The integral \( \int_x^\infty \frac{e^{it}}{tp} \, dt \); the sine and Fresnel integrals

Notes by G.J.O. Jameson

Definitions and elementary facts

We shall consider integrals of the form
\[
i_p(x) = \int_0^x \frac{e^{it}}{tp} \, dt,
\]
with real and imaginary parts
\[
c_p(x) = \int_0^x \frac{\cos t}{tp} \, dt, \quad s_p(x) = \int_0^x \frac{\sin t}{tp} \, dt,
\]
and similarly for the complementary integrals
\[
I_p(x) = \int_x^\infty \frac{e^{it}}{tp} \, dt, \quad C_p(x) = \int_x^\infty \frac{\cos t}{tp} \, dt, \quad S_p(x) = \int_x^\infty \frac{\sin t}{tp} \, dt,
\]
where \( p > 0 \) and \( x \geq 0 \). We write also
\[
I_p(x, y) = \int_x^y \frac{e^{it}}{tp} \, dt,
\]
and similarly \( C_p(x, y) \) and \( S_p(x, y) \). So \( i_p(x) = I_p(0, x) \) and \( I_p(x, y) = I_p(x) - I_p(y) \).

Further, we write \( I_p = I_p(0) \), and similarly \( C_p, S_p \), in the cases where these integrals converge (so \( I_p = i_p(x) + I_p(x) \) for all \( x > 0 \)): these are the “complete” integrals of this kind.

Cases of particular interest are \( p = 1 \), defining the “sine” and “cosine” integrals, and \( p = \frac{1}{2} \), defining the “Fresnel” integrals.

There is no firmly established notation for these integrals for general \( p \), but in the case \( p = 1 \), the usual notation is \( \text{Si}(x) \) for our \( s_1(x) \), and \(-\text{ci}(x)\) and \(-\text{si}(x)\) for our \( C_1(x) \) and \( S_1(x) \).

The integrals defining \( i_p(x) \) and \( c_p(x) \) are convergent at 0, hence well-defined, for \( 0 < p < 1 \), while \( s_p(x) \) is well-defined for \( 0 < p < 2 \).

We now establish convergence of the integral defining \( I_p(x) \) for all \( p > 0 \), together with some useful identities and approximations.

TPE1. The integral defining \( I_p(x) \) is convergent for all \( p > 0 \) and \( x > 0 \), and the following statements hold
\[
I_p(x) = \frac{ie^{ix}}{xp} - ipI_{p+1}(x), \tag{1}
\]
\[
C_p(x) = -\frac{\sin x}{x^p} + pS_{p+1}(x),
\]
\[
S_p(x) = \frac{\cos x}{x^p} - pC_{p+1}(x),
\]
\[
|pI_{p+1}(x)| \leq \frac{1}{x^p},
\]
\[
|I_p(x)| \leq \frac{2}{x^{p+1}},
\]
\[
I_p(x) = \frac{ie^{ix}}{x^p} + r_p(x), \text{ where } |r_p(x)| \leq \frac{2p}{x^{p+1}}.
\]

**Proof.** Integrate by parts:
\[
I_p(x, y) = \left[ -\frac{i}{t^p} e^{it} \right]_x^y - i \int_x^y \frac{p}{t^{p+1}} e^{it} \, dt
\]
\[
= \frac{ie^{ix}}{x^p} - \frac{ie^{iy}}{y^p} - ipI_{p+1}(x, y).
\]
Now
\[
\int_{x}^{\infty} \frac{p}{t^{p+1}} \, dt = \frac{1}{x^p},
\]
so the integral defining \(pI_{p+1}(x)\) is convergent, with absolute value not greater than \(1/x^p\).

