Introduction

The polylogarithm function is defined for $|x| < 1$ and any real $s$ by

$$\text{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}. \quad (1)$$

For particular values of $s$, it can then be extended analytically to a wider range of $x$ (real or complex). It is simultaneously a power series in $x$ and a Dirichlet series in $s$. The first published study of the function was by A. Jonqui`ere in 1889, and it is sometimes called Jonqui`ere’s function. Note that “polylogarithmic functions” means something quite different!

These notes are a summary of some results on polylogarithms that I have seen stated in various places, such as Wikipedia and [BBC], largely with proofs that I have worked out for myself. The methods are mostly elementary. We restrict to integer $s = k \geq 0$, and $x$ usually real. However, in places the reader is invited to accept that formulae established for real $x$ also apply for complex $x$; I can supply a rigorous justification if pressed.

We start with a few immediate facts. Firstly,

$$\text{Li}_0(x) = \frac{x}{1-x},$$

extending meromorphically to the whole plane, and

$$\text{Li}_1(x) = -\log(1-x),$$

extending to all real $x < 1$ (or complex $x$ excluding real $x \geq 1$).

For $|x| < 1$, or $|x| \leq 1$ with $k > 1$, we have

$$\text{Li}_k(x) + \text{Li}_k(-x) = 2^{1-k}\text{Li}_k(x^2). \quad (2)$$

For $k > 1$,

$$\text{Li}_k(1) = \zeta(k), \quad (3)$$

$$\text{Li}_k(-1) = -(1-2^{1-k})\zeta(k). \quad (4)$$

In particular, $\text{Li}_2(-1) = \frac{1}{2}\zeta(2)$ and $\text{Li}_3(-1) = -\frac{3}{4}\zeta(3)$. 

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For $k \geq 2$ and $0 \leq x \leq 1$, we have from the series $0 \leq \text{Li}_k(x) \leq \zeta(k)x$ and $-x \leq \text{Li}_k(x) \leq 0$. Further, $x\text{Li}_k'(x) = \text{Li}_{k-1}(x)$ (also, $\text{Li}_k'(0) = 1$) and

$$
\text{Li}_k(x) = \int_0^x \frac{\text{Li}_{k-1}(t)}{t} \, dt. \tag{5}
$$

(Termwise integration of the series is justified by uniform convergence for $|x| < 1$, and then by continuity of $\text{Li}_k(x)$ at 1 and $-1$.)

We work out various special values of $\text{Li}_2$ and $\text{Li}_3$, which were found by Landen as long ago as 1780. In particular, we evaluate $\text{Li}_3(\phi^{-2})$, where

$$
\phi = \frac{1 + \sqrt{5}}{2}
$$

is the golden ratio. This is used to prove Hjortnaes’ series expression (35) for $\zeta(3)$. My proof of Comtet’s corresponding sum for $\zeta(4)$ involves quantities like $\text{Li}_4(e^{\pi i/3})$.

The dilogarithm

The dilogarithm $\text{Li}_2$ is sometimes called Spence’s function, in tribute to a pioneering study of it by W. Spence in 1809. Beware of the fact that some computer algebra systems denote $\text{Li}_2(1-x)$ by dilog($x$).

By termwise integration of the series

$$
-\frac{\log(1-t)}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n}
$$

we obtain the identity

$$
\text{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t} \, dt, \tag{6}
$$

which we now use to define $\text{Li}_2(x)$ for all $x \leq 1$. Note that

$$
\text{Li}_2(1) = \zeta(2) = -\int_0^1 \frac{\log(1-t)}{t} \, dt
$$

and for $x > 0$,

$$
\text{Li}_2(-x) = -\int_0^{-x} \frac{\log(1-t)}{t} \, dt = -\int_0^x \frac{\log(1+t)}{t} \, dt.
$$

Clearly, $\text{Li}_2'(x) = -\log(1-x)/x$, and hence $\text{Li}_2(x)$ is strictly increasing for all $x < 1$.

For $0 < x < 1$, we have

$$
\text{Li}_2(x) = \zeta(2) + \int_x^1 \frac{\log(1-u)}{u} \, du = \zeta(2) + \int_0^{1-x} \frac{\log t}{1-t} \, dt.
$$
Replacing $x$ by $1 - x$, this says

\[
\text{Li}_2(1 - x) - \zeta(2) = \int_0^x \frac{\log t}{1 - t} dt \\
= \left[ -\log t \log(1 - t) \right]_0^x + \int_0^x \frac{\log(1 - t)}{t} dt \\
= -\log x \log(1 - x) - \text{Li}_2(x),
\]

so we have the following identity, known as Euler’s reflection formula:

\[
\text{Li}_2(x) + \text{Li}_2(1 - x) = \zeta(2) - \log x \log(1 - x). \quad (7)
\]

In particular, using the well-known identity $\zeta(2) = \pi^2/6$, we have

\[
\text{Li}_2(\frac{1}{2}) = \frac{1}{2}(\zeta(2) - \log^2 2) = \frac{\pi^2}{12} - \frac{1}{2} \log^2 2. \quad (8)
\]

We can rewrite (8) as $\zeta(2) = \log^2 2 + 2 \sum_{n=1}^{\infty} (1/n^2 2^n)$. Taking $\log 2$ as known, this can be regarded as a series for $\zeta(2)$ that converges much more rapidly than $\sum_{n=1}^{\infty} (1/n^2)$.

We now give two further identities for $\text{Li}_2$. Firstly, for $x > 0$, we have

\[
\text{Li}_2(-x) = \text{Li}_2(-1) + F(x) = -\frac{i}{2} \zeta(2) + F(x)
\]

where

\[
F(x) = \int_{x}^{1} \frac{\log(1 + t)}{t} dt.
\]

Combined with the same statement for $1/x$, this gives

\[
\text{Li}_2(-x) + \text{Li}_2(-\frac{1}{x}) = -\zeta(2) + F(x) + F(\frac{1}{x}).
\]

But

\[
F(x) = \int_{1}^{1/x} u \log \left( 1 + \frac{1}{u} \right) \frac{1}{u^2} du \\
= \int_{1}^{1/x} \log(1 + u) - \log u \frac{u}{du} \\
= -F(\frac{1}{x}) - \frac{1}{2} \log^2 x,
\]

so we have

\[
\text{Li}_2(-x) + \text{Li}_2(-\frac{1}{x}) = -\zeta(2) - \frac{1}{2} \log^2 x. \quad (9)
\]

So for $x > 1$, $\text{Li}_2(-x) = -\frac{1}{2} \log^2 x - \zeta(2) + r(x)$, where $0 < r(x) \leq 1/x$. 

