POLYLOGARITHMS, MULTIPLE ZETA VALUES, AND THE SERIES OF HJORTNAES AND COMTET

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(edited by Graham Jameson, with references added, March 2014)

Introduction

The polylogarithm function is defined for $|x| < 1$ and any real s by

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}. \tag{1}$$

For particular values of s, it can then be extended analytically to a wider range of x (real or complex). It is simultaneously a power series in x and a Dirichlet series in s. The first published study of the function was by A. Jonquière in 1889, and it is sometimes called Jonquière's function. Note that "polylogarithmic functions" means something quite different!

These notes are a summary of some results on polylogarithms that I have seen stated in various places, such as Wikipedia and [BBC], largely with proofs that I have worked out for myself. The methods are mostly elementary. We restrict to integer $s = k \geq 0$, and x usually real. However, in places the reader is invited to accept that formulae established for real x also apply for complex x; I can supply a rigorous justification if pressed.

We start with a few immediate facts. Firstly,

$$\mathrm{Li}_0(x) = \frac{x}{1-x},$$

extending meromorphically to the whole plane, and

$$\mathrm{Li}_1(x) = -\log(1-x),$$

extending to all real $x < 1$ (or complex x excluding real $x \geq 1$).

For $|x| < 1$, or $|x| \leq 1$ with $k > 1$, we have

$$\mathrm{Li}_k(x) + \mathrm{Li}_k(-x) = 2^{1-k}\mathrm{Li}_k(x^2). \tag{2}$$

For $k > 1$,

$$\mathrm{Li}_k(1) = \zeta(k), \tag{3}$$

$$\mathrm{Li}_k(-1) = -(1 - 2^{1-k})\zeta(k). \tag{4}$$

In particular, $\mathrm{Li}_2(-1) = \frac{1}{2}\zeta(2)$ and $\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3)$.

For $k \geq 2$ and $0 \leq x \leq 1$, we have from the series $0 \leq \mathrm{Li}_k(x) \leq \zeta(k)x$ and $-x \leq \mathrm{Li}_k(x) \leq 0$. Further, $x\mathrm{Li}_k'(x) = \mathrm{Li}_{k-1}(x)$ (also, $\mathrm{Li}_k'(0) = 1$) and

$$\mathrm{Li}_k(x) = \int_0^x \frac{\mathrm{Li}_{k-1}(t)}{t}\, dt. \tag{5}$$

(Termwise integration of the series is justified by uniform convergence for $|x| < 1$, and then by continuity of $\mathrm{Li}_k(x)$ at 1 and -1.)

We work out various special values of Li_2 and Li_3, which were found by Landen as long ago as 1780. In particular, we evaluate $\mathrm{Li}_3(\phi^{-2})$, where

$$\phi = \frac{1+\sqrt{5}}{2}$$

is the golden ratio. This is used to prove Hjortnaes' series expression (35) for $\zeta(3)$. My proof of Comtet's corresponding sum for $\zeta(4)$ involves quantities like $\mathrm{Li}_4(e^{\pi i/3})$.

The dilogarithm

The dilogarithm Li_2 is sometimes called Spence's function, in tribute to a pioneering study of it by W. Spence in 1809. Beware of the fact that some computer algebra systems denote $\mathrm{Li}_2(1-x)$ by $\mathrm{dilog}(x)$.

By termwise integration of the series

$$-\frac{\log(1-t)}{t} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n}$$

we obtain the identity

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\, dt, \tag{6}$$

which we now use to define $\mathrm{Li}_2(x)$ for all $x \leq 1$. Note that

$$\mathrm{Li}_2(1) = \zeta(2) = -\int_0^1 \frac{\log(1-t)}{t}\, dt$$

and for $x > 0$,

$$\mathrm{Li}_2(-x) = -\int_0^{-x} \frac{\log(1-t)}{t}\, dt = -\int_0^x \frac{\log(1+t)}{t}\, dt.$$

Clearly, $\mathrm{Li}_2'(x) = -\log(1-x)/x$, and hence $\mathrm{Li}_2(x)$ is strictly increasing for all $x < 1$.

For $0 < x < 1$, we have

$$\begin{aligned} \mathrm{Li}_2(x) &= \zeta(2) + \int_x^1 \frac{\log(1-u)}{u}\, du \\ &= \zeta(2) + \int_0^{1-x} \frac{\log t}{1-t} dt. \end{aligned}$$

Replacing x by $1-x$, this says

$$\begin{aligned}
\mathrm{Li}_2(1-x)-\zeta(2) &= \int_0^x \frac{\log t}{1-t}dt \\
&= \Big[-\log t\log(1-t)\Big]_0^x + \int_0^x \frac{\log(1-t)}{t}dt \\
&= -\log x\log(1-x)-\mathrm{Li}_2(x),
\end{aligned}$$

so we have the following identity, known as Euler's reflection formula:

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\zeta(2)-\log x\log(1-x). \tag{7}$$

In particular, using the well-known identity $\zeta(2)=\pi^2/6$, we have

$$\mathrm{Li}_2(\tfrac{1}{2})=\tfrac{1}{2}(\zeta(2)-\log^2 2)=\frac{\pi^2}{12}-\tfrac{1}{2}\log^2 2. \tag{8}$$

We can rewrite (8) as $\zeta(2)=\log^2 2+2\sum_{n=1}^{\infty}(1/n^2 2^n)$. Taking $\log 2$ as known, this can be regarded as a series for $\zeta(2)$ that converges much more rapidly than $\sum_{n=1}^{\infty}(1/n^2)$.

We now give two further identities for Li_2. Firstly, for $x>0$, we have

$$\mathrm{Li}_2(-x)=\mathrm{Li}_2(-1)+F(x)=-\tfrac{1}{2}\zeta(2)+F(x)$$

where

$$F(x)=\int_x^1 \frac{\log(1+t)}{t}\,dt.$$

Combined with the same statement for $1/x$, this gives

$$\mathrm{Li}_2(-x)+\mathrm{Li}_2(-\tfrac{1}{x})=-\zeta(2)+F(x)+F(\tfrac{1}{x}).$$

But

$$\begin{aligned}
F(x) &= \int_1^{1/x} u\log\left(1+\frac{1}{u}\right)\frac{1}{u^2}\,du \\
&= \int_1^{1/x} \frac{\log(1+u)-\log u}{u}\,du \\
&= -F(\tfrac{1}{x})-\tfrac{1}{2}\log^2 x,
\end{aligned}$$

so we have

$$\mathrm{Li}_2(-x)+\mathrm{Li}_2(-\tfrac{1}{x})=-\zeta(2)-\tfrac{1}{2}\log^2 x. \tag{9}$$

So for $x>1$, $\mathrm{Li}_2(-x)=-\frac{1}{2}\log^2 x-\zeta(2)+r(x)$, where $0<r(x)\le 1/x$.

