Operator-valued extensions of matrix-norm inequalities

G.J.O. Jameson

1. INTRODUCTION. Let $A = (a_{j,k})$ be a matrix (finite or infinite) of complex numbers. Let $\|A\|$ denote the usual operator norm of A as an operator on ℓ_2^n or ℓ_2. For vectors x, y, we then have the *linear* inequality $\|Ax\| \le \|A\|.\|x\|$ and the *bilinear* inequality $|\langle Ax, y\rangle| \le \|A\|.\|x\|.\|y\|$. Written out explicitly, these inequalities say:

(1) if $x_1, x_2, \ldots$ are scalars and $y_j = \sum_k a_{j,k} x_k$ for each j, then

$$\sum_j |y_j|^2 \le \|A\|^2 \sum_k |x_k|^2;$$

(2) for scalars x_j, y_k,

$$\left| \sum_j \sum_k a_{j,k} \overline{x}_j y_k \right| \le \|A\| \left(\sum_j |x_j|^2 \right)^{1/2} \left(\sum_k |y_k|^2 \right)^{1/2}.$$

Also, if the matrix A is positive definite, then we have $\sum_j \sum_k a_{j,k} \overline{x_j} x_k \ge 0$ for scalars x_j.

We will show that in all these inequalities, the scalars x_j, y_k can be replaced by operators on a Hilbert space H, with the conclusions taking the form of operator inequalities in the usual sense: $C \ge 0$, for self-adjoint C, means $\langle Cu, u\rangle \ge 0$ for $u \in H$. The term $|x_j|^2$ is replaced by $X_j^* X_j$.

In particular, classical inequalities of this type have operator-valued versions. An example is *Hardy's inequality* (one of several bearing his name): if A is the "averaging" operator given by $Ax = y$, where $y_j = \frac{1}{j}(x_1 + x_2 + \cdots + x_j)$, then $\|A\| = 2$, so $\sum_{j=1}^\infty |y_j|^2 \le 4 \sum_{k=1}^\infty |x_k|^2$. The operator-valued version takes the form: $\sum_{j=1}^\infty Y_j^* Y_j \le 4 \sum_{k=1}^\infty X_k^* X_k$.

The bilinear inequality is derived from the linear one with the help of an operator-valued version of the Cauchy-Schwarz inequality. All these results, at least in their finite form, are obtained by simple and elegant methods well within the scope of a basic course on Hilbert spaces. (They can alternatively be obtained by tensor product techniques, but in the author's view, these methods are less simple and decidedly less elegant!)

Though elementary, the results contain as special cases certain inequalities between Schatten ideal norms given in quite recent research papers ([5], [9], [10]).

Throughout the following, H will be a Hilbert space with inner product denoted by $\langle\, , \rangle$, and $L(H)$ denotes the space of bounded linear operators on H. A norm σ on $L(H)$ (or on a subspace) is *monotonic* if $\sigma(B) \leq \sigma(A)$ whenever $0 \leq B \leq A$. The usual operator norm is monotonic.

2. VECTOR-VALUED VERSIONS. First, we give vector-valued extensions of matrix inequalities of the above types. They are very well known, but we include simple proofs, since more elaborate ones have appeared. First, positivity:

Lemma 1. *If $(c_{j,k})$ is a positive definite $n \times n$ matrix and x_j $(1 \leq j \leq n)$ are elements of a Hilbert space H, then*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_{j,k} \langle x_k, x_j \rangle \geq 0.$$

Proof. Choose an orthonormal basis $e_1, e_2, \ldots, e_m$ of the subspace generated by the elements x_j, and write $x_j = \sum_{r=1}^{m} x_j(r) e_r$. For each r we have $\sum_{j=1}^{n} \sum_{k=1}^{n} c_{j,k} \overline{x_j(r)} x_k(r) \geq 0$. Summation over r gives the statement. $\square$

Note. Replacing x_j by $\overline{x}_j$, we see that equally $\sum_{j=1}^{n} \sum_{k=1}^{n} c_{j,k} \langle x_j, x_k \rangle \geq 0$.