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For $0 < x < 1$, we have also

$$\text{Li}_2(1 - x) = -\int_x^1 \frac{\log t}{1 - t} \, dt$$

$$= \int_1^{1/x} \log u \frac{1}{1 - 1/u} \frac{1}{u^2} \, du$$

$$= \int_1^{1/x} \log u \left( \frac{1}{u - 1} - \frac{1}{u} \right) \, du$$

$$= \int_0^{1/x - 1} \frac{\log(1 + v)}{v} \, dv - \frac{1}{2} \log^2 x$$

$$= -\text{Li}_2(1 - \frac{1}{x}) - \frac{1}{2} \log^2 x.$$

Applying this also to $1/x$, we have the following identity for all $x > 0$:

$$\text{Li}_2(1 - x) + \text{Li}_2(1 - \frac{1}{x}) = -\frac{1}{2} \log^2 x. \quad (10)$$

Using (2), (7), (9) and (10), we can reduce the computation of $\text{Li}_2(x)$ for any $x < 1$ to computation of values of at most a few values of $\text{Li}_2(y)$ with $|y| \leq \frac{1}{2}$.

Now consider $\phi$. Note that the fundamental equation satisfied by $\phi$ may be written:

$$\phi^2 = \phi + 1, \quad \phi = 1 + \phi^{-1}, \quad 1 = \phi^{-1} + \phi^{-2}.$$

Putting $x = \phi^{-1}$ in (2), (10), (7) respectively, we obtain three linear equations relating $\text{Li}_2(\phi^{-1})$, $\text{Li}_2(-\phi^{-1})$, and $\text{Li}_2(\phi^{-2})$:

$$\text{Li}_2(\phi^{-1}) + \text{Li}_2(-\phi^{-1}) = \frac{1}{2} \text{Li}_2(\phi^{-2}), \quad (11)$$

$$\text{Li}_2(\phi^{-2}) + \text{Li}_2(-\phi^{-1}) = -\frac{1}{2} \log^2 \phi. \quad (12)$$

$$\text{Li}_2(\phi^{-1}) + \text{Li}_2(\phi^{-2}) = \zeta(2) - 2 \log^2 \phi. \quad (13)$$

Subtracting (12) from (11) gives

$$\text{Li}_2(\phi^{-1}) - \frac{3}{2} \text{Li}_2(\phi^{-2}) = \frac{1}{2} \log^2 \phi. \quad (14)$$

Now subtracting (14) from (13), we have

$$\frac{5}{2} \text{Li}_2(\phi^{-2}) = \zeta(2) - \frac{5}{2} \log^2 \phi,$$

i.e.

$$\text{Li}_2(\phi^{-2}) = \frac{2}{5} \zeta(2) - \log^2 \phi = \frac{\pi^2}{15} - \log^2 \phi. \quad (15)$$

Now (13) or (14) gives

$$\text{Li}_2(\phi^{-1}) = \frac{3}{5} \zeta(2) - \log^2 \phi = \frac{\pi^2}{10} - \log^2 \phi, \quad (16)$$
while (12) gives

\[ \text{Li}_2(-\phi^{-1}) = -\frac{2}{3} \zeta(2) + \frac{1}{2} \log^2 \phi = -\frac{\pi^2}{15} + \frac{1}{2} \log^2 \phi, \]  

hence finally (9) gives

\[ \text{Li}_2(-\phi) = -\frac{3}{5} \zeta(2) - \log^2 \phi = -\frac{\pi^2}{10} - \log^2 \phi. \]  

If we include \( \text{Li}_2(0) = 0 \), then we have found eight special values of \( \text{Li}_2(x) \) for real \( x \). I have seen it stated that these are the only ones known to be expressible in this kind of way.

We now briefly consider complex arguments. Substitution in the series gives

\[ \text{Li}_2(i) = -\frac{1}{8} \zeta(2) + iG, \quad \text{Li}_2(-i) = -\frac{1}{8} \zeta(2) - iG, \]

where

\[ G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots \]

is Catalan’s constant. Assuming (10) valid for complex arguments, take \( x = 1 + i \), so that \( 1 - x = -i \) and \( 1 - x^{-1} = \frac{1}{2} + \frac{1}{2} i \). We obtain

\[ \text{Li}_2 \left( \frac{1 + i}{2} \right) = \frac{1}{8} \zeta(2) + iG - \frac{1}{2} \left( \frac{1}{2} \log 2 + \frac{\pi i}{4} \right)^2 \]
\[ = \frac{5\pi^2}{96} - \frac{1}{8} \log^2 2 + i \left( G - \frac{\pi}{8} \log 2 \right). \]

For \( e^{i\theta} \), we have

\[ \Re \text{Li}_2(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = \pi^2 \tilde{B}_2 \left( \frac{\theta}{2\pi} \right), \]

where \( B_2 \) is the Bernoulli polynomial \( B_2(x) = x^2 - x + \frac{1}{6} \) and \( \tilde{B}_2(x) = B_2(\{x\}) \). Also,

\[ \Im \text{Li}_2(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}, \]

which is called the ‘Clausen’ function in [BBC].

There are many further formulae involving \( \text{Li}_2 \): see, for example, Wikipedia, [Lew] and [Max]. One is ‘Abel’s identity’, which equates a combination of five \( \text{Li}_2 \) values to \( \log(1 - x) \log(1 - y) \).

Note that the ‘Rogers dilogarithm’ is

\[ L(x) = -\frac{1}{2} \int_0^x \left( \log \frac{1 - u}{u} + \log \frac{u}{1 - u} \right) du \]
\[ = \text{Li}_2(x) + \frac{1}{2} \log x \log(1 - x). \]
The trilogarithm

For $0 < |x| \leq 1$, we have by (5)

$$\text{Li}_3(x) = \int_0^x \frac{\text{Li}_2(t)}{t} \, dt.$$  \hspace{1cm} (19)

Since $\text{Li}_2(x)$ has been defined by (6) for all $x \leq 1$, we can use (19) to define $\text{Li}_3(x)$ for all such $x$.

So we have

\[
\text{Li}_3(x) = \int_0^x \frac{\text{Li}_2(u)}{u} \, du
\]

\[= -\int_0^x \frac{1}{u} \int_0^u \frac{\log(1-t)}{t} \, dt \, du\]

\[= -\int_0^x \frac{\log(1-t)}{t} \int_t^x \frac{1}{u} \, du \, dt\]

\[= \int_0^x \frac{\log(1-t)}{t} \log \frac{t}{x} \, dt, \hspace{1cm} (20)\]

valid regardless of the sign of $x$. For $0 < x < 1$, substitution of (6) now gives

$$\text{Li}_3(x) = \int_0^x \frac{\log t \log(1-t)}{t} \, dt + \text{Li}_2(x) \log x,$$  \hspace{1cm} (21)

which can also be derived directly from the series expression. Further, for $x > 0$,

$$\text{Li}_3(-x) = \int_0^{-x} \frac{\log(1-u)}{u} \log \frac{u}{-x} \, du$$

$$= \int_0^{-x} \frac{\log(1+t)}{t} \log \frac{t}{x} \, dt$$

$$= \int_0^{-x} \log t \log(1+t) \, dt + \text{Li}_2(-x) \log x.$$  \hspace{1cm} (22)

