A simple proof of Stirling's formula for the gamma function

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This is a slightly modified version of the article [Jam2]. Stirling's formula for integers states that

$$n! \sim C n^{n+\frac{1}{2}} e^{-n} \quad \text{as } n \to \infty, \tag{1}$$

where $C = (2\pi)^{1/2}$ and the notation $f(n) \sim g(n)$ means that $f(n)/g(n) \to 1$ as $n \to \infty$.

A great deal has been written about Stirling's formula. At this point I will just mention David Fowler's *Gazette* article [Fow], which contains an interesting historical survey.

The continuous extension of factorials is, of course, the gamma function. The established notation, for better or worse, is such that $\Gamma(n)$ equals $(n-1)!$ rather than $n!$. Stirling's formula duly extends to the gamma function, in the form

$$\Gamma(x) \sim C x^{x-\frac{1}{2}} e^{-x} \quad \text{as } x \to \infty. \tag{2}$$

To recapture (1), just state (2) with $x = n$ and multiply by n.

One might expect the proof of (2) to require a lot more work than the proof of (1). However, this is not true! Here, with only a little more effort than what is needed for the integer case, we will prove the following more specific version of (2), incorporating upper and lower bounds.

THEOREM 1. *Let* $S(x) = x^{x-\frac{1}{2}} e^{-x}$ *and* $C = (2\pi)^{1/2}$. *Then for all* $x > 0$,

$$CS(x) \le \Gamma(x) \le CS(x) e^{1/(12x)}. \tag{3}$$

Proofs can be seen in numerous books, e. g. [Art], [AAR], so a compelling excuse is needed for presenting yet another one. Readers can judge whether the measure of simplification achieved by the method given here is sufficiently compelling.

We will not need to assume any knowledge of the gamma function beyond Euler's limit form of its definition and the fundamental identity $\Gamma(x+1) = x\Gamma(x)$.

In common with most proofs of Stirling's formula, we concentrate on showing that (3) holds for *some* constant C. Having done so, one can then use the Wallis product to establish that $C = (2\pi)^{1/2}$. See, for example, [Fow] or [AAR, p. 20]. I am not offering any novelty for this part of the argument.

Also in common with most proofs, we really work with $\log\Gamma(x)$. Clearly, (3) is equivalent to:

$$\log\Gamma(x) = (x - \tfrac{1}{2})\log x - x + c + P(x), \tag{4}$$

where $c = \frac{1}{2}\log(2\pi)$ and $0 \le P(x) \le \frac{1}{12x}$.

The distinctive feature of our method is to estimate $\log\Gamma(x)$ by estimating its derivative. We explain later why this leads to a gain in simplicity. Now $\frac{d}{dx}\log\Gamma(x) = \Gamma'(x)/\Gamma(x)$. Following the usual custom in literature on the gamma function, we denote this function by $\psi(x)$. Many of the statements and formulae relating to the gamma function have a simpler counterpart for $\psi(x)$, and Stirling's formula is no exception. The corresponding statement is:

THEOREM 2. *For all real* $x > 0$,

$$\psi(x) = \log x - \frac{1}{2x} - p(x), \tag{5}$$

where $0 \le p(x) \le 1/(12x^2)$.

Once we know this, Theorem 1 follows in a simple and elegant way, as we now show:

Deduction of Theorem 1 from Theorem 2. Let

$$g(x) = \log\Gamma(x) - (x - \tfrac{1}{2})\log x + x.$$

We have to show that $c \le g(x) \le c + \frac{1}{12x}$ for some constant c. Now

$$g'(x) = \psi(x) - \log x + \frac{1}{2x} = -p(x),$$

so by Theorem 2, $0 \le -g'(x) \le \frac{1}{12x^2}$. Hence $\int_1^\infty g'(t)\,dt$ is convergent (say to I). So

$$g(x) - g(1) = \int_1^x g'(t)\,dt = I - \int_x^\infty g'(t)\,dt,$$

or $g(x) = c - \int_x^\infty g'(t)\,dt$, where $c = I + g(1)$. The required statement follows, since

$$0 \le -\int_x^\infty g'(t)\,dt \le \int_x^\infty \frac{1}{12t^2}\,dt = \frac{1}{12x}. \qquad \square$$