By considering the limit as \(y \to \infty\), we now see that the integral defining \(I_p(x)\) is convergent and satisfies (1). Statements (2) and (3) simply restate (1) in terms of real and imaginary parts. Also, (5) follows. By applying (5) to \(I_{p+1}(x)\) and inserting in (1), we obtain (6). \(\Box\)

By (6), we have \(I_p(x) \sim ie^{ix}/x^p\) as \(x \to \infty\), suggesting that the factor 2 in (5) and (6) might be unnecessary. We shall see later that this is so.

The process can be repeated to deliver asymptotic expressions and inequalities that are effective for large \(x\). These results are better presented in the more general context of integrals of \(f(t)e^{it}\): see [Jam2].

We mention some further elementary facts about \(c_p(x)\) and \(s_p(x)\). By the fundamental theorem of calculus, \(s'_p(x) = \sin x/x^p\). Hence \(s_p(x)\) is increasing on intervals \([2n\pi, (2n + 1)\pi]\) and decreasing on intervals \([(2n-1)\pi, 2n\pi]\), so has maxima at the points \((2n+1)\pi\) and minima at the points \(2n\pi\). Similarly, \(c_p(x)\) has maxima at \((2n + \frac{1}{2})\pi\) and minima at \((2n - \frac{1}{2})\pi\).

**TPE2.** The function \(s_p(x)\) has least value \(0\) and greatest value at \(x = \pi\). The function \(c_p(x)\) has greatest value at \(\pi/2\) and least value either at \(0\) or at \(3\pi/2\).

**Proof.** Write
\[
A_n = \int_{n\pi}^{(n+2)\pi} \frac{\sin t}{t^p} \, dt.
\]
By substituting $t + \pi = u$ on $[n\pi, (n + 1)\pi]$, we see that

$$A_n = \int_{n\pi}^{(n+1)\pi} \left( \frac{1}{tp} - \frac{1}{(t + \pi)^p} \right) \sin t \, dt,$$

in which $t^{-p} - (t + \pi)^{-p} > 0$. If $n$ is even, then $\sin t \geq 0$ on $[n\pi, (n + 1)\pi]$, so $A_n \geq 0$, hence $s_p[(n + 2)\pi] \geq s_p(n\pi)$. It follows that $s_p(2n\pi) \geq s_p(0) = 0$ for all $n$, so, by the preceding remarks, $s_p(x) \geq 0$ for all $x \geq 0$. Meanwhile, if $n$ is odd, then $A_n \leq 0$, so that $s_p(\pi) \geq s_p(3\pi) \geq \ldots$, so the greatest value of $s_p(x)$ occurs at $x = \pi$.

Similarly, if

$$B_n = \int_{(n-\frac{1}{2})\pi}^{(n+\frac{1}{2})\pi} \frac{\cos t}{tp^p} \, dt,$$

then $B_n \geq 0$ for even $n$ and $B_n \leq 0$ for odd $n$. The statements for $c_p(x)$ follow in the same way. \hfill \Box

Note that $\int_0^{3\pi/2} \cos t \, dt = -1$. Simple estimations show that $c_p(3\pi/2) < 0$ for sufficiently small $p$, while (as we see shortly) $c_{1/2}(3\pi/2) > 0$.

From $\cos t \leq 1$ and $\sin t \leq t$, we have the inequalities

$$c_p(x) \leq \frac{x^{1-p}}{1-p}, \quad s_p(x) \leq \frac{x^{2-p}}{2-p}.$$

By inserting the series for $\cos t$ and $\sin t$ and integrating termwise, we obtain explicit series expressions for $c_p(x)$ (for $0 < p < 1$) and $s_p(x)$ (for $0 < p < 2$):

$$c_p(x) = x^{1-p} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!(2n - p + 1)} = x^{1-p} \left( \frac{1}{1-p} - \frac{x^2}{2!(3-p)} + \cdots \right), \quad (7)$$

$$s_p(x) = x^{1-p} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!(2n - p)} = x^{1-p} \left( \frac{x}{2-p} - \frac{x^3}{3!(4-p)} + \cdots \right). \quad (8)$$