In particular,

$$\zeta(3) = \text{Li}_3(1) = \int_0^1 \frac{\log t \log(1-t)}{t} \, dt.$$  \hspace{1cm} (23)

Integrating by parts in (21), we obtain for $0 \leq x \leq 1$,

$$\text{Li}_3(x) = \left[ \frac{1}{2} \log^2 t \log(1-t) \right]_0^x - \int_0^x \frac{1}{2} \log^2 t \frac{-1}{1-t} \, dt + \text{Li}_2(x) \log x,$$

$$= \frac{1}{2} \int_0^x \frac{\log^2 t}{1-t} \, dt + \text{Li}_2(x) \log x + \frac{1}{2} \log^2 x \log(1-x).$$  \hspace{1cm} (24)

In particular,

$$\int_0^1 \frac{\log^2 t}{1-t} \, dt = 2\zeta(3).$$

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Using (9), we can derive a corresponding statement for \( \text{Li}_3 \). [This has been added to Tim’s original version, in which the following identity was deduced from (26) below.] For \( x > 0 \), write
\[
F(x) = \int_x^1 \frac{\log t \log(1 + t)}{t} \, dt.
\]
By (22),
\[
\text{Li}_3(-x) = F(0) - F(x) + \text{Li}_2(x) \log x.
\]
With \( x \) replaced by \(-x\), this says
\[
\text{Li}_3(-\frac{1}{x}) = F(0) - F\left(\frac{1}{x}\right) - \text{Li}_2\left(\frac{1}{x}\right) \log x.
\]
Combining these expressions and using (9), we have
\[
\text{Li}_3(-\frac{1}{x}) - \text{Li}_3(-x) = F\left(\frac{1}{x}\right) - \text{Li}_2\left(\frac{1}{x}\right) + \zeta(2) \log x + \frac{1}{2} \log^3 x.
\]
Substituting \( t = 1/u \), we have
\[
F(x) = \int_{1/x}^1 \log u \log\left(1 + \frac{1}{u}\right) \frac{1}{u} \, du
= - \int_1^{1/x} \frac{1}{u} \log u [\log(u + 1) - \log u] \, du
= \frac{1}{3} \log^3 \frac{1}{x} + F\left(\frac{1}{x}\right)
= -\frac{1}{3} \log^3 x + F\left(\frac{1}{x}\right),
\]
and we conclude that
\[
\text{Li}_3(-\frac{1}{x}) - \text{Li}_3(-x) = \zeta(2) \log x + \frac{1}{6} \log^3 x.
\]
It follows that for \( x > 1 \), \( \text{Li}_3(-x) = -\frac{1}{6} \log^3 x - \zeta(2) \log x - r(x) \), where \( 0 < r(x) \leq \frac{1}{x} \).

We now prove another identity relating \( \text{Li}_3(x) \), \( \text{Li}_3(1-x) \) and \( \text{Li}_3(1-\frac{1}{x}) \). It again goes back to Landen around 1780. [The proof given here was found among Tim’s handwritten papers; it replaces a more complicated proof in Tim’s typed version.]

Let \( 0 < x < 1 \) and
\[
S(x) = \text{Li}_3(x) + \text{Li}_3(1-x) + \text{Li}_3(1-\frac{1}{x}).
\]
We express everything in terms of integrals on \([x, 1]\). The details are kept simpler by expressing the integrands in terms of \( \text{Li}_2 \) rather than \( \log \) terms. Firstly, using (19) and (7), we
have
\[
\text{Li}_3(x) = \int_0^x \frac{\text{Li}_2(t)}{t} \, dt
\]
\[
= \zeta(3) - \int_x^1 \frac{\text{Li}_2(t)}{t} \, dt
\]
\[
= \zeta(3) - \int_x^1 \frac{1}{t} \left[ \zeta(2) - \log t \log(1-t) - \text{Li}_2(1-t) \right] \, dt
\]
\[
= \zeta(3) + \zeta(2) \log x + \int_x^1 \frac{\log t \log(1-t)}{t} \, dt + \int_x^1 \frac{\text{Li}_2(1-t)}{t} \, dt.
\]

Secondly,
\[
\text{Li}_3(1-x) = \int_0^{1-x} \frac{\text{Li}_2(u)}{u} \, du = \int_x^1 \frac{\text{Li}_2(1-t)}{1-t} \, dt.
\]

Thirdly, with obvious substitutions,
\[
\text{Li}_3\left(1 - \frac{1}{x}\right) = \int_0^{1-\frac{1}{x}} \frac{\text{Li}_2(-u)}{u} \, du
\]
\[
= \int_1^{1-\frac{1}{x}} \frac{\text{Li}_2(1-v)}{v-1} \, dv
\]
\[
= \int_x^1 \frac{\text{Li}_2\left(1 - \frac{1}{t}\right)}{1-t} \frac{1}{t^2} \, dt
\]
\[
= \int_x^1 \frac{\text{Li}_2\left(1 - \frac{1}{t}\right)}{t(1-t)} \, dt.
\]

Combining the three terms, we have
\[
S(x) = \zeta(3) + \zeta(2) \log x + \int_x^1 G(t) \, dt,
\]
where
\[
G(t) = \frac{\log t \log(1-t)}{t} + \left( \frac{1}{t} + \frac{1}{1-t} \right) \left[ \text{Li}_2(1-t) + \text{Li}_2(1-\frac{1}{t}) \right].
\]

Now by (10), we have
\[
G(t) = \frac{1}{t} \log t \log(1-t) - \frac{1}{2} \log^2 t \left( \frac{1}{t} + \frac{1}{1-t} \right)
\]
\[
= \frac{d}{dt} \left[ \frac{1}{2} \log^2 t \log(1-t) - \frac{1}{6} \log^3 t \right],
\]
so
\[
\int_x^1 G(t) \, dt = \frac{1}{6} \log^3 x - \frac{1}{2} \log^2 x \log(1-x).
\]

So for 0 < x < 1, we have the identity
\[
\text{Li}_3(x) + \text{Li}_3(1-x) + \text{Li}_3\left(1 - \frac{1}{x}\right) = \zeta(3) + \zeta(2) \log x + \frac{1}{6} \log^3 x - \frac{1}{2} \log^2 x \log(1-x). \quad (26)
\]
One can deduce (25) by writing this with $1 - x$ in place of $x$, taking the difference and putting $x$ for $\frac{1}{2} - 1$.

Putting $x = \frac{1}{2}$ in (26) gives

$$2 \text{Li}_3\left(\frac{1}{2}\right) = -\text{Li}_3(-1) + \zeta(3) + \zeta(2) \log \frac{1}{2} - \frac{1}{2} \log^2 2 \log \frac{1}{2} + \frac{1}{6} \log^3 \frac{1}{2}$$

$$= \frac{3}{4} \zeta(3) + \zeta(3) - \zeta(2) \log 2 + \frac{1}{2} \log^3 2 - \frac{1}{6} \log^3 2$$

$$= \frac{7}{4} \zeta(3) - \zeta(2) \log 2 + \frac{3}{3} \log^3 2,$$

so that

$$\text{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8} \zeta(3) - \frac{\pi^2}{12} \log 2 + \frac{1}{6} \log^3 2.$$ (27)

This gives a series converging rapidly to $\zeta(3)$.