For $0 < x < 1$, we have also

$$\begin{aligned}
\mathrm{Li}_2(1-x) &= -\int_x^1 \frac{\log t}{1-t}\,dt \\
&= \int_1^{\frac{1}{x}} \frac{\log u}{1-1/u}\frac{1}{u^2}\,du \\
&= \int_1^{\frac{1}{x}} \log u \left(\frac{1}{u-1} - \frac{1}{u}\right)\,du \\
&= \int_0^{\frac{1}{x}-1} \frac{\log(1+v)}{v}\,dv - \tfrac{1}{2}\log^2 x \\
&= -\mathrm{Li}_2(1-\tfrac{1}{x}) - \tfrac{1}{2}\log^2 x.
\end{aligned}$$

Applying this also to $1/x$, we have the following identity for all $x > 0$:

$$\mathrm{Li}_2(1-x) + \mathrm{Li}_2(1-\tfrac{1}{x}) = -\tfrac{1}{2}\log^2 x. \tag{10}$$

Using (2), (7), (9) and (10), we can reduce the computation of $\mathrm{Li}_2(x)$ for any $x < 1$ to computation of values of at most a few values of $\mathrm{Li}_2(y)$ with $|y| \le \frac{1}{2}$.

Now consider ϕ. Note that the fundamental equation satisified by ϕ may be written:

$$\phi^2 = \phi + 1, \qquad \phi = 1 + \phi^{-1}, \qquad 1 = \phi^{-1} + \phi^{-2}.$$

Putting $x = \phi^{-1}$ in (2), (10), (7) respectively, we obtain three linear equations relating $\mathrm{Li}_2(\phi^{-1})$, $\mathrm{Li}_2(-\phi^{-1})$, and $\mathrm{Li}_2(\phi^{-2})$):

$$\mathrm{Li}_2(\phi^{-1}) + \mathrm{Li}_2(-\phi^{-1}) = \tfrac{1}{2}\mathrm{Li}_2(\phi^{-2}), \tag{11}$$

$$\mathrm{Li}_2(\phi^{-2}) + \mathrm{Li}_2(-\phi^{-1}) = -\tfrac{1}{2}\log^2\phi. \tag{12}$$

$$\mathrm{Li}_2(\phi^{-1}) + \mathrm{Li}_2(\phi^{-2}) = \zeta(2) - 2\log^2\phi. \tag{13}$$

Subtracting (12) from (11) gives

$$\mathrm{Li}_2(\phi^{-1}) - \tfrac{3}{2}\mathrm{Li}_2(\phi^{-2}) = \tfrac{1}{2}\log^2\phi. \tag{14}$$

Now subtracting (14) from (13), we have

$$\tfrac{5}{2}\mathrm{Li}_2(\phi^{-2}) = \zeta(2) - \tfrac{5}{2}\log^2\phi,$$

i.e.

$$\mathrm{Li}_2(\phi^{-2}) = \tfrac{2}{5}\zeta(2) - \log^2\phi = \frac{\pi^2}{15} - \log^2\phi. \tag{15}$$

Now (13) or (14) gives

$$\mathrm{Li}_2(\phi^{-1}) = \tfrac{3}{5}\zeta(2) - \log^2\phi = \frac{\pi^2}{10} - \log^2\phi, \tag{16}$$

while (12) gives

$$\mathrm{Li}_2(-\phi^{-1}) = -\tfrac{2}{5}\zeta(2) + \tfrac{1}{2}\log^2\phi = -\frac{\pi^2}{15} + \tfrac{1}{2}\log^2\phi, \tag{17}$$

hence finally (9) gives

$$\mathrm{Li}_2(-\phi) = -\tfrac{3}{5}\zeta(2) - \log^2\phi = -\frac{\pi^2}{10} - \log^2\phi. \tag{18}$$

If we include $\mathrm{Li}_2(0) = 0$, then we have found eight special values of $\mathrm{Li}_2(x)$ for real x. I have seen it stated that these are the only ones known to be expressible in this kind of way.

We now briefly consider complex arguments. Substitution in the series gives

$$\mathrm{Li}_2(i) = -\tfrac{1}{8}\zeta(2) + iG, \qquad \mathrm{Li}_2(-i) = -\tfrac{1}{8}\zeta(2) - iG,$$

where

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots$$

is Catalan's constant. Assuming (10) valid for complex arguments, take $x = 1 + i$, so that $1 - x = -i$ and $1 - x^{-1} = \frac{1}{2} + \frac{1}{2}i$. We obtain

$$\begin{aligned} \mathrm{Li}_2\left(\frac{1+i}{2}\right) &= \frac{1}{8}\zeta(2) + iG - \frac{1}{2}\left(\frac{1}{2}\log 2 + \frac{\pi i}{4}\right)^2 \\ &= \frac{5\pi^2}{96} - \frac{1}{8}\log^2 2 + i\left(G - \frac{\pi}{8}\log 2\right). \end{aligned}$$

For $e^{i\theta}$, we have

$$\Re\mathrm{Li}_2(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = \pi^2 \tilde{B}_2\left(\frac{\theta}{2\pi}\right),$$

where B_2 is the Bernoulli polynomial $B_2(x) = x^2 - x + \frac{1}{6}$ and $\tilde{B}_2(x) = B_2(\{x\})$. Also,

$$\Im\mathrm{Li}_2(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2},$$

which is called the 'Clausen' function in [BBC].

There are many further formulae involving Li_2: see, for example, Wikipedia, [Lew] and [Max]. One is 'Abel's identity', which equates a combination of five Li_2 values to $\log(1-x)\log(1-y)$.

Note that the 'Rogers dilogarithm' is

$$\begin{aligned} \mathrm{L}(x) &= -\tfrac{1}{2}\int_0^x \left(\frac{\log(1-u)}{u} + \frac{\log u}{1-u}\right) du \\ &= \mathrm{Li}_2(x) + \tfrac{1}{2}\log x \log(1-x). \end{aligned}$$

The trilogarithm

For $0 < |x| \leq 1$, we have by (5)

$$\mathrm{Li}_3(x) = \int_0^x \frac{\mathrm{Li}_2(t)}{t}\, dt. \tag{19}$$

Since $\mathrm{Li}_2(x)$ has been defined by (6) for all $x \leq 1$, we can use (19) to define $\mathrm{Li}_3(x)$ for all such x.

So we have

$$\begin{aligned} \mathrm{Li}_3(x) &= \int_0^x \frac{\mathrm{Li}_2(u)}{u}\, du \\ &= -\int_0^x \frac{1}{u} \int_0^u \frac{\log(1-t)}{t}\, dt\, du \\ &= -\int_0^x \frac{\log(1-t)}{t} \int_t^x \frac{1}{u}\, du\, dt \\ &= \int_0^x \frac{\log(1-t)}{t} \log \frac{t}{x}\, dt, \end{aligned} \tag{20}$$

valid regardless of the sign of x. For $0 < x < 1$, substitution of (6) now gives

$$\mathrm{Li}_3(x) = \int_0^x \frac{\log t \log(1-t)}{t}\, dt + \mathrm{Li}_2(x) \log x, \tag{21}$$

which can also be derived directly from the series expression. Further, for $x > 0$,

$$\begin{aligned} \mathrm{Li}_3(-x) &= \int_0^{-x} \frac{\log(1-u)}{u} \log \frac{u}{-x}\, du \\ &= \int_0^x \frac{\log(1+t)}{t} \log \frac{t}{x}\, dt \\ &= \int_0^x \frac{\log t \log(1+t)}{t}\, dt + \mathrm{Li}_2(-x) \log x. \end{aligned} \tag{22}$$

