5250(HM)^3.$$

*Proof.* The LHS is

$$\begin{aligned} \sum_{n=-2HM}^{2HM} q(n)^2 &\leq 20^2 \left( H^4 M^2 + 2H^2 M^2 \sum_{n=1}^{2HM} \sigma_{-1}(n)^2 \right) \\ &\leq 20^2 \left( H^4 M^2 + 2H^2 M^2 \cdot \frac{5}{2}\zeta(3) \cdot 2HM \right) \\ &= 20^2 (H^4 M^2 + 10\zeta(3)H^3 M^3) \\ &\leq 20^2(1 + 10\zeta(3))(HM)^3. \end{aligned}$$

Calculation shows that  $20^2(1 + 10\zeta(3)) \approx 5208$ .

LEMMA 7b (irritating variant needed for Lemma 8). *For  $2H \leq M$  we have*

$$\sum_{\substack{h_1, \dots, h_4 = -H \\ h_1 m_1 + h_2 m_2 = h_3 m_3 + h_4 m_4}}^H \sum_{\substack{m_1, \dots, m_4 = -M}}^M 1 \leq 162M^4 + 5250(HM)^3.$$

*Proof.* Let

$$\begin{aligned} Q(n) &= \sum_{h_1, h_2 = -H}^H \sum_{\substack{m_1, m_2 = -M \\ h_1 m_1 + h_2 m_2 = n}}^M 1 \\ &= \sum_{h_1, h_2 = -H}^H q(n; h_1, h_2) \\ &= q(n; 0, 0) + 4 \sum_{\substack{1 \leq h_1 \leq H \\ 0 \leq h_2 \leq H}} q(n; h_1, h_2). \end{aligned}$$

For  $n \neq 0$  we have  $q(n; 0, 0) = 0$  and obtain the same bound for  $Q(n)$  as that for  $q(n)$  given by Lemma 2: In the working of Lemma 2 the  $8H^2$  is replaced by  $8H(H + 1)$ , but since  $H < M$  we can still say  $8H(H + 1) \leq 8HM$ .

However, in the case  $n = 0$  we have an additional term  $q(n; 0, 0) = (2M + 1)^2 \leq 9M^2$ . Thus we have

$$Q(0) \leq 9M^2 + 20H^2M,$$

and so

$$\begin{aligned} Q(0)^2 &\leq 2(9M^2)^2 + 2(20H^2M)^2 \\ &= 162M^4 + 20^2 H^3 M^2 \cdot 2H \\ &\leq 162M^4 + 20^2 (HM)^3. \end{aligned}$$

The working of Lemma 7 now gives

$$\sum_{\substack{h_1, \dots, h_4 = -H \\ h_1 m_1 + h_2 m_2 = h_3 m_3 + h_4 m_4}}^H \sum_{\substack{m_1, \dots, m_4 = -M}}^M 1 = \sum_{n = -2HM}^{2HM} Q(n)^2 \leq 162M^4 + 5250(HM)^3,$$

as required. □

LEMMA 8 (case  $k = 2$  of Theorem 1). *Let*

$$f(n) = a_2 n^2 + a_1 n$$

where  $a_2, a_1$  are integers with

$$0 < |a_2| \leq c_2, \quad |a_1| \leq c_1 N.$$

Then for  $N \geq 1$  we have

$$\int_0^1 \left| \sum_{n=0}^N e(\alpha f(n)) \right|^8 d\alpha \leq CN^6$$

where

$$C = 162(2c_2 + c_1)^4 + 5250(2c_2 + c_1)^3.$$

In particular  $C = 44592$  when  $f(n) = n^2$ ,  $c_2 = 1$ ,  $c_1 = 0$ .