Putting $x = \phi^{-1}$ and using the identities relating $\phi$, $\phi^{-1}$ and $\phi^{-2}$, we get

$$\text{Li}_3(\phi^{-1}) + \text{Li}_3(\phi^{-2}) + \text{Li}_3(-\phi^{-1}) = \zeta(3) - \zeta(2) \log \phi - \frac{1}{2} \log^2 \phi \log \phi^{-2} + \frac{1}{6} \log^3 \phi^{-1}$$

$$= \zeta(3) - \zeta(2) \log \phi + \log^3 \phi - \frac{1}{6} \log^3 \phi$$

$$= \zeta(3) - \zeta(2) \log \phi + \frac{5}{6} \log^3 \phi.$$

But by (2), $\text{Li}_3(\phi^{-1}) + \text{Li}_3(-\phi^{-1}) = \frac{1}{4} \text{Li}_3(\phi^{-2})$, hence

$$\text{Li}_3(\phi^{-2}) = \frac{4}{5} \zeta(3) - \frac{2\pi^2}{15} \log \phi + \frac{2}{3} \log^3 \phi.$$ (28)

It seems that there are no known such special values involving $\phi$ for higher $k$, although there are important relations between various values. This is to do with ‘polylogarithm ladders’ (introduced by Leonard Lewin), which are important in $K$-theory and algebraic geometry, and can be used in conjunction with the BBP algorithm for computing various constants.

A proof of $\zeta(2) = \pi^2/6$ and some power series related to $\arcsin x$

For $0 \leq x < 1$, the substitution $t = \frac{\sqrt{1-x^2}}{x}u$ gives

$$x \int_0^1 \frac{dt}{1 - x^2 + x^2 t^2} = \frac{1}{\sqrt{1 - x^2}} \int_0^\frac{x}{\sqrt{1-x^2}} \frac{du}{1 + u^2}$$

$$= \frac{1}{\sqrt{1 - x^2}} \arctan \frac{x}{\sqrt{1 - x^2}}$$

$$= \frac{\arcsin x}{\sqrt{1 - x^2}}.$$ (29)

Since also

$$\frac{d}{dx} (\arcsin x)^2 = \frac{2 \arcsin x}{\sqrt{1 - x^2}},$$
we now have

\[
\arcsin x \sqrt{x^2 - 1} = \int_0^x \frac{2 \arcsin y}{\sqrt{1 - y^2}} \, dy
\]

\[
= \int_0^x \int_0^1 \frac{2y}{y^2 + 2y^2 t^2} \, dt \, dy
\]

\[
= \int_0^1 \int_0^x \frac{2y}{y^2 + 2y^2 t^2} \, dy \, dt
\]

\[
= -\int_0^1 \left( \frac{1}{1 - t^2} \left[ \log(1 - y^2 + y^2 t^2) \right]^{y=x} \right) \, dt
\]

\[
= -\int_0^1 \frac{\log(1 - x^2 + xt^2)}{1 - t^2} \, dt.
\]  

(30)

In particular, to show that \( \zeta(2) = \pi^2/6 \), we have

\[
\frac{\pi^2}{4} = (\arcsin 1)^2
\]

\[
= -\int_0^1 \frac{2 \log t}{1 - t^2} \, dt
\]

\[
= -2 \sum_{n=0}^{\infty} \int_0^1 t^{2n} \log t \, dt
\]

\[
= 2 \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2}
\]

\[
= \frac{3}{2} \zeta(2).
\]

Alternatively, we can equate the integral to \( \text{Li}_2(1) - \text{Li}_2 \) by (6). This proof has similarities with the one given by Nick Lord [Lo].  

[Tim’s proof has now appeared in Math. Gazette [Jam].]

To derive a power series from (29), observe that for \( |x| < 1 \) and \( 0 \leq t \leq 1 \),

\[
\frac{x}{1 - x^2 + x^2 t^2} = \sum_{n=0}^{\infty} x^{2n+1} (1 - t^2)^n = \sum_{n=1}^{\infty} x^{2n-1} (1 - t^2)^{n-1}
\]

and

\[
\int_0^1 (1 - t^2)^{n-1} \, dt = \int_0^{\pi/2} \cos^{2n-1} \theta \, d\theta = \frac{2 \pi \cdots (2n-2)}{1 \cdots (2n-1)} = \frac{2^{2n-1}}{n!}.
\]

So we have for \( |x| < 1 \),

\[
\arcsin x \sqrt{1 - x^2} = \sum_{n=1}^{\infty} \frac{2^{2n-1} x^{2n-1}}{n!}.
\]  

(31)

This can also be seen as a case of the hypergeometric series \( F(1; 1; 2; x^2) \). For example, the case \( x = \frac{1}{2} \) gives

\[
\frac{\pi}{3 \sqrt{3}} = \sum_{n=1}^{\infty} \frac{1}{n(2n)},
\]
Formulae like this form the basis for various programs which use a spigot algorithm to calculate $\pi$, with time roughly proportional to the square of the number of digits required (e.g. one by Winter and Flimmenkamp).

Now by integrating, or by similar reasoning from (30), we have

$$ (\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{2^{2n-1}x^{2n}}{n^2 \binom{2n}{n}}, \quad (32) $$
equivalently for $|x| < 2$,

$$ \left(\arcsin \frac{x}{2}\right)^2 = \sum_{n=1}^{\infty} \frac{x^{2n}}{2n^2 \binom{2n}{n}}, \quad (33) $$

The case $x = 1$ gives

$$ \frac{\pi^2}{18} = \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}. $$

Either by the identity $\sinh^{-1} y = -i \sin^{-1} iy$, or by similar reasoning with $x^2$ replaced by $-x^2$, we have also

$$ \left(\sinh^{-1} \frac{x}{2}\right)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{2n^2 \binom{2n}{n}}, \quad (34) $$