The linear and bilinear inequalities involving $\|A\|$ extend as follows.

Lemma 2. *Let A be an $m \times n$ matrix, and let H be a Hilbert space. Then:*

(i) if $x_1, \ldots, x_n \in H$ and $y_j = \sum_{k=1}^{n} a_{j,k} x_k$ for $1 \leq j \leq m$, then

$$\sum_{j=1}^{m} \|y_j\|^2 \leq \|A\|^2 \sum_{k=1}^{n} \|x_k\|^2.$$

(ii) for elements x_j, y_k of H, we have

$$\left| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k} \langle y_k, x_j \rangle \right| \leq \|A\| \left(\sum_{j=1}^{m} \|x_j\|^2 \right)^{1/2} \left(\sum_{k=1}^{n} \|y_k\|^2 \right)^{1/2}.$$

Proof. Statement (i) can be proved in similar fashion to Lemma 1. The sum in (ii) can be written as $\sum_{j=1}^{m} \langle z_j, x_j \rangle$, where $z_j = \sum_{k=1}^{n} a_{j,k} y_k$. The statement follows by the Cauchy-Schwarz inequality and (i). $\square$

3. POSITIVITY AND THE LINEAR NORM INEQUALITY. For the moment, we restrict ourselves to finite sequences. In the two cases stated, the desired operator inequalities follow very simply from Lemmas 1 and 2.

Proposition 1. *Let $C = (c_{j,k})$ be a positive definite $n \times n$ matrix. Then for any $X_1, \ldots, X_n \in L(H)$, we have the operator inequality*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_{j,k} X_j^* X_k \geq 0.$$

Proof. Write $\sum_{j=1}^n \sum_{k=1}^n c_{j,k} X_j^* X_k = S$. Choose $u \in H$. By Lemma 1, we have

$$\langle Su, u \rangle = \sum_{j=1}^n \sum_{k=1}^n c_{j,k} \langle X_k u, X_j u \rangle \geq 0. \qquad \square$$

Theorem 2. *Let $A = (a_{j,k})$ be an $m \times n$ matrix. Suppose that $X_1, \ldots, X_n \in L(H)$ and $Y_j = \sum_{k=1}^n a_{j,k} X_k$ for $1 \leq j \leq m$. Then we have the operator inequality*

$$\sum_{j=1}^m Y_j^* Y_j \leq \|A\|^2 \sum_{k=1}^n X_k^* X_k.$$

Proof. The statement is equivalent to the assertion that

$$\sum_{j=1}^m \|Y_j u\|^2 \leq \|A\|^2 \sum_{k=1}^n \|X_k u\|^2$$

for all $u \in H$. But $Y_j u = \sum_{k=1}^n a_{j,k} X_k u$ for each j, so this is true, by Lemma 2(i). $\square$

Corollary 2.1. *Under the same conditions,* $\|\sum_{j=1}^m Y_j^* Y_j\| \leq \|A\|^2 \|\sum_{k=1}^n X_k^* X_k\|$.

Proof. Operator norm is monotonic. $\square$

Corollary 2.2. *Under the same conditions,* $\left(\sum_{j=1}^m Y_j^* Y_j\right)^{1/2} \leq \|A\| \left(\sum_{k=1}^n X_k^* X_k\right)^{1/2}$.

Proof. As is well known, $0 \leq X \leq Y$ implies $X^{1/2} \leq Y^{1/2}$. $\square$

Note. We have equally $\sum_{j=1}^m Y_j Y_j^* \leq \|A\|^2 \sum_{k=1}^n X_k X_k^*$, together with a similar statement for the symmetric modulus defined by $|X|_S^2 = \frac{1}{2} X^* X + \frac{1}{2} X X^*$.