These series can be used to calculate values, though in practice the calculation is only pleasant for fairly small $x$. Some values are

$$s_1(\pi) \approx 1.85194, \quad s_1(2\pi) \approx 1.41816, \quad s_{1/2}(\pi) \approx 1.78967,$$

$$c_{1/2}(\frac{\pi}{2}) \approx 1.95490, \quad c_{1/2}(\frac{3\pi}{2}) \approx 0.80476.$$

The substitution $t = u + x$ gives

$$I_p(x) = e^{ix} \int_0^\infty \frac{e^{iu}}{(u + x)^p} \, du.$$
The “auxiliary functions” $F_p(x), G_p(x)$ are defined by $e^{-ix}I_p(x) = G_p(x) + iF_p(x)$ (we write $F_p$ and $G_p$ this way round as a gesture towards consistency with the usual notation). So

$$F_p(x) = S_p(x) \cos x - C_p(x) \sin x,$$

$$G_p(x) = C_p(x) \cos x + S_p(x) \sin x.$$  

Hence, for example, $F_p(n\pi) = (-1)^n S_p(n\pi)$ and $G_p(n\pi) = (-1)^n C_p(n\pi)$. Some results take on a particularly neat form when expressed in terms of these functions.

We now turn to the evaluation of $I_p$. Simple and elegant methods are available for the sine integral $S_1$ and the Fresnel integral $I_{1/2}$. We present these first, and then describe two methods that apply for general $p$ in $(0, 1)$, respectively by double integration and contour integration.

**Evaluation of the sine integral $S_1$**

We assume the following well-known series identity: for $x \neq k\pi$,

$$\frac{1}{\sin x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{x + n\pi}.$$  

(9)

One proof of (9) [Wa, p. 17–18] is by considering the Fourier series for $\cos ax$ on $[-\pi, \pi]$.

**TPE3 THEOREM.** We have

$$S_1 = \int_{0}^{\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2}.$$  

(10)

**Proof.** (Cf. [Wa, p. 186–187] and [Lo]). Note first that, since $\sin t/t$ is an even function,

$$2S_1 = \int_{-\infty}^{\infty} \frac{\sin t}{t} \, dt.$$  

The substitution $t = x + n\pi$ gives

$$\int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t} \, dt = (-1)^n \int_{0}^{\pi} \frac{\sin x}{x + n\pi} \, dx.$$  

Assuming that termwise integration of the series is valid, we add these identities for all integers $n$ to obtain at once

$$2S_1 = \int_{0}^{\pi} \frac{\sin x}{\sin x} \, dx = \pi.$$  

The termwise integration is easily justified by uniform convergence, as follows. By combining the terms for $n$ and $-n$ and multiplying by $\sin x$, we can rewrite the series (7) as

$$\frac{\sin x}{x} + 2x \sin x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2\pi^2} = 1.$$  

4
For $0 < x < \pi$ and $n \geq 2$,
\[
\left| \frac{2x \sin x}{x^2 - n^2 \pi^2} \right| \leq \frac{2\pi}{(n^2 - 1)\pi^2}.
\]
Since $\sum_{n=2}^{\infty} 1/(n^2 - 1)$ is convergent, it follows, by Weierstrass’s “M-test”, that the series converges uniformly on the open interval $(0, \pi)$: this is all we need.

Of course, $C_1$ is undefined. However, by considering the function
\[
c_1^*(x) = \int_0^x \frac{1 - \cos t}{t} \, dt,
\]
one can prove the following result for $\cos t$ corresponding, in a sense, to (10):
\[
C_1(x) = c_1^*(x) - \log x - \gamma,
\]
so that $C_1(x) + \log x \to -\gamma$ as $x \to 0^+$. For a proof, see, for example, [Jam1].