In particular,

$$\zeta(3) = \mathrm{Li}_3(1) = \int_0^1 \frac{\log t \log(1-t)}{t}\, dt. \tag{23}$$

Integrating by parts in (21), we obtain for $0 \leq x \leq 1$,

$$\begin{aligned} \mathrm{Li}_3(x) &= \Big[\tfrac{1}{2} \log^2 t \log(1-t)\Big]_0^x - \int_0^x \tfrac{1}{2} \log^2 t \frac{-1}{1-t}\, dt + \mathrm{Li}_2(x) \log x, \\ &= \tfrac{1}{2} \int_0^x \frac{\log^2 t}{1-t}\, dt + \mathrm{Li}_2(x) \log x + \tfrac{1}{2} \log^2 x \log(1-x). \end{aligned} \tag{24}$$

In particular,

$$\int_0^1 \frac{\log^2 t}{1-t}\, dt = 2\zeta(3).$$

Using (9), we can derive a corresponding statement for Li_3. [*This has been added to Tim's original version, in which the following identity was deduced from (26) below.*] For $x > 0$, write

$$F(x) = \int_x^1 \frac{\log t \log(1+t)}{t}\, dt.$$

By (22),

$$\mathrm{Li}_3(-x) = F(0) - F(x) + \mathrm{Li}_2(-x)\log x.$$

With x replaced by $\frac{1}{x}$, this says

$$\mathrm{Li}_3(-\tfrac{1}{x}) = F(0) - F(\tfrac{1}{x}) - \mathrm{Li}_2(-\tfrac{1}{x})\log x.$$

Combining these expressions and using (9), we have

$$\begin{aligned} \mathrm{Li}_3(-\tfrac{1}{x}) - \mathrm{Li}_3(-x) &= F(x) - F(\tfrac{1}{x}) - \log x \left[\mathrm{Li}_2(-x) + \mathrm{Li}_2(-\tfrac{1}{x})\right] \\ &= F(x) - F(\tfrac{1}{x}) + \zeta(2)\log x + \tfrac{1}{2}\log^3 x. \end{aligned}$$

Substituting $t = 1/u$, we have

$$\begin{aligned} F(x) &= \int_{1/x}^1 \log u \log\left(1 + \frac{1}{u}\right)\frac{1}{u}\, du \\ &= -\int_1^{1/x} \frac{1}{u}\log u\left[\log(u+1) - \log u\right]\, du \\ &= \tfrac{1}{3}\log^3 \tfrac{1}{x} + F(\tfrac{1}{x}) \\ &= -\tfrac{1}{3}\log^3 x + F(\tfrac{1}{x}), \end{aligned}$$

and we conclude that

$$\mathrm{Li}_3(-\tfrac{1}{x}) - \mathrm{Li}_3(-x) = \zeta(2)\log x + \tfrac{1}{6}\log^3 x. \tag{25}$$

It follows that for $x > 1$, $\mathrm{Li}_3(-x) = -\frac{1}{6}\log^3 x - \zeta(2)\log x - r(x)$, where $0 < r(x) \le \frac{1}{x}$.

We now prove another identity relating $\mathrm{Li}_3(x)$, $\mathrm{Li}_3(1-x)$ and $\mathrm{Li}_3(1-\frac{1}{x})$. It again goes back to Landen around 1780. [*The proof given here was found among Tim's handwritten papers; it replaces a more complicated proof in Tim's typed version.*]

Let $0 < x < 1$ and

$$S(x) = \mathrm{Li}_3(x) + \mathrm{Li}_3(1-x) + \mathrm{Li}_3(1-\tfrac{1}{x}).$$

We express everything in terms of integrals on $[x, 1]$. The details are kept simpler by expressing the integrands in terms of Li_2 rather than log terms. Firstly, using (19) and (7), we have

$$\begin{aligned}
\mathrm{Li}_3(x) &= \int_0^x \frac{\mathrm{Li}_2(t)}{t}\,dt \\
&= \zeta(3) - \int_x^1 \frac{\mathrm{Li}_2(t)}{t}\,dt \\
&= \zeta(3) - \int_x^1 \frac{1}{t}\left[\zeta(2) - \log t\log(1-t) - \mathrm{Li}_2(1-t)\right]\,dt \\
&= \zeta(3) + \zeta(2)\log x + \int_x^1 \frac{\log t\log(1-t)}{t}\,dt + \int_x^1 \frac{\mathrm{Li}_2(1-t)}{t}\,dt.
\end{aligned}$$

Secondly,

$$\mathrm{Li}_3(1-x) = \int_0^{1-x} \frac{\mathrm{Li}_2(u)}{u}\,du = \int_x^1 \frac{\mathrm{Li}_2(1-t)}{1-t}\,dt.$$

Thirdly, with obvious substitutions,

$$\begin{aligned}
\mathrm{Li}_3(1-\tfrac{1}{x}) &= \int_0^{1/x-1} \frac{\mathrm{Li}_2(-u)}{u}\,du \\
&= \int_1^{1/x} \frac{\mathrm{Li}_2(1-v)}{v-1}\,dv \\
&= \int_x^1 \frac{\mathrm{Li}_2(1-\frac{1}{t})}{\frac{1}{t}-1}\,\frac{1}{t^2}\,dt \\
&= \int_x^1 \frac{\mathrm{Li}_2(1-\frac{1}{t})}{t(1-t)}\,dt.
\end{aligned}$$

Combining the three terms, we have

$$S(x) = \zeta(3) + \zeta(2)\log x + \int_x^1 G(t)\,dt,$$

where

$$G(t) = \frac{\log t\log(1-t)}{t} + \left(\frac{1}{t} + \frac{1}{1-t}\right)\left[\mathrm{Li}_2(1-t) + \mathrm{Li}_2(1-\tfrac{1}{t})\right].$$

Now by (10), we have

$$\begin{aligned}
G(t) &= \frac{1}{t}\log t\log(1-t) - \tfrac{1}{2}\log^2 t\left(\frac{1}{t} + \frac{1}{1-t}\right) \\
&= \frac{d}{dt}\left[\tfrac{1}{2}\log^2 t\log(1-t) - \tfrac{1}{6}\log^3 t\right],
\end{aligned}$$

so

$$\int_x^1 G(t)\,dt = \tfrac{1}{6}\log^3 x - \tfrac{1}{2}\log^2 x\log(1-x).$$

So for $0 < x < 1$, we have the identity

$$\mathrm{Li}_3(x) + \mathrm{Li}_3(1-x) + \mathrm{Li}_3(1-\tfrac{1}{x}) = \zeta(3) + \zeta(2)\log x + \tfrac{1}{6}\log^3 x - \tfrac{1}{2}\log^2 x\log(1-x). \quad (26)$$

One can deduce (25) by writing this with $1-x$ in place of x, taking the difference and putting x for $\frac{1}{x}-1$.