*Proof.* We have

$$\begin{aligned} \int_0^1 \left| \sum_{n=0}^N e(\alpha f(n)) \right|^8 d\alpha &= \sum_{n_1, \dots, n_8=0}^N \int_0^1 e(\alpha(f(n_1) + \dots + f(n_4) - f(n_5) - \dots - f(n_8))) d\alpha \\ &= \sum_{\substack{n_1, \dots, n_8=0 \\ f(n_1) + \dots + f(n_4) = f(n_5) + \dots + f(n_8)}}^N 1. \end{aligned}$$

We may write the equation here as

$$\begin{aligned} \sum_{i=1}^4 (f(n_i) - f(n_{i+4})) &= \sum_{i=1}^4 (a_2(n_i^2 - n_{i+4}^2) + a_1(n_i - n_{i+4})) \\ &= \sum_{i=1}^4 h_i m_i \\ &= 0, \end{aligned}$$

where

$$\begin{aligned} h_i &= n_i - n_{i+4} \\ m_i &= a_2(n_i + n_{i+4}) + a_1. \end{aligned}$$

Note that  $(h_i, m_i)$  uniquely determines  $(n_i, n_{i+4})$  since

$$\begin{pmatrix} 1 & -1 \\ a_2 & a_2 \end{pmatrix} \begin{pmatrix} n_i \\ n_{i+4} \end{pmatrix} = \begin{pmatrix} h_i \\ m_i - a_1 \end{pmatrix}$$

has the inverse

$$\begin{pmatrix} n_i \\ n_{i+4} \end{pmatrix} = \frac{1}{2a_2} \begin{pmatrix} a_2 & 1 \\ -a_2 & 1 \end{pmatrix} \begin{pmatrix} h_i \\ m_i - a_1 \end{pmatrix}$$

for  $a_2 \neq 0$ . Clearly we have

$$|h_i| \leq N$$

and

$$|m_i| \leq M, \text{ where } M = (2c_2 + c_1)N.$$

Noting that  $M \geq 2N$  since  $c_2 \geq |a_2| \geq 1$ , the result follows from Lemma 6b.

**THEOREM 1.** *Let  $k \geq 2$  and*

$$f(n) = a_k n^k + \cdots + a_1 n$$

where  $a_1, \dots, a_k$  are integers with  $a_k \neq 0$  and

$$|a_j| \leq c_{j,k} N^{k-j}.$$

Then for  $N \geq 1$  we have

$$\int_0^1 \left| \sum_{n=0}^N e(\alpha f(n)) \right|^{8^{k-1}} d\alpha \ll_{k, c_{1,k}, \dots, c_{k,k}} N^{8^{k-1}-k}.$$

*Proof.* The statement is more than we need for the application. It has been elaborated to make its proof by induction on  $k$  work. The case  $k = 2$  is given by Lemma 8. We will omit the suffices in the  $\ll$  notation. Suppose the statement is true with  $k - 1$  in place of  $k$ . We have

$$\left| \sum_{n=0}^N e(\alpha f(n)) \right|^2 = \sum_{\substack{m, n=0 \\ m-n=h}}^N e(\alpha f(m) - \alpha f(n)) = N + 1 + \sum_{\substack{h=-N \\ h \neq 0}}^N b_h, \quad (1)$$

where

$$b_h = \sum_{\substack{m, n=0 \\ m-n=h}}^N e(\alpha f(m) - \alpha f(n)) = \sum_{n=\max(0, -h)}^{\min(N, N-h)} e(\alpha h \phi(n, h)),$$

where

$$\begin{aligned} \phi(n, h) &= \frac{1}{h} (f(n+h) - f(n)) \\ &= \frac{1}{h} \sum_{j=1}^k a_j ((n+h)^j - n^j) \\ &= \sum_{j=1}^k a_j \sum_{r=0}^{j-1} \binom{j}{r} h^{j-r-1} n^r \\ &= \sum_{r=0}^{k-1} \left( \sum_{j=r+1}^k \binom{j}{r} a_j h^{j-r-1} \right) n^r \end{aligned}$$

is a degree  $k - 1$  polynomial in  $n$ . From the definition we see that  $\phi(n, h) \ll N^{k-1}$ . The coefficient of  $n^{k-1}$  in  $\phi(n, h)$  is

$$\binom{k}{k-1} a_k h^{k-(k-1)-1} = k a_k \neq 0,$$

and the coefficient of  $n^r$  is

$$\begin{aligned} \sum_{j=r+1}^k \binom{j}{r} a_j h^{j-r-1} &\ll \sum_{j=r+1}^k \binom{j}{r} N^{k-j} N^{j-r-1} \\ &\ll N^{k-r-1}. \end{aligned}$$