So we have to prove (5). We start from Euler's limit definition of the gamma function: $\Gamma(x) = \lim_{n\to\infty} G_n(x)$, where

$$G_n(x) = \frac{n^x (n-1)!}{x(x+1)\dots(x+n-1)}. \tag{6}$$

(An alternative version, clearly equivalent in the limit, has a further n at the top and $(x+n)$ at the bottom.) Note that $G_n(1) = 1$ for all n, so the definition immediately gives $\Gamma(1) = 1$.

Of course, it needs to be shown that $\lim_{n\to\infty} G_n(x)$ exists for general x. This can be seen in any account of the gamma function; a simple method was presented in [Jam1]. The identity $\Gamma(x+1) = x\Gamma(x)$, and hence $\Gamma(n) = (n-1)!$, follows at once from

$$G_n(x+1) = \frac{n}{n+x}\, xG_n(x).$$

Standard accounts also include the equivalence of (6) with Euler's other definition of the gamma function, the integral $\int_0^\infty t^{x-1}e^{-t}\,dt$, but this is not needed for our purposes.

Now

$$\log G_n(x) = x\log n + \log[(n-1)!] - \sum_{r=0}^{n-1} \log(x+r), \tag{7}$$

so $\psi(x) = \lim_{n\to\infty} \psi_n(x)$, where

$$\psi_n(x) = \frac{d}{dx}\log G_n(x) = \log n - \sum_{r=0}^{n-1} \frac{1}{r+x}. \tag{8}$$

(Here, of course, we are taking it for granted that the derivative of the limit is the limit of the derivatives; for purists, this is justified by uniform convergence of $(\psi_n(x))$ on bounded intervals, which they may care to prove as an exercise.)

To put (5) into perspective,, we digress briefly to mention some more elementary facts about $\psi(x)$, though they are not strictly needed for Theorem 2.

PROPOSITION 1. *The function $\psi(x)$ is increasing. Further:*

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{9}$$

$$\log x - \frac{1}{x} \le \psi(x) \le \log x. \tag{10}$$

Proof. From (8), it is clear that $\psi_n(x)$ increases with x, hence so does $\psi(x)$. Since $\Gamma(x+1) = x\Gamma(x)$, we have $\log\Gamma(x+1) = \log\Gamma(x) + \log x$. Differentiation gives (9). Also, by the mean-value theorem, it follows that $\log x = \psi(\xi)$ for some ξ in $(x, x+1)$. Since $\psi(x)$ is increasing, this shows that $\psi(x) \le \log x \le \psi(x+1)$. With (9), this gives (10). □

Note. As the reader may know, a function with increasing derivative is *convex* (informally, this means curving upwards). So $\log\Gamma(x)$ is convex. The celebrated Bohr-Mollerup theorem states that the gamma function is the unique function $f(x)$ with the property that $\log f(x)$ is convex, together with $f(x+1) = xf(x)$ and $f(1) = 1$. For a proof, see [Jam1].

Clearly, (5) is a greatly enhanced version of (10). We now embark on its proof. Write

$$S_n(x) = \sum_{r=0}^{n-1} \frac{1}{r+x},$$

so that $\psi_n(x) = \log n - S_n(x)$. Also, write

$$S_n^*(x) = \frac{1}{2x} + \sum_{r=1}^{n-1} \frac{1}{r+x} + \frac{1}{2(n+x)} = S_n(x) - \frac{1}{2x} + \frac{1}{2(n+x)}.$$

Clearly, $\psi(x) = \lim_{n\to\infty} \psi_n^*(x)$, where

$$\psi_n^*(x) = \log n - S_n^*(x) - \frac{1}{2x}. \tag{11}$$

Note that $S_n^*(x) = \sum_{r=0}^{n-1} T(r+x)$, where $T(x)$ is defined by

$$T(x) = \frac{1}{2x} + \frac{1}{2(x+1)}.$$

Now $T(x)$ is the trapezium-rule approximation to $\int_x^{x+1} t^{-1}\,dt = \log(x+1) - \log x$. The key step is the following result, giving bounds for the error in this approximation.