Putting $x=\frac{1}{2}$ in (26) gives

$$\begin{aligned}
2\mathrm{Li}_3(\tfrac{1}{2}) &= -\mathrm{Li}_3(-1)+\zeta(3)+\zeta(2)\log\tfrac{1}{2}-\tfrac{1}{2}\log^2 2\log\tfrac{1}{2}+\tfrac{1}{6}\log^3\tfrac{1}{2} \\
&= \tfrac{3}{4}\zeta(3)+\zeta(3)-\zeta(2)\log 2+\tfrac{1}{2}\log^3 2-\tfrac{1}{6}\log^3 2 \\
&= \tfrac{7}{4}\zeta(3)-\zeta(2)\log 2+\tfrac{1}{3}\log^3 2,
\end{aligned}$$

so that

$$\mathrm{Li}_3(\tfrac{1}{2})=\tfrac{7}{8}\zeta(3)-\tfrac{\pi^2}{12}\log 2+\tfrac{1}{6}\log^3 2. \tag{27}$$

This gives a series converging rapidly to $\zeta(3)$. A more direct proof of (27) is given in [Mel].

Putting $x=\phi^{-1}$ and using the identities relating ϕ, ϕ^{-1} and ϕ^{-2}, we get

$$\begin{aligned}
\mathrm{Li}_3(\phi^{-1})+\mathrm{Li}_3(\phi^{-2})+\mathrm{Li}_3(-\phi^{-1}) &= \zeta(3)-\zeta(2)\log\phi-\tfrac{1}{2}\log^2\phi\log\phi^{-2}+\ \tfrac{1}{6}\log^3\phi^{-1} \\
&= \zeta(3)-\zeta(2)\log\phi+\log^3\phi-\frac{1}{6}\log^3\phi \\
&= \zeta(3)-\zeta(2)\log\phi+\tfrac{5}{6}\log^3\phi.
\end{aligned}$$

But by (2), $\mathrm{Li}_3(\phi^{-1})+\mathrm{Li}_3(-\phi^{-1})=\frac{1}{4}\mathrm{Li}_3(\phi^{-2})$, hence

$$\mathrm{Li}_3(\phi^{-2})=\tfrac{4}{5}\zeta(3)-\tfrac{2\pi^2}{15}\log\phi+\tfrac{2}{3}\log^3\phi. \tag{28}$$

It seems that there are no known such special values involving ϕ for higher k, although there are important relations between various values. This is to do with 'polylogarithm ladders' (introduced by Leonard Lewin), which are important in K-theory and algebraic geometry, and can be used in conjunction with the BBP algorithm for computing various constants.

A proof of $\zeta(2)=\pi^2/6$ and some power series related to $\arcsin x$

For $0\le x<1$, the substutution $t=\frac{\sqrt{1-x^2}}{x}u$ gives

$$\begin{aligned}
x\int_0^1\frac{dt}{1-x^2+x^2t^2} &= \frac{1}{\sqrt{1-x^2}}\int_0^{\frac{x}{\sqrt{1-x^2}}}\frac{du}{1+u^2} \\
&= \frac{1}{\sqrt{1-x^2}}\ \arctan\frac{x}{\sqrt{1-x^2}} \\
&= \frac{\arcsin x}{\sqrt{1-x^2}}.
\end{aligned} \tag{29}$$

Since also

$$\frac{d}{dx}(\arcsin x)^2=\frac{2\arcsin x}{\sqrt{1-x^2}},$$

we now have

$$\begin{aligned}
(\arcsin x)^2 &= \int_0^x \frac{2\arcsin y}{\sqrt{1-y^2}}\,dy \\
&= \int_0^x \int_0^1 \frac{2y}{1-y^2+y^2t^2}\,dt\,dy \\
&= \int_0^1 \int_0^x \frac{2y}{1-y^2+y^2t^2}\,dy\,dt \\
&= -\int_0^1 \frac{1}{1-t^2}\left[\log(1-y^2+y^2t^2)\right]_{y=0}^{y=x}\,dt \\
&= -\int_0^1 \frac{\log(1-x^2+x^2t^2)}{1-t^2}\,dt. \qquad (30)
\end{aligned}$$

In particular, to show that $\zeta(2)=\pi^2/6$, we have

$$\begin{aligned}
\frac{\pi^2}{4} &= (\arcsin 1)^2 \\
&= -\int_0^1 \frac{2\log t}{1-t^2}\,dt \\
&= -2\sum_{n=0}^{\infty}\int_0^1 t^{2n}\log t\,dt \\
&= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\
&= \tfrac{3}{2}\zeta(2).
\end{aligned}$$

Alternatively, we can equate the integral to $\mathrm{Li}_2(1)-\mathrm{Li}_2(-1)$ by (6). This proof has similarities with the one given by Nick Lord [Lo]. [*Tim's proof has now appeared in Math. Gazette* [Jam].]

To derive a power series from (29), observe that for $|x|<1$ and $0\le t\le 1$,

$$\frac{x}{1-x^2+x^2t^2}=\sum_{n=0}^{\infty}x^{2n+1}(1-t^2)^n=\sum_{n=1}^{\infty}x^{2n-1}(1-t^2)^{n-1}$$

and

$$\int_0^1 (1-t^2)^{n-1}\,dt=\int_0^{\pi/2}\cos^{2n-1}\theta\,d\theta=\frac{2.4.\ldots(2n-2)}{1.3.\ldots(2n-1)}=\frac{2^{2n-1}}{n\binom{2n}{n}}.$$

So we have for $|x|<1$,

$$\frac{\arcsin x}{\sqrt{1-x^2}}=\sum_{n=1}^{\infty}\frac{2^{2n-1}x^{2n-1}}{n\binom{2n}{n}}. \qquad (31)$$

This can also be seen as a case of the hypergeometric series $F(1,1;\frac{3}{2};x^2)$. For example, the case $x=\frac{1}{2}$ gives

$$\frac{\pi}{3\sqrt{3}}=\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}.$$

Formulae like this form the basis for various programs which use a spigot algorithm to calculate π, with time roughly proportional to the square of the number of digits required (e.g. one by Winter and Flimmenkamp).

Now by integrating, or by similar reasoning from (30), we have

$$(\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{2^{2n-1}x^{2n}}{n^2\binom{2n}{n}}, \tag{32}$$

equivalently for $|x| < 2$,

$$\left(\arcsin \frac{x}{2}\right)^2 = \sum_{n=1}^{\infty} \frac{x^{2n}}{2n^2\binom{2n}{n}}. \tag{33}$$

The case $x = 1$ gives

$$\frac{\pi^2}{18} = \sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}}.$$

Either by the identity $\sinh^{-1} y = -i \sin^{-1} iy$, or by similar reasoning with x^2 replaced by $-x^2$, we have also

$$\left(\sinh^{-1} \frac{x}{2}\right)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{2n^2\binom{2n}{n}}. \tag{34}$$

Hjortnaes' series for $\zeta(3)$

We give a proof of this identity using (34) and the values of $\mathrm{Li}_2(\phi^{-2})$ and $\mathrm{Li}_3(\phi^{-2})$. The connection with ϕ arises from the fact that $\sinh^{-1} \frac{1}{2} = \log \phi$. Let