Raising (1) to the power  $8^{k-2}$  using Hölder's inequality gives

$$\begin{aligned} \left| \sum_{n=0}^N e(\alpha f(n)) \right|^{2 \cdot 8^{k-2}} &\ll N^{8^{k-2}} + \left| \sum_{\substack{h=-N \\ h \neq 0}}^N b_h \right|^{8^{k-2}} \\ &\ll N^{8^{k-2}} + \left( \sum_{\substack{h=-N \\ h \neq 0}}^N 1 \right)^{8^{k-2}-1} \sum_{\substack{h=-N \\ h \neq 0}}^N |b_h|^{8^{k-2}} \\ &\ll N^{8^{k-2}} + N^{8^{k-2}-1} \sum_{\substack{h=-N \\ h \neq 0}}^N |b_h|^{8^{k-2}}. \end{aligned}$$

Raising this to a further fourth power and integrating over  $\alpha$  then gives

$$\int_0^1 \left| \sum_{n=0}^N e(\alpha f(n)) \right|^{8^{k-1}} d\alpha \ll N^{4 \cdot 8^{k-2}} + N^{4 \cdot 8^{k-2}-4} \int_0^1 \left( \sum_{\substack{h=-N \\ h \neq 0}}^N |b_h|^{8^{k-2}} \right)^4 d\alpha. \quad (2)$$

As a function of  $\alpha$ ,  $b_h$  has period  $1/|h|$ . Let  $|b_h|^{8^{k-2}}$  have the Fourier series

$$|b_h|^{8^{k-2}} = \sum_{m=-\infty}^{\infty} A(m, h) e(\alpha h m).$$

This is finite really because

$$A(m, h) \neq 0 \Rightarrow m \ll \max_{0 \leq n \leq N} |\phi(n, h)| \ll N^{k-1},$$

so we may write the range for  $m$  as  $|m| \leq CN^{k-1}$  (where  $C$  is independent of  $h$ ). The coefficients are given by

$$\begin{aligned} A(m, h) &= \int_0^1 |b_h|^{8^{k-2}} e(-\alpha h m) d\alpha \\ &= \int_0^{|h|} \left| \sum_{n=\max(0, -h)}^{\min(N, N-h)} e((\text{sgn } h)\beta\phi(n, h)) \right|^{8^{k-2}} e(-(\text{sgn } h)\beta m) \frac{d\beta}{|h|} \\ &= \int_0^1 \left| \sum_{n=\max(0, -h)}^{\min(N, N-h)} e(\beta\phi(n, h)) \right|^{8^{k-2}} e(-(\text{sgn } h)\beta m) d\beta. \end{aligned}$$



Thus

$$|A(m, h)| \leq \int_0^1 \left| \sum_{n=\max(0, -h)}^{\min(N, N-h)} e(\beta\phi(n, h)) \right|^{8^{k-2}} d\beta \\ \ll N^{8^{k-2}-(k-1)},$$

by the inductive hypothesis. (We have trivially translated the region of summation over  $n$ . This should be written in as part of the official statement or perhaps done away with by restricting to  $h > 0$  using a  $2\mathfrak{R}$ .) Now we have