The case $p = \frac{1}{2}$: the Fresnel integrals

The substitution $t = u^2$ gives
\[
i_{1/2}(x) = \int_0^x e^{it} \frac{dt}{t^{1/2}} = 2 \int_0^{x^{1/2}} e^{iu^2} \, du, \quad I_{1/2}(x) = 2 \int_{x^{1/2}}^{\infty} e^{iu^2} \, du,
\]
\[
I_{1/2} = 2 \int_0^{\infty} e^{iu^2} \, du = \int_{-\infty}^{\infty} e^{iu^2} \, du. \tag{11}
\]
These integrals are often quoted in this form, and it is the form we will use for the evaluation.

We assume the following elementary form of the theorem on pointwise convergence of Fourier series. Suppose that $f$ is differentiable on $[0, 1]$ and $f(0) = f(1)$. Let $c_n = \int_0^1 f(x)e^{-2n\pi ix} \, dx$. Then for all $x$ in $[0, 1]$, we have $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2n\pi ix}$, where the notation $\sum_{n=-\infty}^{\infty} a_n$ means $\lim_{N \to \infty} \sum_{n=-N}^{N} a_n$. (For readers more familiar with real Fourier series, observe that the substitution $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$ translates the series into the form $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos 2n\pi x + b_n \sin 2n\pi x)$.)

TPE4 THEOREM. We have
\[
I_{1/2} = \left( \frac{\pi}{2} \right)^{1/2} (1 + i), \tag{12}
\]
equivalently
\[
\int_{-\infty}^{\infty} e^{2\pi ix^2} \, dx = \frac{1}{2} (1 + i), \tag{13}
\]
so that
\[
\int_{-\infty}^{\infty} \cos 2\pi x^2 \, dx = \int_{-\infty}^{\infty} \sin 2\pi x^2 \, dx = \frac{1}{2}.
\]
Proof. The equivalence of (12) and (13) is clear from (11) and the further substitution $u = (2\pi)^{1/2}x$. Let $f(x) = e^{2\pi ix^2}$ and $I = \int_{-\infty}^{\infty} f(x) \, dx$. Consider the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i \tau}$ for $f(x)$ on $[0, 1]$. Note that $f(0) = f(1) = 1$. Convergence at the points 0 and $\frac{1}{2}$ gives
\[ \sum_{n=-\infty}^{\infty} c_n = f(0) = 1, \]
\[ \sum_{n=-\infty}^{\infty} (-1)^n c_n = f(\frac{1}{2}) = e^{\pi i/2} = i, \]
hence
\[ \sum_{n=-\infty}^{\infty} c_{2n} = \frac{1}{2} (1 + i). \]
This sum can equally be written as $\sum_{n=-\infty}^{\infty} c_{-2n}$. Now
\[ c_{-2n} = \int_{0}^{1} e^{2\pi i (x^2 + 2nx)} \, dx, \]
and $x^2 + 2nx = (x + n)^2 - n^2$, so
\[ c_{-2n} = \int_{0}^{1} e^{2\pi i (x + n)^2} \, dx = \int_{n}^{n+1} e^{2\pi i y} \, dy. \]
Hence
\[ \sum_{n=-\infty}^{\infty} c_{-2n} = I. \]

Alternatively, this proof can be presented in terms of real Fourier series, resulting in separate evaluations of the cosine and sine integrals.

Evaluation for general $p$: double integral method

We will establish an integral expression for $I_p(x)$, which has other applications as well as the evaluation of $I_p$ for $0 < p < 1$. We assume familiarity with the gamma function. The starting point is the observation that for any $p > 0$, the substitution $tu = v$ gives
\[ \int_{0}^{\infty} u^{p-1} e^{-tu} \, du = \frac{1}{t^p} \int_{0}^{\infty} v^{p-1} e^{-v} \, dv = \frac{\Gamma(p)}{t^p}. \]
This gives an integral expression for $1/t^p$ which we substitute in $I_p(x)$. In the case $p = 1$, it simplifies to $\int_{0}^{\infty} e^{-tu} \, du = 1/t$.