Some particular cases. We record the operator-valued generalizations of some famous classical inequalities of this type (for the moment, in the finite form). First, let A be the averaging operator: if $x = (x_k)$, then $Ax = y$, where $y_j = \frac{1}{j}(x_1 + x_2 + \cdots + x_j)$ for all j. *Hardy's inequality* [1] states that the norm of A, as an operator on ℓ_2, is 2 (as an operator on ℓ_p, its norm is the conjugate index $p^* = p/(p-1)$). Hence we have:

Proposition 3. *Let $X_k \in L(H)$ $(1 \leq k \leq n)$ and let $Y_j = \frac{1}{j}(X_1 + X_2 + \cdots + X_j)$ for $1 \leq j \leq n$. Then $\sum_{j=1}^n Y_j^* Y_j \leq 4 \sum_{k=1}^n X_k^* X_k$.* $\square$

The best constant (valid for all n) is 4, since this is already the case for scalars.

It is interesting to note that the method of [2] for scalars does not adapt to give a direct proof of the operator version. A method that does adapt is the less well known one of [6].

Copson's inequality is the dual of Hardy's inequality. We go straight to the statement of the operator-valued version:

Proposition 4. *Let* $X_k \in L(H)$ $(1 \le k \le n)$ *and let* $Y_j = \sum_{k=j}^n X_k/k$ $(1 \le j \le n)$. *Then* $\sum_{j=1}^n Y_j^* Y_j \le 4 \sum_{k=1}^n X_k^* X_k$. $\square$

Hilbert's inequality (cf. [11, section 13.4]) is best stated in terms of two-sided sequences, as follows. For $n \in \mathbb{Z}$, write

$$c_n = \begin{cases} 1/n & \text{if } n \neq 0 \\ 0 & \text{if } n = 0. \end{cases}$$

Then the doubly infinite matrix (c_{j+k}) has norm π. So we have, for the finite case:

Proposition 5. *Let* $X_k \in L(H)$ $(-n \le k \le n)$, *and let* $Y_j = \sum_{k=-n}^n c_{j+k} X_k$ $(-m \le j \le m)$. *Then* $\sum_{j=-m}^m Y_j^* Y_j \le \pi^2 \sum_{k=-n}^n X_k^* X_k$. $\square$

Remark on norm inequalities. Let $Y_j = \sum_{k=1}^n a_{j.k} X_k$. Consider inequalities of the form

$$\sum_{j=1}^m \|Y_j\|^2 \le C^2 \sum_{k=1}^n \|X_k\|^2, \tag{1}$$

another possible generalization from the scalar case. Clearly, $\|Y_j\| \le \sum_{k=1}^n |a_{j,k}|.\|X_k\|$, so (1) holds with C equal to the norm of the matrix $(|a_{j,k}|)$. However, (1) does not hold with $C = \|A\|$, as the following simple example shows.

Example 1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $\|A\| = \sqrt{2}$ and $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Now take the 2×2 matrices $X_1 = \mathrm{diag}(1,1)$ and $X_2 = \mathrm{diag}(1,-1)$. Then $\|X_1\| = \|X_2\| = 1$, and clearly $\|Y_1\| = \|Y_2\| = 2$.

3. THE CAUCHY-SCHWARZ INEQUALITY.

The operator form of this inequality will be needed both for the bilinear operator inequality and for extensions to infinite sequences. There are many ways to prove it, e.g. see [4] and [7, Proposition 1.1]). Here we outline two particularly simple ways.