**Hjortnaes’ series for $\zeta(3)$**

We give a proof of this identity using (34) and the values of $\text{Li}_2(\phi^{-2})$ and $\text{Li}_3(\phi^{-2})$. The connection with $\phi$ arises from the fact that $\sinh^{-1} \frac{1}{2} = \log \phi$. Let

$$ S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}. $$

Then by (34),

$$ S = \int_0^1 \frac{4}{x} \left(\sinh^{-1} \frac{x}{2}\right)^2 dx $$
$$ = \int_0^{\sinh^{-1} \frac{1}{2}} \frac{4}{2 \sinh u} u^2 \cosh u \, du \quad (x = 2 \sinh u) $$
$$ = 4 \int_0^{\log \phi} u^2 \coth u \, du $$
$$ = 4 \int_0^{\log \phi} u^2 \frac{e^{2u} + 1}{e^{2u} - 1} \, du $$
$$ = 4 \int_0^{2 \log \phi} \left(\frac{v}{2}\right)^2 \frac{e^v + 1}{e^v - 1} \, dv $$
$$ = \frac{1}{2} \int_0^{2 \log \phi} v^2 \frac{e^v + 1}{e^v - 1} \, dv $$
\[ S = \frac{1}{2} \int_0^{2 \log \phi} v^2 \left( 1 + \frac{2}{e^v - 1} \right) dv \\
= \frac{4}{3} \log^3 \phi + \int_0^{2 \log \phi} \frac{v^2}{e^v - 1} dv \\
= \frac{4}{3} \log^3 \phi + \int_1^{\phi^{-2}} \frac{\log^2 u - du}{u - 1} \\
= \frac{4}{3} \log^3 \phi + \int_1^{1 \log u} \frac{\log^2 u du}{1 - u} \\
= 2 \zeta(3) - \int_0^{\phi^{-2}} \frac{\log^2 u}{1 - u} du + \frac{4}{3} \log^3 \phi. \]

Now substituting from (24), and then the values of \( \text{Li}_2(\phi^{-2}) \) and \( \text{Li}_3(\phi^{-2}) \) from (15) and (28), we have

\[
S = 2 \zeta(3) - 2 \text{Li}_3(\phi^{-2}) + 2 \text{Li}_2(\phi^{-2}) \log \phi^{-2} + \log^2 \phi^{-2} \log(1 - \phi^{-2}) + \frac{4}{3} \log^3 \phi \\
= 2 \zeta(3) - 2 \text{Li}_3(\phi^{-2}) - 4 \text{Li}_2(\phi^{-2}) \log \phi + 4 \log^2 \phi \log \phi^{-1} + \frac{4}{3} \log^3 \phi \\
= 2 \zeta(3) - 2 \text{Li}_3(\phi^{-2}) - 4 \text{Li}_2(\phi^{-2}) \log \phi - \frac{8}{3} \log^3 \phi \\
= 2 \zeta(3) - 2 \text{Li}_3(\phi^{-2}) - 4 \left( \frac{2}{5} \zeta(2) - \log^2 \phi \right) \log \phi - \frac{8}{3} \log^3 \phi \\
= 2 \zeta(3) - 2 \text{Li}_3(\phi^{-2}) - \frac{8}{5} \zeta(2) \log \phi + \frac{4}{3} \log^3 \phi \\
= 2 \zeta(3) - 2 \left( \frac{4}{5} \zeta(3) - \frac{4}{5} \zeta(2) \log \phi + \frac{2}{3} \log^3 \phi \right) - \frac{8}{5} \zeta(2) \log \phi + \frac{4}{3} \log^3 \phi \\
= \frac{2}{5} \zeta(3),
\]

by a neat cancellation. So

\[
\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3(2n)}}. \tag{35}
\]

This series was found by Hjortnaes in 1953 [Hj]. It was used by Apéry in his proof that \( \zeta(3) \) is irrational [Ap], and has sometimes been wrongly attributed to Apéry. However, it is not used in the simpler proof by Beukers [Beu]. Somehwere I have seen a proof of (35) that does not involve polylogarithms. [Note added by Graham Jameson: a generalized version is proved in [AG].]

**Some series involving \( H_n \)**

We denote the harmonic sum by \( H_n = \sum_{r=1}^{n} \frac{1}{r} \). For \( |x| < 1 \), we have easily

\[
- \frac{\log(1 - x)}{1 - x} = \sum_{j=0}^{\infty} x^j \sum_{k=1}^{\infty} \frac{x^k}{k} = \sum_{n=1}^{\infty} H_n x^n.
\]
Similarly (or by integration of the previous case),

\[ \log^2(1 - x) = \sum_{j=1}^{\infty} \frac{x^j}{j} \sum_{k=1}^{\infty} \frac{x^k}{k}. \]

The coefficient of \( x^n \), for \( n \geq 2 \), is

\[ \sum_{j=1}^{n-1} \frac{1}{j(n-j)} = \frac{1}{n} \sum_{j=1}^{n-1} \left( \frac{1}{j} + \frac{1}{n-j} \right) = \frac{2H_{n-1}}{n}, \]

so

\[ \log^2(1 - x) = \sum_{n=2}^{\infty} \frac{2H_{n-1}}{n} x^n. \]  

(36)

Integration gives

\[ \sum_{n=2}^{\infty} \frac{2H_{n-1}}{n^2} x^n = \int_0^x \frac{\log^2(1 - t)}{t} \, dt \]  

(37)

and

\[ \sum_{n=2}^{\infty} \frac{2H_{n-1}}{n^3} x^n = \int_0^x \frac{1}{t} \int_0^t \frac{\log^2(1 - u)}{u} \, du \, dt \]

\[ = \int_0^x \frac{\log^2(1 - u)}{u} \int_u^x \frac{dt}{t} \, du \]

\[ = \int_0^x \frac{\log^2(1 - u)}{u} \log \frac{x}{u} \, du, \]  

(38)

which we’ll use with \( x = e^{\pi i/3} \) later.

Combining (37) with (24), with \( 1 - x \) replacing \( x \), we obtain the following identity, which can be regarded as the expansion of \( \text{Li}_3 \) about 1: for \( 0 < x < 1 \),

\[ \text{Li}_3(1 - x) = \zeta(3) - \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^2} x^n + \text{Li}_2(1 - x) \log(1 - x) + \frac{1}{2} \log x \log^2(1 - x). \]

Some multiple zeta values

We define the ‘multiple zeta value’ function as

\[ \zeta(s_1, \ldots, s_k) = \sum_{n_1 \cdots n_k > 0, n_1, \ldots, n_k} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}. \]

(Annoyingly this is sometimes written \( \zeta(s_k, \ldots, s_1) \) instead!) Here I will only consider \( \zeta(s, t) \).

We have

\[ \zeta(s, t) = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=m+1}^{\infty} \frac{1}{n^t} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{1}{(m+n)^t}. \]
Also, reversing the order in the first expression, we have
\[\zeta(s, t) = \sum_{n=2}^{\infty} \frac{1}{n^t} \sum_{m=1}^{n-1} \frac{1}{m^s}.\]

in particular,
\[\zeta(1, t) = \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^t}.\] \hspace{1cm} (39)

Most obviously, we have for \(\Re s > 1\)
\[
\zeta(s, s) = \sum_{m,n \text{ even}} (mn)^{-s} \nonumber
\]
\[
= \frac{1}{2} \sum_{m,n \text{ odd}} (mn)^{-s} \nonumber
\]
\[
= \frac{1}{2} \left( \sum_{m,n} (mn)^{-s} - \sum_{n} n^{-2s} \right) \nonumber
\]
\[
= \frac{1}{2} (\zeta(s)^2 - \zeta(2s)). \hspace{1cm} (40)
\]