PROPOSITION 2. *For all $x > 0$, we have*

$$T(x) = \log(x+1) - \log x + \Delta(x), \tag{12}$$

where

$$0 \le \Delta(x) \le \frac{1}{12x^2} - \frac{1}{12(x+1)^2}. \tag{13}$$

Hence

$$S_n^*(x) = \log(n+x) - \log x + p_n(x), \tag{14}$$

where $p_n(x)$ increases with n and $0 \le p_n(x) \le 1/(12x^2)$ for all n.

We give two alternative proofs. The first method is based on the power series for $(1+y)^{-1}$, $(1+y)^{-2}$ and $\log(1+y)$; it can be traced to [Art, p. 21].

Proof 1. We have $\log(x+1) - \log x = \log(1+1/x)$, and

$$1 + \frac{1}{x} = \frac{1+y}{1-y},$$

where $y = 1/(2x+1)$. Note that $0 < y < 1$. By the power series for $\log(1+y)$,

$$\log(1+1/x) = \log(1+y) - \log(1-y) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\right).$$

Also,

$$\frac{1}{2x} = \frac{y}{1-y}, \qquad \frac{1}{2(x+1)} = \frac{y}{1+y},$$

so by the geometric series

$$T(x) = 2(y + y^3 + y^5 + \cdots),$$

hence $\log(1 + 1/x) < T(x)$ and

$$\Delta(x) = T(x) - \log(1 + 1/x) \le \tfrac{4}{3}y^3 + 2(y^5 + y^7 + \cdots).$$

Meanwhile, $(1 + y)^{-2} = \sum_{n=0}^{\infty}(n + 1)y^n$, so

$$\frac{1}{4x^2} - \frac{1}{4(x+1)^2} = \frac{y^2}{(1-y)^2} - \frac{y^2}{(1+y)^2} = 2y^2(2y + 4y^3 + 6y^5 + \cdots) = \sum_{n=1}^{\infty} 4ny^{2n+1}.$$

Denote this by $U(x)$. Term-by-term comparison shows that $\Delta(x) \le \frac{1}{3}U(x)$.

By addition and cancellation, we now have

$$S_n^*(x) = \sum_{r=0}^{n-1} T(r + x) = \log(n + x) - \log x + p_n(x),$$

where $p_n(x) = \sum_{r=0}^{n-1} \Delta(r + x)$, so $p_n(x)$ is non-negative and increasing with n, also

$$p_n(x) \le \frac{1}{12}\sum_{r=0}^{n-1}\left(\frac{1}{(x+r)^2} - \frac{1}{(x+r+1)^2}\right) = \frac{1}{12}\left(\frac{1}{x^2} - \frac{1}{n+x)^2}\right). \qquad \square$$

The second proof is by Euler-Maclaurin summation (see, for example, the companion article [Jam3]). For a function f and integers m, n, write

$$S_{m,n}^*(f) = \tfrac{1}{2}f(m) + \sum_{r=m+1}^{n-1} f(r) + \tfrac{1}{2}f(n).$$

One says that f is *completely monotonic* on $[0, \infty)$ if $(-1)^k f^{(k)}(t) \ge 0$ for $k \ge 0$ and $t \ge 0$, so the even-numbered derivatives are non-negative and the odd-numbered ones non-positive. This condition is satisfied by $f(t) = 1/(t+x)$, where $x > 0$. The first stage of Euler-Maclaurin summmation states that for such functions f,

$$S_{m,n}^*(f) - \int_m^n f(t)\, dt = p(m, n),$$

where $p(m, n)$ increases with n and

$$0 \le p(m, n) \le \frac{1}{12}\Big(f'(n) - f'(m)\Big).$$

Applied to $f(t) = 1/(t+x)$, this gives (14).