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Then by (34),

$$\begin{aligned}
S &= \int_0^1 \frac{4}{x}\left(\sinh^{-1}\frac{x}{2}\right)^2 dx \\
&= \int_0^{\sinh^{-1}\frac{1}{2}} \frac{4}{2\sinh u} u^2 2\cosh u \, du \qquad (x = 2\sinh u) \\
&= 4\int_0^{\log\phi} u^2 \coth u \, du \\
&= 4\int_0^{\log\phi} u^2 \frac{e^{2u}+1}{e^{2u}-1} du \\
&= 4\int_0^{2\log\phi} \left(\frac{v}{2}\right)^2 \frac{e^v+1}{e^v-1}\frac{dv}{2} \\
&= \frac{1}{2}\int_0^{2\log\phi} v^2 \frac{e^v+1}{e^v-1} dv
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^{2\log\phi} v^2\left(1+\frac{2}{e^v-1}\right)dv \\
&= \frac{4}{3}\log^3\phi + \int_0^{2\log\phi}\frac{v^2}{e^v-1}dv \\
&= \frac{4}{3}\log^3\phi + \int_1^{\phi^{-2}}\frac{\log^2 u}{\frac{1}{u}-1}\frac{-du}{u} \qquad (v=-\log u) \\
&= \frac{4}{3}\log^3\phi + \int_{\phi^{-2}}^1\frac{\log^2 u}{1-u}du \\
&= 2\zeta(3) - \int_0^{\phi^{-2}}\frac{\log^2 u}{1-u}du + \frac{4}{3}\log^3\phi.
\end{aligned}
$$

Now substituting from (24), and then the values of $\mathrm{Li}_2(\phi^{-2})$ and $\mathrm{Li}_3(\phi^{-2})$ from (15) and (28), we have

$$
\begin{aligned}
S &= 2\zeta(3) - 2\mathrm{Li}_3(\phi^{-2}) + 2\mathrm{Li}_2(\phi^{-2})\log\phi^{-2} + \log^2\phi^{-2}\log(1-\phi^{-2}) + \tfrac{4}{3}\log^3\phi \\
&= 2\zeta(3) - 2\mathrm{Li}_3(\phi^{-2}) - 4\mathrm{Li}_2(\phi^{-2})\log\phi + 4\log^2\phi\log\phi^{-1} + \tfrac{4}{3}\log^3\phi \\
&= 2\zeta(3) - 2\mathrm{Li}_3(\phi^{-2}) - 4\mathrm{Li}_2(\phi^{-2})\log\phi - \tfrac{8}{3}\log^3\phi \\
&= 2\zeta(3) - 2\mathrm{Li}_3(\phi^{-2}) - 4\left(\tfrac{2}{5}\zeta(2) - \log^2\phi\right)\log\phi - \tfrac{8}{3}\log^3\phi \\
&= 2\zeta(3) - 2\mathrm{Li}_3(\phi^{-2}) - \tfrac{8}{5}\zeta(2)\log\phi + \tfrac{4}{3}\log^3\phi \\
&= 2\zeta(3) - 2\left(\tfrac{4}{5}\zeta(3) - \tfrac{4}{5}\zeta(2)\log\phi + \tfrac{2}{3}\log^3\phi\right) - \tfrac{8}{5}\zeta(2)\log\phi + \tfrac{4}{3}\log^3\phi \\
&= \tfrac{2}{5}\zeta(3),
\end{aligned}
$$

by a neat cancellation. So

$$
\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}. \tag{35}
$$

This series was found by Hjortnaes in 1953 [Hj]. It has sometimes been wrongly attributed to Apéry, because it was used in his proof that $\zeta(3)$ is irrational [Ap]. However, it is not used in the simpler proof of Apéry's theorem by Beukers [Beu]. A proof of (35) that does not involve polylogarithms is outlined in [VDP]; it is reproduced in [BB, p. 378–379]. A generalized version is proved in [AG].

Some series involving H_n

We denote the harmonic sum by $H_n = \sum_{r=1}^n \frac{1}{r}$. For $|x|<1$, we have easily

$$
-\frac{\log(1-x)}{1-x} = \sum_{j=0}^{\infty}x^j\sum_{k=1}^{\infty}\frac{x^k}{k} = \sum_{n=1}^{\infty}H_n x^n.
$$

Similarly (or by integration of the previous case),

$$\log^2(1-x) = \sum_{j=1}^{\infty} \frac{x^j}{j} \sum_{k=1}^{\infty} \frac{x^k}{k}.$$

The coefficient of x^n, for $n \geq 2$, is

$$\sum_{j=1}^{n-1} \frac{1}{j(n-j)} = \frac{1}{n} \sum_{j=1}^{n-1} \left(\frac{1}{j} + \frac{1}{n-j} \right) = \frac{2H_{n-1}}{n},$$

so

$$\log^2(1-x) = \sum_{n=2}^{\infty} \frac{2H_{n-1}}{n} x^n. \tag{36}$$

Integration gives

$$\sum_{n=2}^{\infty} \frac{2H_{n-1}}{n^2} x^n = \int_0^x \frac{\log^2(1-t)}{t} dt \tag{37}$$

and

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{2H_{n-1}}{n^3} x^n &= \int_0^x \frac{1}{t} \int_0^t \frac{\log^2(1-u)}{u} \, du \, dt \\ &= \int_0^x \frac{\log^2(1-u)}{u} \int_u^x \frac{dt}{t} \, du \\ &= \int_0^x \frac{\log^2(1-u)}{u} \log \frac{x}{u} \, du, \end{aligned} \tag{38}$$

which we'll use with $x = e^{\pi i/3}$ later.

Combining (37) with (24), with $1-x$ replacing x, we obtain the following identity, which can be regarded as the expansion of Li_3 about 1: for $0 < x < 1$,

$$\mathrm{Li}_3(1-x) = \zeta(3) - \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^2} x^n + \mathrm{Li}_2(1-x) \log(1-x) + \tfrac{1}{2} \log x \log^2(1-x).$$

Some multiple zeta values

We define the 'multiple zeta value' function as

$$\zeta(s_1, \ldots, s_k) = \sum_{\substack{n_1, \ldots, n_k \\ 0 < n_1 < \cdots < n_k}} n_1^{-s_1} \cdots n_k^{-s_k}.$$

(Annoyingly this is sometimes written $\zeta(s_k, \ldots, s_1)$ instead!) Here I will only consider $\zeta(s,t)$. We have

$$\zeta(s,t) = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=m+1}^{\infty} \frac{1}{n^t} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{1}{(m+n)^t}.$$

Also, reversing the order in the first expression, we have

$$\zeta(s,t) = \sum_{n=2}^{\infty} \frac{1}{n^t} \sum_{m=1}^{n-1} \frac{1}{m^s}.$$

in particular,

$$\zeta(1,t) = \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^t}. \tag{39}$$

Most obviously, we have for $\Re s > 1$

$$\begin{aligned}
\zeta(s,s) &= \sum_{\substack{m,n \\ m<n}} (mn)^{-s} \\
&= \frac{1}{2} \sum_{\substack{m,n \\ m\neq n}} (mn)^{-s} \\
&= \frac{1}{2} \left(\sum_{m,n} (mn)^{-s} - \sum_{n} n^{-2s} \right) \\
&= \frac{1}{2} (\zeta(s)^2 - \zeta(2s)). \qquad (40)
\end{aligned}$$

The case $s = 2$ gives

$$\zeta(2,2) = \frac{\pi^4}{2} \left(\frac{1}{36} - \frac{1}{90} \right) = \frac{\pi^4}{120} = \frac{3}{4} \zeta(4). \tag{41}$$