Theorem 6. *Let* X_j, Y_j $(1 \le j \le n)$ *be operators on a Hilbert space* H. *Let* $S = \sum_{j=1}^n X_j^* Y_j$. *Then*

$$S^* S \le \left\| \sum_{j=1}^n X_j^* X_j \right\| \left(\sum_{j=1}^n Y_j^* Y_j \right).$$

Proof 1. Let $\| \sum_{j=1}^n X_j^* X_j \| = M$. The statement is equivalent to $\|Sv\|^2 \le M \sum_{j=1}^n \|Y_j v\|^2$ for $v \in H$. With the substitution $Y_j v = y_j$, this becomes $\| \sum_{j=1}^n X_j^* y_j \|^2 \le M (\sum_{j=1}^n \|y_j\|^2)$. To prove it, choose u with $\|u\| = 1$ such that $\langle u, \sum_{j=1}^n X_j^* y_j \rangle = \| \sum_{j=1}^n X_j^* y_j \|)$. By the ordinary Cauchy-Schwarz inequality,

$$\left\langle u, \sum_{j=1}^{n} X_j^* y_j \right\rangle^2 = \left\langle \sum_{j=1}^{n} X_j u, y_j \right\rangle^2 \leq \sum_{j=1}^{n} \|X_j u\|^2 \sum_{j=1}^{n} \|y_j\|^2,$$

and the result follows because $\sum_{j=1}^n \|X_j u\|^2 = \langle \sum_{j=1}^n X_j^* X_j u, u\rangle \leq M$. $\square$

Proof 2 (Cf. [8]). Define $A : H \to H^n$ by $Au = (X_1 u, X_2 u, \ldots, X_n u)$ (and B similarly from Y_j). Then $A^* : H^n \to H$ is given by: $A^*(u_1, u_2, \ldots, u_n) = \sum_{j=1}^n X_j^* u_j$, and hence we have $A^*A = \sum_{j=1}^n X_j^* X_j$ and $A^*B = S$. The result now follows from the well-known fact that $B^*AA^*B \leq \|AA^*\| B^*B$, together with $\|AA^*\| = \|A^*A\|$. $\square$

Corollary 6.1 (Cf. [5]) *We have* $\| \sum_{j=1}^n X_j^* Y_j \|^2 \leq \| \sum_{j=1}^n X_j^* X_j \| . \| \sum_{j=1}^n Y_j^* Y_j \|$. $\square$

Corollary 6.2. *Given operators X_j and scalars λ_j $(1 \leq j \leq n)$, let $S = \sum_{j=1}^n \lambda_j X_j$. Then $S^*S \leq (\sum_{j=1}^n |\lambda_j|^2)(\sum_{j=1}^n X_j^* X_j)$.* $\square$

Note. The asymmetry in Theorem 6 is not avoidable. If X_j and Y_j are interchanged on the right-hand side, the resulting expression is a bound for SS^*, but not for S^*S, even when $n = 1$. Also, $\sum Y_j^* Y_j$ cannot be replaced by $\sum Y_j Y_j^*$. Of course, the statement can be rewritten as follows: if $T = \sum_{j=1}^n X_j Y_j$, then $T^*T \leq \| \sum_{j=1}^n X_j X_j^* \| (\sum_{j=1}^n Y_j^* Y_j)$.

5. THE BILINEAR MATRIX-NORM INEQUALITY. The way is now clear to formulate the operator version of this inequality.

Theorem 7. *Let $A = (a_{j,k})$ be an $m \times n$ matrix. Let X_j $(1 \leq j \leq m)$ and Y_k $(1 \leq k \leq n)$ be elements of $L(H)$, and let*

$$T = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k} X_j^* Y_k.$$

Then

$$T^*T \leq \|A\|^2 \left\| \sum_{j=1}^{m} X_j^* X_j \right\| \left(\sum_{k=1}^{n} Y_k^* Y_k \right).$$

Proof. For each j, let $Z_j = \sum_{k=1}^n a_{j,k} Y_k$. Then $T = \sum_{j=1}^n X_j^* Z_j$, and by Theorem 2,

$$\sum_{j=1}^{m} Z_j^* Z_j \leq \|A\|^2 \sum_{k=1}^{n} Y_k^* Y_k.$$

By Theorem 6,

$$T^*T \leq \left\| \sum_{j=1}^{m} X_j^* X_j \right\| \left(\sum_{j=1}^{m} Z_j^* Z_j \right).$$

The statement follows. $\square$

Corollary 7.1. *Under the same conditions,*

$$\|T\|^2 \le \|A\|^2 \left\| \sum_{j=1}^{m} X_j^* X_j \right\| . \left\| \sum_{k=1}^{n} Y_k^* Y_k \right\| .$$

6. EXTENSION TO INFINITE MATRICES AND SEQUENCES. We start with the extension of Theorem 6, since it is needed at the very first step in the extension of the other results. It follows easily from Corollary 6.1.