TPE5 THEOREM. For $x > 0$ and $p > 0$, also for $x = 0$ and $0 < p < 1$, we have
\[ I_p(x) = \frac{e^{ix}}{\Gamma(p)} \int_{0}^{\infty} \frac{u + i}{u^2 + 1} u^{p-1} e^{-ux} \, du. \]
Further, for $x > 0$, 

$$|I_p(x)| \leq \frac{1}{x^p}. \quad (16)$$

(Note that (16) amounts to the removal of the factor 2 in (5).)

Proof. By (14), we have

$$\Gamma(p)I_p(x, y) = \int_x^y e^{it} \int_0^\infty u^{p-1}e^{-ut} \, du \, dt.$$ 

Assuming reversal of the integral is valid, we have

$$\Gamma(p)I_p(x, y) = \int_0^\infty u^{p-1} \int_x^y e^{-(u-i)t} \, dt \, du = \int_0^\infty \frac{u^{p-1}}{u-i} (e^{-(u-i)x} - e^{-(u-i)y}) \, du = H(x) - H(y),$$

where

$$H(x) = e^{ix} \int_0^\infty \frac{u^{p-1}}{u-i} e^{-ux} \, du.$$ 

Since $|u-i| \geq 1$, we have

$$|H(x)| \leq \int_0^\infty u^{p-1}e^{-ux} \, du = \frac{\Gamma(p)}{x^p}$$

for $x > 0$. Hence $H(y) \to 0$ as $y \to \infty$. Taking the limit as $y \to \infty$, we conclude that $\Gamma(p)I_p(x) = H(x)$, hence both our statements.

To justify reversal of the double integral, observe that the modulus of the integrand is no greater than $u^{p-1}e^{-ut}$, and we have

$$\int_x^y \int_0^\infty u^{p-1}e^{-ut} \, du \, dt = \Gamma(p) \int_x^y \frac{1}{t^p} \, dt,$$

which is finite for the stated combinations of $x$ and $p$. \[\square\]

Note. A variant of the proof is as follows. Assuming its validity, differentiation under the integral sign gives

$$H'(x) = -\int_0^\infty u^{p-1}e^{-(u-i)x} \, du = -\Gamma(p) \frac{e^{ix}}{x^p} = \Gamma(p)I'_p(x),$$

so $\Gamma(p)I_p(x) - H(x)$ is constant. Both $I_p(x)$ and $H(x)$ tend to 0 as $x \to \infty$, so the constant is 0.

As a corollary, we have the following expressions for the auxiliary functions. For the case $p = 1$, they can be seen stated without proof in compilations such as Wikipedia and [DLMF, chapter 6].
TPE6 COROLLARY. The auxiliary functions \( F_p(x) \), \( G_p(x) \) satisfy
\[
F_p(x) = \frac{1}{\Gamma(p)} \int_0^\infty \frac{u^{p-1}}{u^2 + 1} e^{-ux} \, du,
\]
\[
G_p(x) = \frac{1}{\Gamma(p)} \int_0^\infty \frac{u^p}{u^2 + 1} e^{-ux} \, du.
\]
Both are positive, decreasing functions of \( x \).

Both the inequality (16) and the positive, decreasing nature of \( F_p(x) \) and \( G_p(x) \) generalise to integrals of the form \( \int_x^\infty f(t) e^{it} \, dt \) for functions that are completely monotonic, that is, \((-1)^n f^{(n)}(t) \geq 0 \) for all \( n \geq 1 \) (note that \( 1/t^p \) satisfies this condition). The absolute value of the integral is not greater than \( f(x) \). See [Jam2] or [Jam3]. For the special case \( S_1(x) \), the stronger bound \( \frac{\pi}{2} - \tan^{-1}(x) \) was established in [JLM].