The case \(s = 2\) gives
\[\zeta(2, 2) = \frac{\pi^4}{2} \left( \frac{1}{36} - \frac{1}{90} \right) = \frac{\pi^4}{120} = \frac{3}{4} \zeta(4). \hspace{1cm} (41)
\]

Taking \(x = 1\) in (37) and applying (24), we have
\[
\zeta(1, 2) = \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^2} \nonumber
\]
\[
= \frac{1}{2} \int_{0}^{1} \log^2(1-t) \frac{1}{t} \, dt \nonumber
\]
\[
= \zeta(3). \nonumber
\]

an identity already known to Euler. Later I found the following direct proof avoiding integrals. \[\text{[Note added by Graham Jameson: this method can be seen, for example, in [BB, p. 7–8], where it is attributed to R. Steinberg]}\]

Since
\[\zeta(1, 2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m(m+n)^2}.\]

we have
\[2\zeta(1, 2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^2} \left( \frac{1}{m} + \frac{1}{n} \right) \nonumber
\]
\[
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)}. \hspace{1cm} (42)
\]
Now
\[ \frac{1}{mn(m+n)} = \frac{1}{m^2} \frac{m}{n(m+n)} = \frac{1}{m^2} \left( \frac{1}{n} - \frac{1}{m+n} \right) \]
and by cancellation
\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{m+n} \right) = 1 + \frac{1}{2} + \cdots + \frac{1}{m} = H_m, \]
so
\[ 2\zeta(1, 2) = \sum_{m=1}^{\infty} \frac{H_m}{m^2} \]
\[ = \zeta(3) + \sum_{m=2}^{\infty} \frac{H_{m-1}}{m^2} \]
\[ = \zeta(3) + \zeta(1, 2). \]

A similar method delivers an identity for \( \zeta(1, 3) \), which will be used for Comtet’s series.
Tim’s original method was by manipulation of the series for \( \text{Li}_2(x)^2 \); the following is slightly shorter and more like the previous proof.

\[ 2\zeta(1, 3) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^3} \left( \frac{1}{m} + \frac{1}{n} \right) \]
\[ = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)^2} \]
and
\[ \frac{1}{mn(m+n)^2} = \frac{1}{m^2(m+n)} \left( \frac{1}{n} - \frac{1}{m+n} \right) \]
\[ = \frac{1}{m^2n(m+n)} - \frac{1}{m^2(m+n)^2} \]
\[ = \frac{1}{m^3} \left( \frac{1}{n} - \frac{1}{m+n} \right) - \frac{1}{m^2(m+n)^2}, \]
so, as before,
\[ 2\zeta(1, 3) = \sum_{m=1}^{\infty} \frac{H_m}{m^3} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2(m+n)^2} \]
\[ = \zeta(4) + \sum_{m=2}^{\infty} \frac{H_{m-1}}{m^3} - \zeta(2, 2) \]
\[ = \zeta(4) + \zeta(1, 3) - \zeta(2, 2). \]

Hence, by (41),
\[ \zeta(1, 3) = \zeta(4) - \zeta(2, 2) = \frac{1}{4}\zeta(4). \]
Comtet’s series for $\zeta(4)$

For this, [BBC] gives a reference to Comtet’s book [Com], but I have not seen this or any other proof. Here is my proof (completed 4-8-07). Perhaps it could be substantially simplified! Let

$$S = \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}}.$$  

From (33), we have

$$\sum_{n=1}^{\infty} \frac{y^{2n}}{n^3 \binom{2n}{n}} = \int_{0}^{y} \frac{4}{z} \left( \arcsin \frac{z}{2} \right)^2 \, dz$$

so

$$S = 2 \int_{0}^{1} \frac{1}{y} \int_{0}^{y} \frac{4}{z} \left( \arcsin \frac{z}{2} \right)^2 \, dz \, dy = 8 \int_{0}^{1} \frac{1}{z} \left( \arcsin \frac{z}{2} \right)^2 \int_{z}^{1} \frac{dy}{y} \, dz = -8 \int_{0}^{1} \frac{1}{z} \left( \arcsin \frac{z}{2} \right)^2 \log z \, dz = -8 \int_{0}^{\frac{\pi}{2}} \frac{1}{2 \sin \theta} \theta^2 \log(2 \sin \theta) \cdot 2 \cos \theta \, d\theta \quad (z = 2 \sin \theta)$$

$$= -8 \int_{0}^{\frac{\pi}{2}} \theta^2 \cot \theta \log(2 \sin \theta) \, d\theta = -4 \int_{0}^{\frac{\pi}{2}} \theta^2 \frac{d}{d\theta} \left( \log^2(2 \sin \theta) \right) \, d\theta = -4 \left[ \theta^2 \log^2(2 \sin \theta) \right]_{0}^{\frac{\pi}{2}} + 4 \int_{0}^{\frac{\pi}{2}} 2 \theta \log^2(2 \sin \theta) \, d\theta = 8 \int_{0}^{\frac{\pi}{2}} \theta \log^2(2 \sin \theta) \, d\theta.$$

Now expand as follows (maybe something circular is happening here?):

$$\log(2 \sin \theta) = \log \left( \frac{e^{i\theta} - e^{-i\theta}}{i} \right) = \left( \theta - \frac{\pi}{2} \right) i + \log \left( 1 - e^{-2i\theta} \right).$$

so

$$S = 8 \int_{0}^{\frac{\pi}{2}} \theta \left( - \left( \theta - \frac{\pi}{2} \right)^2 + 2i \left( \theta - \frac{\pi}{2} \right) \log \left( 1 - e^{-2i\theta} \right) + \log^2 \left( 1 - e^{-2i\theta} \right) \right) \, d\theta = -8I_1 + 16I_2 + 8I_3,$$

where

$$I_1 = \int_{0}^{\frac{\pi}{2}} \theta \left( \theta - \frac{\pi}{2} \right)^2 \, d\theta$$
\begin{align*}
I_2 &= i \int_0^\pi \theta \left( \theta - \frac{\pi}{2} \right) \log(1 - e^{-2i\theta}) \, d\theta \\
&= -i \sum_{n=1}^\infty \frac{1}{n} \int_0^\pi \left( \theta^2 - \frac{\pi \theta}{2} \right) e^{-2in\theta} \, d\theta \\
&= -i \sum_{n=1}^\infty \frac{1}{n} \left( \left[ \left( \theta^2 - \frac{\pi \theta}{2} \right) \frac{e^{-2in\theta}}{-2in} \right]_0^\pi - \left[ \left( 2\theta - \frac{\pi}{2} \right) \frac{e^{-2in\theta}}{-2in} \right]_0^\pi \right) \\
&= -i \sum_{n=1}^\infty \frac{1}{n} \left( \left( \frac{\pi^2}{36} - \frac{\pi^2}{12} \right) e^{-\pi in^3/3} - \left( \frac{\pi}{2} \right) \frac{e^{-\pi in^3/3}}{-4n^2} + \left( \frac{\pi}{2} \right) \frac{1}{4n^2} + \left( 2 \frac{e^{-\pi in^3}}{-2in} \right) \right) \\
&= -i \sum_{n=1}^\infty \frac{1}{n} \left( -\frac{\pi^2}{36n} e^{-\pi in^3/3} - \frac{\pi i}{24n^2} e^{-\pi in^3/3} + \frac{\pi i}{8n^2} - \frac{\pi i}{4n^4} (e^{-\pi in^3/3} - 1) \right) \\
&= \sum_{n=1}^\infty \left( -\frac{\pi^2}{36n^2} e^{-\pi in^3/3} + \frac{\pi i}{24n^3} e^{-\pi in^3/3} - \frac{\pi i}{8n^3} - \frac{1}{4n^4} e^{-\pi in^3/3} + \frac{1}{4n^4} \right),
\end{align*}