Taking $x = 1$ in (37) and applying (24), we have

$$\begin{aligned}
\zeta(1,2) &= \sum_{n=2}^{\infty} \frac{H_{n-1}}{n^2} \\
&= \frac{1}{2} \int_0^1 \frac{\log^2(1-t)}{t} dt \\
&= \zeta(3).
\end{aligned}$$

an identity already known to Euler. Later I found the following direct proof avoiding integrals. [*Note added by Graham Jameson: this method can be seen, for example, in [BBr, p. 7–8], where it is attributed to R. Steinberg*]. Since

$$\zeta(1,2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m(m+n)^2}.$$

we have

$$\begin{aligned}
2\zeta(1,2) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^2} \left(\frac{1}{m} + \frac{1}{n} \right) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(m+n)}.
\end{aligned}$$

Now

$$\frac{1}{mn(m+n)} = \frac{1}{m^2}\,\frac{m}{n(m+n)} = \frac{1}{m^2}\left(\frac{1}{n} - \frac{1}{m+n}\right)$$

and by cancellation

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{m+n}\right) = 1 + \frac{1}{2} + \cdots + \frac{1}{m} = H_m,$$

so

$$\begin{aligned} 2\zeta(1,2) &= \sum_{m=1}^{\infty} \frac{H_m}{m^2} \\ &= \zeta(3) + \sum_{m=2}^{\infty} \frac{H_{m-1}}{m^2} \\ &= \zeta(3) + \zeta(1,2). \end{aligned}$$

A similar method delivers an identity for $\zeta(1,3)$, which will be used for Comtet's series. [*Tim's original method was by manipulation of the series for* $\mathrm{Li}_2(x)^2$*; the following is slightly shorter and more like the previous proof.*]

$$\begin{aligned} 2\zeta(1,3) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m+n)^3}\left(\frac{1}{m} + \frac{1}{n}\right) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn(m+n)^2} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{mn(m+n)^2} &= \frac{1}{m^2(m+n)}\left(\frac{1}{n} - \frac{1}{m+n}\right) \\ &= \frac{1}{m^2 n(m+n)} - \frac{1}{m^2(m+n)^2} \\ &= \frac{1}{m^3}\left(\frac{1}{n} - \frac{1}{m+n}\right) - \frac{1}{m^2(m+n)^2}, \end{aligned}$$

so, as before,

$$\begin{aligned} 2\zeta(1,3) &= \sum_{m=1}^{\infty} \frac{H_m}{m^3} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^2(m+n)^2} \\ &= \zeta(4) + \sum_{m=2}^{\infty} \frac{H_{m-1}}{m^3} - \zeta(2,2) \\ &= \zeta(4) + \zeta(1,3) - \zeta(2,2). \end{aligned}$$

Hence, by (41),

$$\zeta(1,3) = \zeta(4) - \zeta(2,2) = \tfrac{1}{4}\zeta(4). \tag{42}$$

Comtet's series for $\zeta(4)$

For this, [BBC] gives a reference to Comtet's book [Com], but I have not seen this or any other proof. Here is my proof (completed 4-8-07). Perhaps it could be substantially simplified! Let

$$S = \sum_{n=1}^{\infty} \frac{1}{n^4\binom{2n}{n}}.$$

From (33), we have

$$\sum_{n=1}^{\infty} \frac{y^{2n}}{n^3\binom{2n}{n}} = \int_0^y \frac{4}{z}\left(\arcsin\frac{z}{2}\right)^2 dz$$

so

$$\begin{aligned}
S &= 2\int_0^1 \frac{1}{y}\int_0^y \frac{4}{z}\left(\arcsin\frac{z}{2}\right)^2 dz\, dy \\
&= 8\int_0^1 \frac{1}{z}\left(\arcsin\frac{z}{2}\right)^2 \int_z^1 \frac{dy}{y}\, dz \\
&= -8\int_0^1 \frac{1}{z}\left(\arcsin\frac{z}{2}\right)^2 \log z\, dz \\
&= -8\int_0^{\frac{\pi}{6}} \frac{1}{2\sin\theta}\theta^2 \log(2\sin\theta)\cdot 2\cos\theta\, d\theta \qquad (z = 2\sin\theta) \\
&= -8\int_0^{\frac{\pi}{6}} \theta^2 \cot\theta\, \log(2\sin\theta)\, d\theta \\
&= -4\int_0^{\frac{\pi}{6}} \theta^2 \frac{d}{d\theta}(\log^2(2\sin\theta))\, d\theta \\
&= -4\left[\theta^2\log^2(2\sin\theta)\right]_0^{\frac{\pi}{6}} + 4\int_0^{\frac{\pi}{6}} 2\theta\log^2(2\sin\theta)\, d\theta \\
&= 8\int_0^{\frac{\pi}{6}} \theta\log^2(2\sin\theta)\, d\theta.
\end{aligned}$$

Now expand as follows (maybe something circular is happening here?):

$$\log(2\sin\theta) = \log\left(\frac{e^{i\theta}-e^{-i\theta}}{i}\right) = \left(\theta - \frac{\pi}{2}\right)i + \log\left(1-e^{-2i\theta}\right).$$

so

$$\begin{aligned}
S &= 8\int_0^{\frac{\pi}{6}} \theta\left(-\left(\theta-\frac{\pi}{2}\right)^2 + 2i\left(\theta-\frac{\pi}{2}\right)\log\left(1-e^{-2i\theta}\right) + \log^2\left(1-e^{-2i\theta}\right)\right) d\theta \\
&= -8I_1 + 16I_2 + 8I_3,
\end{aligned}$$

where

$$I_1 = \int_0^{\frac{\pi}{6}} \theta\left(\theta-\frac{\pi}{2}\right)^2 d\theta$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{6}} \left(\theta^3 - \pi\theta^2 + \frac{\pi^2}{4}\theta\right) d\theta \\
&= \pi^4 \left(\frac{1}{4\cdot 6^4} - \frac{1}{3\cdot 6^3} + \frac{1}{2\cdot 4\cdot 6^2}\right) \\
&= \frac{\pi^4}{4\cdot 6^4}(1-8+18) \\
&= \frac{11\pi^4}{4\cdot 6^4},
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= i\int_0^{\frac{\pi}{6}} \theta\left(\theta - \frac{\pi}{2}\right)\log(1-e^{-2i\theta})\, d\theta \\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{6}}\left(\theta^2 - \frac{\pi\theta}{2}\right) e^{-2in\theta}\, d\theta \\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\left[\left(\theta^2 - \frac{\pi\theta}{2}\right)\frac{e^{-2in\theta}}{-2in}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\left(2\theta - \frac{\pi}{2}\right)\frac{e^{-2in\theta}}{-2in}\, d\theta\right) \\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\left(\frac{\pi^2}{36} - \frac{\pi^2}{12}\right)\frac{e^{-\pi in/3}}{-2in} - \left[\left(2\theta - \frac{\pi}{2}\right)\frac{e^{-2in\theta}}{(-2in)^2}\right]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} 2\frac{e^{-2in\theta}}{(-2in)^2}\, d\theta\right) \\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\left(-\frac{\pi^2}{18}\right)\frac{ie^{-\pi in/3}}{2n} - \left(\frac{\pi}{3} - \frac{\pi}{2}\right)\frac{e^{-\pi in/3}}{-4n^2} + \left(-\frac{\pi}{2}\right)\frac{1}{-4n^2} + \left[2\frac{e^{-2in\theta}}{(-2in)^3}\right]_0^{\frac{\pi}{6}}\right) \\
&= -i\sum_{n=1}^{\infty}\frac{1}{n}\left(-\frac{\pi^2 i}{36n}e^{-\pi in/3} - \frac{\pi}{24n^2}e^{-\pi in/3} + \frac{\pi}{8n^2} - \frac{i}{4n^3}(e^{-\pi in/3} - 1)\right) \\
&= \sum_{n=1}^{\infty}\left(-\frac{\pi^2}{36n^2}e^{-\pi in/3} + \frac{\pi i}{24n^3}e^{-\pi in/3} - \frac{\pi i}{8n^3} - \frac{1}{4n^4}e^{-\pi in/3} + \frac{1}{4n^4}\right),
\end{aligned}
$$