Theorem 8. *Suppose that* $\| \sum_{j=1}^{n} X_j^* X_j \| \le M$ *for all* n*, and that* $\sum_{j=1}^{\infty} Y_j^* Y_j$ *is norm-convergent. Then* $\sum_{j=1}^{\infty} X_j^* Y_j$ *is norm-convergent, say to* S*, and* $S^* S \le M \sum_{j=1}^{\infty} Y_j^* Y_j$*.*

Proof. Let $\varepsilon > 0$ be given. For sufficiently large m we have $\| \sum_{j=m+1}^{n} Y_j^* Y_j \| \le \varepsilon$ for all $n > m$. By Corollary 6.1, $\| \sum_{j=m+1}^{n} X_j^* Y_j \|^2 \le 2M\varepsilon$. Norm-convergence of $\sum_{j=1}^{\infty} X_j^* Y_j$ follows, by the Cauchy principle. The stated inequality follows from Theorem 6 by taking limits. $\square$

Corollary 8.1. *If* $\| \sum_{j=1}^{n} X_j^* X_j \| \le M$ *for all* n *and* (λ_j) *is a sequence of scalars such that* $\sum_{j=1}^{\infty} |\lambda_j|^2$ *is convergent, then* $\sum_{j=1}^{\infty} \lambda_j X_j$ *is norm-convergent.* $\square$

Recall that a sequence (X_j) of operators is "strongly convergent" to X if $\lim_{j\to\infty} X_j u = Xu$ for all $u \in H$. Norm-convergence implies strong convergence. Boundedness of $\sum_{j=1}^{n} X_j^* X_j$ is actually equivalent to strong convergence of $\sum_{j=1}^{\infty} X_j^* X_j$, since any norm-bounded, increasing sequence of self-adjoint operators converges strongly [1, Theorem 9.6.14]. In Theorem 8, norm-convergence of $\sum_{j=1}^{\infty} X_j^* Y_j$ follows equally if we have $\sum_{j=1}^{\infty} X_j^* X_j$ norm-convergent and $\sum_{j=1}^{\infty} Y_j^* Y_j$ strongly convergent; the inequality for $S^* S$ still holds with M defined as before.

A bit more work is needed to extend Theorem 2. We show that either norm-convergence or strong convergence is inherited by the transformed sequence (strong convergence is easier, which is why we mention it). The problem is to establish convergence; the inequality itself then follows automatically by taking limits.

Theorem 9. *Let* $A = (a_{j,k})$ *be an infinite matrix with finite norm* $\|A\|$ *as an operator on* ℓ_2*. Let* (X_k) *be a sequence in* $L(H)$ *such that* $\sum_{k=1}^{\infty} X_k^* X_k$ *is either (i) strongly convergent, or (ii) norm-convergent. Then* $\sum_{k=1}^{\infty} a_{j,k} X_k$ *is norm-convergent (say to* Y_j*) for each* j*. Further,* $\sum_{j=1}^{\infty} Y_j^* Y_j$ *is strongly convergent in case (i), norm-convergent in case (ii), and*

$$\sum_{j=1}^{\infty} Y_j^* Y_j \le \|A\|^2 \sum_{k=1}^{\infty} X_k^* X_k .$$

Proof. Write $\|A\| = M$. Note that $\sum_{j=1}^{\infty} |a_{j,k}|^2$ is convergent for each k and $\sum_{k=1}^{\infty} |a_{j,k}|^2$ is convergent for each j (these conditions are necessary, but not sufficient, for A to be a bounded operator on ℓ_2). So by Corollary 8.1, the series defining Y_j is norm-convergent under either hypothesis. Write $Y_{j,K} = \sum_{k=1}^{K} a_{j,k} X_k$. Since the norm of any truncated matrix $(a_{j,k})_{(j \le J, k \le K)}$ is not greater than M, it follows from Theorem 2 that for any J and K,