$$\int_0^1 \left( \sum_{\substack{h=-N \\ h \neq 0}}^N |b_h|^{8^{k-2}} \right)^4 d\alpha = \int_0^1 \left( \sum_{\substack{h=-N \\ h \neq 0}}^N \sum_{|m| \leq CN^{k-1}} A(m, h) e(\alpha hm) \right)^4 d\alpha \\ = \sum_{\substack{h_1, \dots, h_4 = -N \\ h_1, \dots, h_4 \neq 0}}^N \sum_{|m_1|, \dots, |m_4| \leq CN^{k-1}} \left( \prod_{i=1}^4 A(m_i, h_i) \right) \int_0^1 e \left( \alpha \sum_{i=1}^4 h_i m_i \right) d\alpha \\ = \sum_{\substack{h_1, \dots, h_4 = -N \\ h_1, \dots, h_4 \neq 0}}^N \sum_{\substack{|m_1|, \dots, |m_4| \leq CN^{k-1} \\ h_1 m_1 + h_2 m_2 + h_3 m_3 + h_4 m_4 = 0}} \prod_{i=1}^4 A(m_i, h_i) \\ \ll N^{4(8^{k-2}-(k-1))} \sum_{\substack{h_1, \dots, h_4 = -N \\ h_1, \dots, h_4 \neq 0}}^N \sum_{\substack{|m_1|, \dots, |m_4| \leq CN^{k-1} \\ h_1 m_1 + h_2 m_2 + h_3 m_3 + h_4 m_4 = 0}} 1 \\ \ll N^{4(8^{k-2}-(k-1))} N^{3k} \\ = N^{4 \cdot 8^{k-2} - k + 4},$$

by Lemma 7 (note that we may assume  $C \geq 1$ ). So finally (2) gives

$$\int_0^1 \left| \sum_{n=0}^N e(\alpha f(n)) \right|^{8^{k-1}} d\alpha \ll N^{4 \cdot 8^{k-2}} + N^{4 \cdot 8^{k-2} - 4} N^{4 \cdot 8^{k-2} - k + 4} \\ \ll N^{4 \cdot 8^{k-2}} + N^{8^{k-1} - k}.$$

The fact that the second term dominates is equivalent to  $8^k \geq 16k$ , which is certainly true for  $k \geq 3$  so completing the proof.  $\square$

**THEOREM 2.** *Let  $k \geq 2$  and  $s = 8^{k-1}$ . Then the set  $A = \{n \geq 1 : r_s^{(k)}(n) > 0\}$  has positive Schnirelmann density.*

**Proof.** Write  $r(n)$  for  $r_s^{(k)}(n)$ . We have

$$\sum_{n=0}^N r(n) = \sum_{\substack{m_1, \dots, m_s \geq 0 \\ m_1^k + \dots + m_s^k \leq N}} 1$$

$$\begin{aligned}
&\geq \sum_{0 \leq m_1, \dots, m_s \leq (N/s)^{1/k}} 1 \\
&\geq (N/s)^{s/k} \\
&\gg_k N^{s/k},
\end{aligned}$$

But on the other hand, for  $n \geq 1$ ,

$$\begin{aligned}
r(n) &= \sum_{0 \leq m_1, \dots, m_s \leq n^{1/k}} \int_0^1 e(\alpha(m_1^k + \dots + m_s^k - n)) d\alpha \\
&= \int_0^1 \left( \sum_{0 \leq m \leq n^{1/k}} e(\alpha m^k) \right)^s e(-\alpha n) d\alpha \\
&\leq \int_0^1 \left| \sum_{0 \leq m \leq n^{1/k}} e(\alpha m^k) \right|^s d\alpha \\
&\ll_k (n^{1/k})^{s-k} \quad (\text{by Theorem 1}) \\
&= n^{s/k-1},
\end{aligned}$$

so that

$$\sum_{n=0}^N r(n) \ll_k 1 + N^{s/k-1} A(N).$$

The statement clearly follows. □

By Lemma 1, we can deduce at once:

**THEOREM 3** (the Waring-Hilbert theorem). *For each  $k \geq 2$ , there exists  $s(k)$  such that  $r_s(n) > 0$  for all  $n \geq 0$ .*

The mistake in [Hua] is the claim that  $Q(0) \ll \min(H^2M, M^2H) \ll (HM)^{3/2}$ , which is wrong because there are  $(2M+1)^2$  solutions with  $h_1 = h_2 = 0$ . This leads to the incorrect bound  $\ll (HM)^3$  (regardless of the relative sizes of  $H$  and  $M$ ) for the quantity in Lemma 6b. This is wrong if  $M$  is much bigger than  $H$  (as it is in the application to the inductive step in Theorem 1) since  $Q(0)^2$  then dominates. The way I've corrected it is simply to note that  $h_1, \dots, h_4 = 0$  does not occur in Theorem 1 so we can use Lemma 7 instead.

### References *(added by Graham Jameson)*

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