and by (36)

$$
\begin{aligned}
I_3 &= \int_0^{\frac{\pi}{6}} \theta\log^2(1-e^{-2i\theta})\, d\theta \\
&= \sum_{n=2}^{\infty}\frac{2H_{n-1}}{n}\int_0^{\frac{\pi}{6}} \theta e^{-2in\theta}\, d\theta \\
&= \sum_{n=2}^{\infty}\frac{2H_{n-1}}{n}\left(\left[\theta\frac{e^{-2in\theta}}{-2in}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}}\frac{e^{-2in\theta}}{-2in}\, d\theta\right) \\
&= \sum_{n=2}^{\infty}\frac{2H_{n-1}}{n}\left(\frac{\pi i}{12n}e^{-\pi in/3} - \left[\frac{e^{-2in\theta}}{(-2in)^2}\right]_0^{\frac{\pi}{6}}\right) \\
&= \sum_{n=2}^{\infty}\frac{2H_{n-1}}{n}\left(\frac{\pi i}{12n}e^{-\pi in/3} + \frac{1}{4n^2}(e^{-\pi in/3} - 1)\right) \\
&= \sum_{n=2}^{\infty} H_{n-1}\left(\frac{\pi i}{6n^2}e^{-\pi in/3} + \frac{1}{2n^3}e^{-\pi in/3} - \frac{1}{2n^3}\right).
\end{aligned}
$$

Hence

$$\begin{aligned} S &= -\frac{22\pi^4}{6^4} + \sum_{n=1}^{\infty} \left(-\frac{4\pi^2}{9n^2} e^{-\pi i n/3} + \frac{2\pi i}{3n^3} e^{-\pi i n/3} - \frac{2\pi i}{n^3} - \frac{4}{n^4} e^{-\pi i n/3} + \frac{4}{n^4} \right) \\ &\quad + \sum_{n=2}^{\infty} H_{n-1} \left(\frac{4\pi i}{3n^2} e^{-\pi i n/3} + \frac{4}{n^3} e^{-\pi i n/3} - \frac{4}{n^3} \right) \\ &= -\frac{11\pi^4}{2^3 \cdot 3^4} - 2\pi i \zeta(3) + 4\zeta(4) - 4\zeta(1,3) \\ &\quad + \sum_{n=1}^{\infty} \left(-\frac{4\pi^2}{9n^2} + \frac{2\pi i}{3n^3} - \frac{4}{n^4} \right) e^{-\pi i n/3} + \sum_{n=2}^{\infty} H_{n-1} \left(\frac{4\pi i}{3n^2} + \frac{4}{n^3} \right) e^{-\pi i n/3}. \end{aligned}$$

By (42), $4\zeta(1,3) = \zeta(4)$, so the first four terms equate to

$$\begin{aligned} -\frac{11\pi^4}{2^3 \cdot 3^4} + 3\zeta(4) - 2\pi i \zeta(3) &= \left(\frac{1}{2 \cdot 3 \cdot 5} - \frac{11}{2^3 \cdot 3^4} \right) \pi^4 - 2\pi i \zeta(3) \\ &= \frac{2^2 \cdot 3^3 - 5 \cdot 11}{2^3 \cdot 3^4 \cdot 5} \pi^4 - 2\pi i \zeta(3) \\ &= \frac{108 - 55}{2^3 \cdot 3^4 \cdot 5} \pi^4 - 2\pi i \zeta(3) \\ &= \frac{53\pi^4}{2^3 \cdot 3^4 \cdot 5} - 2\pi i \zeta(3). \end{aligned}$$

So our formula becomes

$$S = \frac{53\pi^4}{2^3 \cdot 3^4 \cdot 5} - 2\pi i \zeta(3) + \sum_{n=1}^{\infty} \left(-\frac{4\pi^2}{9n^2} + \frac{2\pi i}{3n^3} - \frac{4}{n^4} \right) e^{-\pi i n/3} + \sum_{n=2}^{\infty} H_{n-1} \left(\frac{4\pi i}{3n^2} + \frac{4}{n^3} \right) e^{-\pi i n/3}.$$

The imaginary part of S is zero, so we have proved

$$\begin{aligned} 2\pi\zeta(3) &= \sum_{n=1}^{\infty} \left(\frac{4\pi^2}{9n^2} \sin\frac{\pi n}{3} + \frac{2\pi}{3n^3} \cos\frac{\pi n}{3} + \frac{4}{n^4} \sin\frac{\pi n}{3} \right) \\ &\quad + \sum_{n=2}^{\infty} H_{n-1} \left(\frac{4\pi}{3n^2} \cos\frac{\pi n}{3} - \frac{4}{n^3} \sin\frac{\pi n}{3} \right). \end{aligned}$$

Incidentally, I cannot evaluate the separate contributions of any of these terms. The real terms give

$$\begin{aligned} S &= \frac{53\pi^4}{2^3 \cdot 3^4 \cdot 5} + \sum_{n=1}^{\infty} \left(-\frac{4\pi^2}{9n^2} \cos\frac{\pi n}{3} + \frac{2\pi}{3n^3} \sin\frac{\pi n}{3} - \frac{4}{n^4} \cos\frac{\pi n}{3} \right) \\ &\quad + \sum_{n=2}^{\infty} H_{n-1} \left(\frac{4\pi}{3n^2} \sin\frac{\pi n}{3} + \frac{4}{n^3} \cos\frac{\pi n}{3} \right). \end{aligned} \tag{43}$$

The two sums here will turn out to give a total of $-\zeta(4)$. We now evaluate the first sum. The contribution of each of the three terms is of the form $CB_k(\frac{1}{6})$, by the Fourier series for the periodified Bernoulli polynomial

$$\begin{aligned}\tilde{B}_k(x) &= B_k(\{x\}_0)\\ &= -k!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^k}\\ &= -\frac{k!}{(2\pi i)^k}(F(k,x)+(-1)^kF(k,-x)),\end{aligned}$$

where

$$F(s,\alpha)=\mathrm{Li}_s(e^{2\pi i\alpha})=\sum_{n=1}^{\infty}\frac{e^{2\pi in\alpha}}{n^s}.$$

is the Lerch zeta function. Splitting according to parity gives

$$\sum_{n=1}^{\infty}\frac{\cos 2\pi nx}{n^{2k}}=(-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}\tilde{B}_{2k}(x),$$

and

$$\sum_{n=1}^{\infty}\frac{\sin 2\pi nx}{n^{2k+1}}=(-1)^{k-1}\frac{(2\pi)^{2k+1}}{2(2k+1)!}\tilde{B}_{2k+1}(x).$$