$$\sum_{j=1}^{J} Y_{j,K}^* Y_{j,K} \le M^2 \sum_{k=1}^{K} X_k^* X_k,$$

and hence by taking limits that

$$\sum_{j=1}^{J} Y_j^* Y_j \le M^2 \sum_{k=1}^{\infty} X_k^* X_k.$$

The statement for case (i) now follows by the remark after Corollary 8.1.

Now consider case (ii). Let $\varepsilon > 0$ be chosen, and let N be such that $\| \sum_{k=N+1}^{\infty} X_k^* X_k \| \le \varepsilon$. Write $Y_j = S_j + Z_j$, where

$$S_j = \sum_{k=1}^{N} a_{j,k} X_k, \qquad Z_j = \sum_{k=N+1}^{\infty} a_{j,k} X_k.$$

Let $n > m > N$. By the above, with X_k replaced by 0 for $1 \le k \le N$, we have $\| \sum_{j=m+1}^{n} Z_j^* Z_j \| \le M^2 \varepsilon$, so that for $u \in H$, we have $\sum_{j=m+1}^{n} \|Z_j u\|^2 \le \varepsilon M^2 \|u\|^2$. Also, by Corollary 6.1, $\|S_j\|^2 \le C \sum_{k=1}^{N} |a_{j,k}|^2$, where $C = \| \sum_{k=1}^{\infty} X_k^* X_k \|$. Since $\sum_{j=1}^{\infty} |a_{j,k}|^2$ is convergent for each k, $\sum_{j=1}^{\infty} \|S_j\|^2$ is convergent. Take N_1 such that $\sum_{j=N_1}^{\infty} \|S_j\|^2 \le \varepsilon$, and suppose that m also satisfies $m \ge N_1$. Then for $u \in H$, we have

$$\sum_{j=m+1}^{n} \|Y_j u\|^2 \le 2 \sum_{j=m+1}^{n} \|S_j u\|^2 + 2 \sum_{j=m+1}^{n} \|Z_j u\|^2 \le \varepsilon (M^2 + 1) \|u\|^2,$$

so that $\| \sum_{j=m+1}^{n} Y_j^* Y_j \| \le \varepsilon (M^2 + 1)$. Norm-convergence of $\sum_{j=1}^{\infty} Y_j^* Y_j$ follows. $\square$

So the specific inequalities of Propositions 3, 4 and 5 extend to the infinite case in the way stated in Theorem 9.

We can now repeat the steps of the proof of Theorem 7 to obtain the following extension:

Theorem 10. *Let A be as in Theorem 9. Let $X_j, Y_k \in L(H)$ be such that $\| \sum_{j=1}^{n} X_j^* X_j \| \le M$ for all n and $\sum_{k=1}^{\infty} Y_k^* Y_k$ is norm-convergent. Then the repeated sum $\sum_{j=1}^{\infty} X_j^* \sum_{k=1}^{\infty} a_{j,k} Y_k$ is norm-convergent (say to T), and*

$$T^* T \le \|A\|^2 M \sum_{k=1}^{\infty} Y_k^* Y_k. \qquad \square$$

Again, the same conclusion holds (with M defined as before) if $\sum_{j=1}^{\infty} X_j^* X_j$ is norm-convergent and $\sum_{k=1}^{\infty} Y_k^* Y_k$ is strongly convergent.

7. DERIVATION OF CERTAIN NORM INEQUALITIES. We now show that certain norm inequalities given in [9] and [10], even in a strengthened form, are immediate consequences of our operator inequalities.