and by (36)

\begin{align*}
I_3 &= \int_0^\pi \theta \log^2(1 - e^{-2i\theta}) \, d\theta \\
&= \sum_{n=2}^\infty \frac{2H_{n-1}}{n} \int_0^\pi \theta e^{-2in\theta} \, d\theta \\
&= \sum_{n=2}^\infty \frac{2H_{n-1}}{n} \left( \left[ \theta e^{-2in\theta} \right]_0^\pi - \left[ \int_0^\pi e^{-2in\theta} \, d\theta \right] \right) \\
&= \sum_{n=2}^\infty \frac{2H_{n-1}}{n} \left( \frac{\pi i}{12n} e^{-\pi in^3/3} - \left[ \frac{e^{-2in\theta}}{-2in} \right]_0^\pi \right) \\
&= \sum_{n=2}^\infty \frac{2H_{n-1}}{n} \left( \frac{\pi i}{12n} e^{-\pi in^3/3} + \frac{1}{4n^2} (e^{-\pi in^3/3} - 1) \right) \\
&= \sum_{n=2}^\infty H_{n-1} \left( \frac{\pi i}{6n^2} e^{-\pi in^3/3} + \frac{1}{2n^3} e^{-\pi in^3/3} - \frac{1}{2n^3} \right).
\end{align*}
Hence
\[
S = -\frac{22\pi^4}{64} + \sum_{n=1}^{\infty} \left( \frac{4\pi^2}{9n^2} e^{-\pi\text{i}n/3} + \frac{2\pi i}{3n^3} e^{-\pi\text{i}n/3} - \frac{2\pi i}{n^3} - \frac{4}{n^4} e^{-\pi\text{i}n/3} + \frac{4}{n^4} \right) \\
+ \sum_{n=2}^{\infty} H_{n-1} \left( \frac{4\pi i}{3n^2} e^{-\pi\text{i}n/3} + \frac{4}{n^3} e^{-\pi\text{i}n/3} - \frac{4}{n^3} \right)
\]
\[
= -\frac{11\pi^4}{2^3 \cdot 3^4} - 2\pi i\zeta(3) + 4\zeta(4) - 4\zeta(1,3)
\]
\[
+ \sum_{n=1}^{\infty} \left( -\frac{4\pi^2}{9n^2} + \frac{2\pi i}{3n^3} - \frac{4}{n^4} \right) e^{-\pi\text{i}n/3} + \sum_{n=2}^{\infty} H_{n-1} \left( \frac{4\pi i}{3n^2} + \frac{4}{n^3} \right) e^{-\pi\text{i}n/3}.
\]

By (42), \(4\zeta(1,3) = \zeta(4)\), so the first four terms equate to
\[
-\frac{11\pi^4}{2^3 \cdot 3^4} + 3\zeta(4) - 2\pi i\zeta(3) = \left( \frac{1}{2 \cdot 3 \cdot 5} - \frac{11}{2^4 \cdot 3^4} \right) \pi^4 - 2\pi i\zeta(3)
\]
\[
= \frac{2^2 \cdot 3^3 - 5 \cdot 11}{2^4 \cdot 3^4} \pi^4 - 2\pi i\zeta(3)
\]
\[
= \frac{108 - 55}{2^4 \cdot 3^4} \pi^4 - 2\pi i\zeta(3)
\]
\[
= \frac{53\pi^4}{2^4 \cdot 3^4} - 2\pi i\zeta(3).
\]

So our formula becomes
\[
S = \frac{53\pi^4}{2^4 \cdot 3^4} - 2\pi i\zeta(3) + \sum_{n=1}^{\infty} \left( -\frac{4\pi^2}{9n^2} + \frac{2\pi i}{3n^3} - \frac{4}{n^4} \right) e^{-\pi\text{i}n/3} + \sum_{n=2}^{\infty} H_{n-1} \left( \frac{4\pi i}{3n^2} + \frac{4}{n^3} \right) e^{-\pi\text{i}n/3}.
\]

The imaginary part of \(S\) is zero, so we have proved
\[
2\pi\zeta(3) = \sum_{n=1}^{\infty} \left( \frac{4\pi^2}{9n^2} \sin \frac{\pi n}{3} + \frac{2\pi}{3n^3} \cos \frac{\pi n}{3} + \frac{4}{n^4} \sin \frac{\pi n}{3} \right) \\
+ \sum_{n=2}^{\infty} H_{n-1} \left( \frac{4\pi}{3n^2} \cos \frac{\pi n}{3} - \frac{4}{n^3} \sin \frac{\pi n}{3} \right).
\]

Incidentally, I cannot evaluate the separate contributions of any of these terms. The real terms give
\[
S = \frac{53\pi^4}{2^4 \cdot 3^4} \cdot 5 - 2\pi i\zeta(3) + \sum_{n=1}^{\infty} \left( -\frac{4\pi^2}{9n^2} + \frac{2\pi i}{3n^3} - \frac{4}{n^4} \right) e^{-\pi\text{i}n/3} + \sum_{n=2}^{\infty} H_{n-1} \left( \frac{4\pi i}{3n^2} + \frac{4}{n^3} \right) e^{-\pi\text{i}n/3}.
\]

The two sums here will turn out to give a total of \(-\zeta(4)\). We now evaluate the first sum. The contribution of each of the three terms is of the form \(CB_k(\frac{1}{3})\), by the Fourier series for
the periodified Bernoulli polynomial

\[ \tilde{B}_k(x) = B_k(\{x\}_0) \]

\[ = -k! \sum_{n=-\infty}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^k} \]

\[ = -\frac{k!}{(2\pi i)^k} (F(k, x) + (-1)^k F(k, -x)), \]

where

\[ F(s, \alpha) = \text{Li}_s(e^{2\pi i \alpha}) = \sum_{n=1}^{\infty} \frac{e^{2\pi in\alpha}}{n^s}. \]

is the Lerch zeta function. Splitting according to parity gives

\[ \sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{n^{2k}} = (-1)^{k-1} \frac{(2\pi)^{2k}}{2(2k)!} \tilde{B}_{2k}(x), \]

and

\[ \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{n^{2k+1}} = (-1)^{k-1} \frac{(2\pi)^{2k+1}}{2(2k + 1)!} \tilde{B}_{2k+1}(x). \]