Proof of Theorem 2. By (11) and (14),

$$\psi_n^*(x) = \log n - \log(n+x) + \log x - \frac{1}{2x} - p_n(x).$$

Also, $p_n(x)$ tends to $p(x)$ (say) as $n \to \infty$, where $0 \le p(x) \le 1/(12x^2)$. Now $\log(n+x) - \log n = \log(1 + \frac{x}{n}) \to 0$ as $n \to \infty$, so by taking the limit as $n \to \infty$, we obtain (5). □

A further degree of accuracy. A further step of Euler-Maclaurin summation gives, for completely monotonic functions,

$$S_{m,n}^*(f) - \int_m^n f(t)\,dt = \tfrac{1}{12}\Big(f'(n) - f'(m)\Big) - r(m,n),$$

where $r(m,n)$ increases with n and

$$0 \le r(m,n) \le \tfrac{1}{720}\Big(f^{(3)}(n) - f^{(3)}(m)\Big).$$

So we have

$$S_n^*(x) = \log(n+x) - \log x + \frac{1}{12x^2} - \frac{1}{12(n+x)^2} - r_n(x),$$

where $r_n(x)$ increases with n and $0 \le r_n(x) \le 1/(120x^4)$. With sufficient effort, this estimate can also be established by the power series method. Fed into the proofs of Theorem 2 and Theorem 1, it leads to

$$\psi(x) = \log x - \frac{1}{2x} - \frac{1}{12x^2} + r(x),$$

where

$$0 \le r(x) \le \frac{1}{120x^4},$$

and

$$\log\Gamma(x) = (x - \tfrac{1}{2})\log x - x + c + \frac{1}{12x} - R(x),$$

where

$$0 \le R(x) \le \frac{1}{360x^3}.$$

The Euler-Maclaurin process delivers further terms if desired: the next term for $\psi(x)$ is $-1/(252x^6)$. (Compare [AAR, p. 22], where the estimation of $\psi(x)$ is derived in a rather different way.) For $\Gamma(x)$ itself, this leads to an asymptotic expansion of the form

$$\Gamma(x) = CS(x)\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right).$$

Comparison with direct estimation of $\log\Gamma(x)$. Wouldn't it be more direct to estimate $\log\Gamma(x)$ directly, using (7)? One needs the following analogue of Proposition 2, which again can be proved either from power series or by Euler-Maclaurin summation:

$$\tfrac{1}{2}\log x + \tfrac{1}{2}\log(x+1) = (x+1)\log(x+1) - x\log x - 1 - \Delta_1(x), \tag{15}$$

where $0 < \Delta_1(x) \le 1/[12x(x+1)]$. If we only want Stirling's formula for *integers*, we have simply $\log[(n-1)!] = \sum_{r=0}^{n-1} \log r$ instead of (7), and this method is indeed highly efficient. It can be seen, for example, in [Fe, p. 52–53], or in a more accurate form in [Jam3]. But, for the gamma function, (7) contains both $\log[(n-1)!]$ and $\sum_{r=0}^{n-1} \log(x+r)$. One has to apply (14) to each of these and combine the results. In a sense, this doubles the work. The estimation of $\psi(x)$ was simpler because of the disappearance of the term $\log[(n-1)!]$ under differentiation.

The complex case. Euler's limit definition (6) applies equally for a complex variable z. By a suitable development of the method (e.g. [AAR] or [Cop, chap. 9]), one can derive the following variant of (4): *Let $z = re^{i\theta}$, where $|\theta| \le \pi - \delta$ for some $\delta > 0$. Then there is a logarithm $L(z)$ of $\Gamma(z)$ satisfying*

$$L(z) = (z - \tfrac{1}{2}) \log z - z + c + P(z),$$

where $|P(z)| \le A/r$ for some constant A depending on δ.

References

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