For an operator A, we define the right and left moduli by

$$|A|_R = (A^*A)^{1/2}, \qquad |A|_L = (AA^*)^{1/2}.$$

Also, for operators $X_1, X_2, \ldots, X_n$, we define $X_R = \left(\sum_{j=1}^{n} X_j^* X_j\right)^{1/2}$, and X_L similarly.

We consider a monotonic norm σ such that $\sigma(|A|_R) = \sigma(|A|_L) = \sigma(A)$ for all A. These conditions are satisfied by the Schatten ideal (alias trace class) norms, which we denote by σ_p.

Proposition 11. *Let σ be a monotonic norm that satisfies $\sigma(|A|_R) = \sigma(|A|_L) = \sigma(A)$ for all A. Let $X_j,\ Y_j \in L(H)$ $(1 \le j \le n)$, and let $S = \sum_{j=1}^{n} X_j^* Y_j$. Then*

$$\sigma(S) \le \|X_R\|\, \sigma(Y_R).$$

If $A = (a_{j,k})$ and $T = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j,k} X_j^* Y_k$, as in Theorem 7, then

$$\sigma(T) \le \|A\|.\|X_R\|\, \sigma(Y_R).$$

Proof. With the above notation, Theorem 6 states that $|S|_R^2 \le \|X_R\|^2 Y_R^2$, so that $|S|_R \le \|X_R\| Y_R$. The statement follows, by the assumptions about σ. Similarly for T, with $\|A\|$ inserted. □

The norm σ is *2-convex* if for any $X_1, X_2, \ldots, X_n$, we have also

$$\sigma(X_R)^2 \le \sum_{j=1}^{n} \sigma(X_j)^2$$

(a similar statement then holds for X_L). Operator norm is trivially 2-convex, and σ_p is 2-convex for $p \ge 2$ [3].

So if σ is 2-convex, Proposition 11 implies that

$$\sigma(S) \le \left(\sum_{j=1}^{n} \|X_j\|^2\right)^{1/2} \left(\sum_{j=1}^{n} \sigma(Y_j)^2\right)^{1/2},$$

and similarly for T. This statement for σ_p (for $p \ge 2$) is essentially the main result (8) of [9] and the case (p, ∞) of [10]. In the same way, Corollary 7.1 implies the case (∞, ∞) of [10].

REFERENCES

1. A.L. Brown and A. Page, *Elements of Functional Analysis*, Van Nostrand Reinhold, 1970.
2. G.H. Hardy, J. Littlewood and G. Polya, *Inequalities*, 2nd ed., Cambridge Univ. Press, 1967.
3. G.J.O. Jameson, 2-convexity and 2-concavity in Schatten ideals, *Math. Proc. Cambridge Phil. Soc.* **120** (1996), 697–701.
4. G.J.O. Jameson, Khinchin's inequality for operators, *Glasgow Math. J.* **38** (1996), 327–336.
5. D.R. Jocić, Cauchy-Schwartz inequality for elementary operators in Schatten ideals, *J. London Math. Soc.* **60** (1999), 925–934.
6. K. Knopp, Über Reihen mit positiven Gliedern, *J. London Math. Soc.* **3** (1928), 205–211.
7. E.C. Lance, *Hilbert C*-modules*, London Math. Soc. Lecture Notes no. 210, Cambridge Univ. Press, 1995.
8. F. Lust-Piquard and G. Pisier, Non-commutative Khintchine and Paley inequalities, *Arkiv för Matematik* **29** (1991), 241–260.
9. R.M. Redheffer and P. Volkmann, Bilineare Funktionen mit Hilbertraum-Operatoren als Veränderlichen, *Math. Ann.* **262** (1983), 133–143.
10. A.M. Tonge, Bilinear mappings and trace class ideals, *Math. Ann.* **275** (1986), 281–290.
11. N.J. Young, *An Introduction to Hilbert Space*, Cambridge Univ. Press (1988).

Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4LN,

Great Britain.

g.jameson@lancaster.ac.uk