The Bernoulli polynomials up to $B_4$ are

\[
\begin{align*}
B_0(x) & = 1, \\
B_1(x) & = x - \frac{1}{2}, \\
B_2(x) & = x^2 - x + \frac{1}{6}, \\
B_3(x) & = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\
B_4(x) & = x^4 - 2x^3 + x^2 - \frac{1}{30},
\end{align*}
\]

so we find

\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{\cos(\pi n/3)}{n^2} & = \pi^2 B_2(\frac{1}{6}) = \frac{\pi^2}{36}, \\
\sum_{n=1}^{\infty} \frac{\sin(\pi n/3)}{n^3} & = \frac{(2\pi)^3}{2 \cdot 3!} B_3(\frac{1}{6}) \\
& = \frac{2\pi^3}{3 \cdot 6^3} \left( 1 - \frac{3 \cdot 6}{2} + \frac{6^2}{2} \right) \\
& = \frac{2\pi^3}{3 \cdot 6^3} (1 - 9 + 18) \\
& = \frac{20\pi^3}{3 \cdot 6^3}, \\
\sum_{n=1}^{\infty} \frac{\cos(\pi n/3)}{n^4} & = -\frac{(2\pi)^4}{2 \cdot 4!} B_4(\frac{1}{6})
\end{align*}
\]
\[
\begin{align*}
&= -\frac{\pi^4}{3\cdot 6^4} \left( 1 - 2 \cdot 6 + 6^2 - \frac{6^4}{30} \right) \\
&= -\frac{\pi^4}{3\cdot 5\cdot 6^4} (125 - 216) \\
&= \frac{91\pi^4}{3\cdot 5\cdot 6^4}.
\end{align*}
\]

So the first sum in (43) is
\[
\sum_{n=1}^{\infty} \left( -\frac{4\pi^2}{9} \cdot \frac{\cos(\pi n/3)}{n^2} + \frac{2\pi}{3} \cdot \frac{\sin(\pi n/3)}{n^3} - 4\frac{\cos(\pi n/3)}{n^4} \right)
= -\frac{4\pi^2}{9} \cdot \frac{\pi^2}{6^2} + \frac{2\pi}{3} \cdot \frac{20\pi^3}{3\cdot 6^3} - 4\frac{91\pi^4}{3 \cdot 5 \cdot 6^4}
= \frac{\pi^4}{3\cdot 5\cdot 6^4} \left( -\frac{4}{9} \cdot 3 \cdot 5 \cdot 6^2 + \frac{2}{3} \cdot 20 \cdot 5 \cdot 6 - 4 \cdot 91 \right)
= \frac{\pi^4}{3\cdot 5\cdot 6^4} \left( -\frac{20}{3} \cdot 6^2 + \frac{40}{3} \cdot 5 \cdot 6 - 4 \cdot 91 \right)
= \frac{\pi^4}{3\cdot 5\cdot 6^4} (-240 + 400 - 364)
= -\frac{204\pi^4}{3\cdot 5\cdot 6^4}
= -\frac{34\pi^4}{3\cdot 5\cdot 6^3}.
\]

Since \(3 \cdot 5 \cdot 6^3 = 2^3 \cdot 3^4 \cdot 5\) and \(53 - 34 = 19\), we have proved
\[
S = \frac{19\pi^4}{2^3 \cdot 3^4 \cdot 5} + 4 \sum_{n=2}^{\infty} H_{n-1} \left( \frac{\pi \sin(\pi n/3)}{3n^2} + \frac{\cos(\pi n/3)}{n^3} \right). \tag{44}
\]

We now define
\[
f(x) = \int_0^x \frac{\log^2(1-u) \log u}{u} \, du. \tag{45}
\]

By (38), we have
\[
f(x) = \left( \sum_{n=2}^{\infty} \frac{2H_{n-1}x^n}{n^2} \right) \log x - \sum_{n=2}^{\infty} \frac{2H_{n-1}x^n}{n^3},
\]

so that (44) becomes
\[
S = \frac{19\pi^4}{2^3 \cdot 3^4 \cdot 5} + 2\Re(f(e^{\pi i/3})). \tag{46}
\]

The obvious integration by parts gives a reflection relation between \(f(x)\) and \(f(1-x)\):
\[
f(x) = \left[ \log^2(1-u) \cdot \frac{1}{2} \log^2 u \right]_0^x - \int_0^x \left( -\frac{2 \log(1-u)}{1-u} \right) \frac{1}{2} \log^2 u \, du
= \frac{1}{2} \log^2(1-x) \log^2 x + \int_0^x \frac{\log^2 u \log(1-u)}{1-u} \, du
= \frac{1}{2} \log^2(1-x) \log^2 x + f(1) - \int_x^1 \frac{\log^2 u \log(1-u)}{1-u} \, du.
\]
By (39) and (42), \( f(1) = -2\zeta(1,3) = -\frac{1}{2}\zeta(4) \). Also,

\[
\int_1^x \frac{\log^2 u \log(1-u)}{1-u} du = \int_0^{1-x} \frac{\log^2(1-u) \log u}{u} du = f(1-x),
\]

so

\[
f(x) + f(1-x) = -\frac{1}{2}\zeta(4) + \frac{1}{2}\log^2(1-x) \log^2 x.
\]

(47)

Since \( f(\overline{z}) = \overline{f(z)} \) (this occurs when \( f \) is analytic and real on the real axis), the case \( x = e^{\pi i/3} \) gives us the value we want:

\[
2\Re f(e^{\pi i/3}) = f(e^{\pi i/3}) + f(e^{-\pi i/3})
\]

\[
= f(e^{\pi i/3}) + f(1 - e^{\pi i/3})
\]

\[
= -\frac{1}{2}\zeta(4) + \frac{1}{2}\left(-\frac{\pi i}{3}\right)^2 \left(\frac{\pi i}{3}\right)^2
\]

\[
= -\frac{1}{2}\zeta(4) + \frac{1}{2} \left(\frac{\pi}{3}\right)^4
\]

\[
= \frac{\pi^4}{2}\left(-\frac{1}{2 \cdot 3^2 \cdot 5} + \frac{1}{3^4}\right)
\]

\[
= \frac{\pi^4}{2^2 \cdot 3^4 \cdot 5}(-3^2 + 2 \cdot 5)
\]

\[
= \frac{\pi^4}{2^2 \cdot 3^4 \cdot 5}.
\]

Inserting this into (46) gives

\[
S = \frac{17\pi^4}{2^3 \cdot 3^4 \cdot 5} = \frac{17}{36\zeta(4)},
\]

so finally we have Comtet’s series

\[
\zeta(4) = \frac{36}{17} \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}}.
\]

(48)

We have had two occurrences of \( \zeta(1,3) = \frac{1}{3}\zeta(4) \) in our calculation of \( S \). These did not simply cancel out: the first occurrence was \(-4\zeta(1,3)\), while the second was \(-f(1) = 2\zeta(1,3)\).
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