The Bernoulli polynomials up to B_4 are

$$\begin{aligned}B_0(x) &= 1,\\ B_1(x) &= x-\tfrac{1}{2},\\ B_2(x) &= x^2-x+\tfrac{1}{6},\\ B_3(x) &= x^3-\tfrac{3}{2}x^2+\tfrac{1}{2}x,\\ B_4(x) &= x^4-2x^3+x^2-\tfrac{1}{30},\end{aligned}$$

so we find

$$\sum_{n=1}^{\infty}\frac{\cos(\pi n/3)}{n^2}=\pi^2B_2(\tfrac{1}{6})=\frac{\pi^2}{36},$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin(\pi n/3)}{n^3} &= \frac{(2\pi)^3}{2\cdot 3!}B_3(\tfrac{1}{6})\\ &= \frac{2\pi^3}{3\cdot 6^3}\left(1-\frac{3\cdot 6}{2}+\frac{6^2}{2}\right)\\ &= \frac{2\pi^3}{3\cdot 6^3}(1-9+18)\\ &= \frac{20\pi^3}{3\cdot 6^3},\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{\cos(\pi n/3)}{n^4} = -\frac{(2\pi)^4}{2\cdot 4!}B_4(\tfrac{1}{6})$$

$$
\begin{aligned}
&= -\frac{\pi^4}{3\cdot 6^4}\left(1-2\cdot 6+6^2-\frac{6^4}{30}\right)\\
&= -\frac{\pi^4}{3\cdot 5\cdot 6^4}(125-216)\\
&= \frac{91\pi^4}{3\cdot 5\cdot 6^4}.
\end{aligned}
$$

So the first sum in (43) is

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(-\frac{4\pi^2}{9}\cdot\frac{\cos(\pi n/3)}{n^2}+\frac{2\pi}{3}\cdot\frac{\sin(\pi n/3)}{n^3}-4\frac{\cos(\pi n/3)}{n^4}\right)\\
&= -\frac{4\pi^2}{9}\cdot\frac{\pi^2}{6^2}+\frac{2\pi}{3}\cdot\frac{20\pi^3}{3\cdot 6^3}-4\frac{91\pi^4}{3\cdot 5\cdot 6^4}\\
&= \frac{\pi^4}{3\cdot 5\cdot 6^4}\left(-\frac{4}{9}\cdot 3\cdot 5\cdot 6^2+\frac{2}{3}\cdot 20\cdot 5\cdot 6-4\cdot 91\right)\\
&= \frac{\pi^4}{3\cdot 5\cdot 6^4}\left(-\frac{20\cdot 6^2}{3}+\frac{40\cdot 5\cdot 6}{3}-4\cdot 91\right)\\
&= \frac{\pi^4}{3\cdot 5\cdot 6^4}(-240+400-364)\\
&= -\frac{204\pi^4}{3\cdot 5\cdot 6^4}\\
&= -\frac{34\pi^4}{3\cdot 5\cdot 6^3}.
\end{aligned}
$$

Since $3\cdot 5\cdot 6^3=2^3\cdot 3^4\cdot 5$ and $53-34=19$, we have proved

$$
S=\frac{19\pi^4}{2^3\cdot 3^4\cdot 5}+4\sum_{n=2}^{\infty}H_{n-1}\left(\frac{\pi\sin(\pi n/3)}{3n^2}+\frac{\cos(\pi n/3)}{n^3}\right). \tag{44}
$$

We now define

$$
f(x)=\int_0^x\frac{\log^2(1-u)\log u}{u}\,du. \tag{45}
$$

By (38), we have

$$
f(x)=\left(\sum_{n=2}^{\infty}\frac{2H_{n-1}}{n^2}x^n\right)\log x-\sum_{n=2}^{\infty}\frac{2H_{n-1}}{n^3}x^n,
$$

so that (44) becomes

$$
S=\frac{19\pi^4}{2^3\cdot 3^4\cdot 5}-2\Re f(e^{\pi i/3}). \tag{46}
$$

The obvious integration by parts gives a reflection relation between $f(x)$ and $f(1-x)$:

$$
\begin{aligned}
f(x) &= \left[\log^2(1-u)\cdot\tfrac{1}{2}\log^2 u\right]_0^x-\int_0^x\left(-\frac{2\log(1-u)}{1-u}\right)\tfrac{1}{2}\log^2 u\,du\\
&= \tfrac{1}{2}\log^2(1-x)\log^2 x+\int_0^x\frac{\log^2 u\log(1-u)}{1-u}\,du\\
&= \tfrac{1}{2}\log^2(1-x)\log^2 x+f(1)-\int_x^1\frac{\log^2 u\log(1-u)}{1-u}\,du.
\end{aligned}
$$

By (39) and (42), $f(1) = -2\zeta(1,3) = -\frac{1}{2}\zeta(4)$. Also,

$$\int_x^1 \frac{\log^2 u \log(1-u)}{1-u}\, du. = \int_0^{1-x} \frac{\log^2(1-u)\log u}{u}\, du = f(1-x),$$

so

$$f(x) + f(1-x) = -\tfrac{1}{2}\zeta(4) + \tfrac{1}{2}\log^2(1-x)\log^2 x. \tag{47}$$

Since $f(\overline{z}) = \overline{f(z)}$ (this occurs when f is analytic and real on the real axis), the case $x = e^{\pi i/3}$ gives us the value we want:

$$\begin{aligned}
2\Re f(e^{\pi i/3}) &= f(e^{\pi i/3}) + f(e^{-\pi i/3}) \\
&= f(e^{\pi i/3}) + f(1 - e^{\pi i/3}) \\
&= -\tfrac{1}{2}\zeta(4) + \tfrac{1}{2}\left(-\frac{\pi i}{3}\right)^2 \left(\frac{\pi i}{3}\right)^2 \\
&= -\tfrac{1}{2}\zeta(4) + \tfrac{1}{2}\left(\frac{\pi}{3}\right)^4 \\
&= \frac{\pi^4}{2}\left(-\frac{1}{2\cdot 3^2 \cdot 5} + \frac{1}{3^4}\right) \\
&= \frac{\pi^4}{2^2\cdot 3^4\cdot 5}(-3^2 + 2\cdot 5) \\
&= \frac{\pi^4}{2^2\cdot 3^4\cdot 5}.
\end{aligned}$$

Inserting this into (46) gives

$$S = \frac{17\pi^4}{2^3\cdot 3^4\cdot 5} = \frac{17}{36}\zeta(4),$$

so finally we have Comtet's series

$$\zeta(4) = \frac{36}{17}\sum_{n=1}^{\infty}\frac{1}{n^4\binom{2n}{n}}. \tag{48}$$

We have had two occurences of $\zeta(1,3) = \frac{1}{4}\zeta(4)$ in our calculation of S. These did not simply cancel out: the first occurrence was $-4\zeta(1,3)$, while the second was $-f(1) = 2\zeta(1,3)$.

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updated with minor corrections, April 2018