# Orbits made easy by complex numbers

## Tim Jameson

Complex numbers are equivalent to two-dimensional vectors, with the bonus that multiplication represents rotation. Consequently they lend themselves rather well to the description of motion in a plane. The position of a particle is represented by $z = re^{i\theta}$, where $r$ and $\theta$ are real-valued functions of $t$. Note that at this point, $e^{i\theta}$ represents the radial unit vector and $ie^{i\theta}$ the transverse one.

The expressions for radial and transverse components of velocity and acceleration can be derived using the ordinary rules for differentiation (which, of course, apply equally to complex-valued functions). With the customary dot notation denoting differentiation with respect to $t$, we have

$$\dot{z} = (\dot{r} + ir\dot{\theta})e^{i\theta}, \tag{1}$$

and

$$
\begin{aligned}
\ddot{z} &= (\ddot{r} + i\dot{r}\dot{\theta} + ir\ddot{\theta})e^{i\theta} + i\dot{\theta}(\dot{r} + ir\dot{\theta})e^{i\theta} \\
&= (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + i(2\dot{r}\dot{\theta} + r\ddot{\theta})e^{i\theta} \\
&= (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + \frac{i}{r}\frac{d}{dt}(r^2\dot{\theta})e^{i\theta},
\end{aligned} \tag{2}
$$

an expression of the form $(A + iB)e^{i\theta}$, in which $A$ is the radial component of acceleration, and $B$ the transverse component. So far, this is only slightly different from the usual method based on differentiation of the radial and transverse unit vectors.

If the particle is moving under the action of a central force, directed towards (or away from) the origin, then $\ddot{z}$ is of the form $F(z)e^{i\theta}$ for some real-valued function $F(z)$. In (2), it follows that $B = 0$, hence $r^2\dot{\theta}$ is constant, say

$$r^2\dot{\theta} = h. \tag{3}$$

This embodies the fact that angular momentum is constant, and equally the fact that area is swept out at a constant rate, i.e. Kepler's second law. This much is true for *any* central force.

Now suppose further that the force is an attraction obeying the inverse square law, so that for some (real) constant $\mu$,

$$\ddot{z} = -\frac{\mu}{r^2}e^{i\theta}. \tag{4}$$

By (3) and (4), we have

$$\ddot{z} = -\frac{\mu}{h}\dot{\theta}e^{i\theta},$$

hence
$$ih\ddot{z} = -i\mu\dot{\theta}e^{i\theta} = -\mu\frac{d}{dt}(e^{i\theta}),$$

so that
$$\frac{d}{dt}(ih\dot{z} + \mu e^{i\theta}) = 0.$$

So there is a complex constant, which we write in polar form as $-Ce^{i\delta}$, such that

$$ih\dot{z} + \mu e^{i\theta} = -Ce^{i\delta}.$$

Now use (1) to substitute for $\dot{z}$ and multiply by $e^{-i\theta}$. We obtain

$$ih\dot{r} - hr\dot{\theta} + \mu = -Ce^{i(\delta-\theta)}.$$

By (3), $hr\dot{\theta} = h^2/r$. Taking the real part, we have

$$\frac{h^2}{r} = \mu + C\cos(\theta - \delta),$$

or

$$r[\mu + C\cos(\theta - \delta)] = h^2. \tag{5}$$

Well-known geometrical arguments show that this represents a conic with eccentricity $e = C/\mu$ and semi-latus rectum $\ell = h^2/\mu$. This is Kepler's first law (though in fact Kepler only stated the case of an ellipse).

For comparison, we summarise briefly two of the methods commonly found in books. The first one, e.g. [1, chapter 4], is as follows. By (2) and (4), $\ddot{r} - r\dot{\theta}^2 = -\mu/r^2$. Use (3) to substitute for $\dot{\theta}$. One then takes two rather non-obvious steps: we substitute $u = 1/r$ and use the chain rule to express the derivatives in terms of $\theta$ instead of $t$. After some calculations, one arrives at the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}.$$

The solution of this equation can be rewritten as (5). We remark that our method made no use of $\ddot{r} - r\dot{\theta}^2$.

The second method (see, e.g. [2, pp. 42–46]) is more closely related to ours. It proceeds by showing that $\boldsymbol{h} \times \dot{\boldsymbol{r}} + \mu\boldsymbol{e}_r$ is a constant vector, where $\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$ and $\boldsymbol{e}_r$ is the unit vector in radial direction. This (strictly, its negative) is known as the Runge-Lenz vector, and with a bit of effort one can verify that it equates to our $ih\dot{z} + \mu e^{i\theta}$. In fact, our method is basically an adaptation of the same idea with vectors replaced by complex numbers. The steps correspond in a recognisable way, but the details of the vector method are distinctly more intricate, involving a good deal of manipulation of vector cross products. Arguably,

these cross products introduce a spurious third dimension into a purely two-dimensional problem.

Thanks to Nick Lord for pointing out that a version of our complex-number method was given in [3]. Readers can compare this version with ours. The general idea is the same, but the details are presented in a rather different way.

*References*

1. T. Heard, *Mechanics and Vectors*, Cambridge Univ. Press (1986).
2. B. P. Cowan, *Classical Mechanics*, Routledge Kegan Paul (1984).
3. R. Shail, A note on Newtonian central orbits, *Math. Gaz.* **68** (1984), 290–292.

*Sadly, Tim Jameson died in September 2013. The original version of this note was written in 1997, but never submitted.*