To deduce the value of \( I_p \), we will use the well-known integral
\[
\int_0^\infty \frac{1}{y^n(1+y)} \, dy = \frac{\pi}{\sin \pi a} \quad (0 < a < 1),
\]
which can be proved using the series (9), e.g. [Wa, p. 187]. Also, we will use Euler’s reflection formula for the gamma function:
\[
\Gamma(q)\Gamma(1-q) = \frac{\pi}{\sin \pi q}
\]
(18)
The value of \( I_p \) is actually stated more neatly in terms of \( q = 1 - p \), as follows:

TPE7 THEOREM. Let \( 0 < p < 1 \) and \( q = 1 - p \). Then
\[
I_p = \Gamma(q)e^{\frac{1}{2}\pi q i}, \quad C_p = \Gamma(q) \cos \frac{1}{2}\pi q, \quad S_p = \Gamma(q) \sin \frac{1}{2}\pi q.
\]
(19)

Proof. By the case \( x = 0 \) in (15),
\[
I_p = \frac{1}{\Gamma(p)} \int_0^\infty \frac{u^p + iu^{p-1}}{u^2 + 1} \, du \frac{1}{\Gamma(1-q)} \int_0^\infty \frac{u^{1-q} + iu^{-q}}{u^2 + 1} \, du.
\]
Substituting \( u^2 = y \), we obtain from (17)
\[
\int_0^\infty \frac{u^{1-q}}{1+u^2} \, du = \int_0^\infty \frac{1}{2y^{1/2}(1+y)} \, dy = \frac{\pi}{2 \sin \frac{1}{2}\pi q}.
\]
Replacing \( q \) by \( q + 1 \) in this, we have also
\[
\int_0^\infty \frac{u^{-q}}{1+u^2} \, du = \frac{\pi}{2 \cos \frac{1}{2}\pi q}.
\]
Now substituting for $\Gamma(1-q)$ from (18) and using $\sin \pi q = 2 \sin \frac{1}{2} \pi q \cos \frac{1}{2} \pi q$, we obtain

$$C_p = \frac{\Gamma(q) \sin \pi q}{\pi} \frac{\pi}{2 \sin \frac{1}{2} \pi q} = \Gamma(q) \cos \frac{1}{2} \pi q$$

and similarly $S_p = \Gamma(q) \sin \frac{1}{2} \pi q$.

Of course, (12) is the special case $p = \frac{1}{2}$. For this case, the required integrals $\int_0^\infty u^{\pm 1/2}/(1 + u^2) \, du$ can be evaluated by more elementary means.

Note. Formally, the method of Theorem TPE5 appears to give a very quick proof of (10). Recall that $\int_0^\infty e^{-ut} \sin t \, dt = 1/(u^2 + 1)$. So reversal of the double integral gives

$$\int_0^\infty \frac{\sin t}{t} \, dt = \int_0^\infty \sin t \int_0^\infty e^{-ut} \, du \, dt = \int_0^\infty \int_0^\infty e^{-ut} \sin t \, dt \, du = \int_0^\infty \frac{1}{u^2 + 1} \, du = \frac{\pi}{2}.$$  

However, the conditions for reversal are decidedly not satisfied! One way to satisfy these conditions is to start instead from

$$\int_a^b e^{-ut} \, du = \frac{1}{t}(e^{-at} - e^{-bt}).$$  

We leave it to the reader to verify that reversal (now valid) leads to

$$\int_0^\infty (e^{-at} - e^{-bt}) \frac{\sin t}{t} \, dt = \tan^{-1} b - \tan^{-1} a. \quad (20)$$

By consideration of the limits as $a \to 0^+$ and $b \to \infty$ (not entirely trivial in the case of $a$), one can derive (10).

Evaluation for general $p$: contour integral method

We give a second proof of (19) by contour integration. We will actually prove the statement in the conjugate form

$$\int_0^\infty t^{q-1} e^{-it} \, dt = \Gamma(q) e^{-\frac{1}{2} \pi q i}. \quad (21)$$

Let $0 < r < R$. Let $C_R$ denote the circular arc of radius $R$ in the positive quadrant, represented by $z = Re^{i\theta}$ for $0 \leq \theta \leq \frac{\pi}{2}$, similarly $C_r$. Denote by $\Delta$ the closed contour consisting of the real interval $[r, R]$, $C_R$ described anticlockwise, the imaginary axis from $iR$ to $ir$ and $C_r$ described clockwise. Let $f(z) = z^{q-1} e^{-z}$, where the principal value of $\log z$
is used to define $z^{p-1}$. In particular, $\log it$ (for $t > 0$) is expressed as $\log t + i\frac{\pi}{2}$, so $(it)^{q-1}$ is expressed as $t^{q-1}\exp\left(\frac{1}{2}(q-1)\pi i\right)$. By Cauchy’s integral theorem, $\int_{\Delta} f(z) \, dz = 0$. The contribution of the real axis converges to $\Gamma(q)$ when $r \to 0$ and $R \to \infty$. The contribution of the imaginary axis, taken towards the origin, is

$$-\int_{r}^{R} (it)^{q-1}e^{-it} \, dt = -e^{\frac{1}{2}q\pi i} \int_{r}^{R} t^{q-1}e^{-it} \, dt.$$ 

The conclusion will follow if we can show that the contributions of the circular arcs tend to 0 when $r \to 0$ and $R \to \infty$.

The arc $C_r$ is given by $z = re^{i\theta}$ for $0 \leq \theta \leq \frac{\pi}{2}$. For such $z$, we have $|e^{-z}| = e^{-r \cos \theta} \leq 1$, hence $|f(z)| \leq r^{q-1}$ and

$$\left|\int_{C_r} f(z) \, dz\right| \leq r^{q-1} \frac{\pi r}{2} = \frac{\pi r^q}{2},$$

which tends to 0 as $r \to 0^+$, since $q > 0$.

The integral on $C_R$ is

$$I_R =: \int_{0}^{\pi/2} R^{q-1}e^{i(q-1)\theta}e^{-Re^{i\theta}}iRe^{i\theta} \, d\theta.$$ 

Since $|e^{-Re^{i\theta}}| = e^{-R \cos \theta}$,

$$|I_R| \leq R^q \int_{0}^{\pi/2} e^{-R \cos \theta} \, d\theta.$$

TPES LEMMA. We have

$$0 \leq \int_{0}^{\pi/2} e^{-R \cos \theta} \, d\theta \leq \frac{\pi}{2R}.$$ 

Proof. Denote this integral by $J_R$. The substitution $\theta = \frac{\pi}{2} - \phi$ shows that $J_R = \int_{0}^{\pi/2} e^{-Rs\sin \theta} \, d\theta$. By the concavity of the sine function, $\sin \theta \geq \frac{2a}{\pi}$ on $[0, \frac{\pi}{2}]$. Hence

$$J_R \leq \int_{0}^{\pi/2} e^{-2R\theta/\pi} \, d\theta = \left[ -\frac{\pi}{2R} e^{-2R\theta/\pi} \right]_{0}^{\pi/2} = \frac{\pi}{2R} (1 - e^{-R}).$$

So we have

$$|I_R| \leq R^q J_r \leq \frac{\pi}{2R^{1-q}},$$

which tends to 0 as $R \to \infty$, since $q < 1$. This concludes the proof.

A variation of this method can be used to evaluate $S_1$. An efficient version is as follows. Let $f(z) = \frac{1}{z}(e^{iz} - 1)$ (which has no pole at 0). We consider the integral of $f(z)$ around the contour consisting of the real interval $[-R, R]$ and the semicircular arc $z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$. Lemma TPES is again needed to show that the contribution of the arc tends to zero.
References


[Jam2] G. J. O. Jameson, Integrals of the form $\int_x^\infty f(t)e^{it} dt$, at www.maths.lancs.ac.uk/~jameson/


