# Integrals of the form $\int_x^\infty f(t)e^{it}\,dt$

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*Basic results*

We consider integrals of the form

$$I_f(x) = \int_x^\infty f(t)e^{it}\,dt,$$

with real and imaginary parts

$$C_f(x) = \int_x^\infty f(t)\cos t\,dt, \qquad S_f(x) = \int_x^\infty f(t)\sin t\,dt.$$

where $x \geq 0$ and $f(t)$ is real, continuous and non-negative on $(0,\infty)$. Our conditions will ensure that $I_f(x)$ exists for all $x > 0$. If $I_f(0)$ exists (it will in some cases, not others) it is called the "complete" integral of this kind.

First, some elementary observations, presupposing the existence of $I_f(x)$ for $x > 0$:

(E1) By the fundamental theorem of calculus, $I_f'(x) = -f(x)e^{ix}$.

(E2) $I_f(x) = e^{ix}\int_0^\infty f(x+u)e^{iu}\,du$.

(E3) $I_f(x + 2k\pi) = I_g(x)$, where $g(t) = f(t + 2k\pi)$.

(E4) For $\lambda > 0$, we have $\int_x^\infty f(t)e^{i\lambda t}\,dt = \frac{1}{\lambda}I_g(\lambda x)$, where $g(t) = f(t/\lambda)$.

Our general assumption is that $f(t)$ is defined and non-negative for all $t > 0$ (and possibly also for $t = 0$), and that $f(t)$ is *completely monotonic*, that is:

(CM) $\lim_{t\to\infty} f(t) = 0$ and $(-1)^n f^{(n)}(t) \geq 0$ for all $n \geq 0$ and $t > 0$.

This implies automatically that $(-1)^n f^{(n)}(t)$ is decreasing and tends to 0 as $t \to \infty$. So $(-1)^n f^{(n)}(t)$ also satisfies (CM).

The basic example is $f(t) = 1/t^p$ for any $p > 0$: by a slight abuse of notation, we use the notation $I_p(x)$, $C_p(x)$, $S_p(x)$ for this case. Cases of particular interest are $p = 1$, defining the "sine" and "cosine" integrals and $p = \frac{1}{2}$, defining the "Fresnel integrals". There are well-known values for the complete integrals $S_1(0) = \pi/2$ and $I_p(0) = \Gamma(q)e^{\frac{1}{2}q\pi i}$, where $0 < p < 1$ and $q = 1 - p$. Methods for these evaluations, and other results specific to these cases, are presented in companion notes [Jam1], [Jam2].

Another example satisfying (CM) is $f(t) = e^{-at}$, where $a > 0$. In this case, we can evaluate $I_f(x)$ explicitly:

$$I_f(x) = \int_x^\infty e^{(i-a)t}\,dt = \frac{e^{(i-a)x}}{a-i} = \frac{a+i}{a^2+1}e^{-ax}e^{ix}.$$

If $f$ is completely monotonic, then so are $f(t+a)$ and $f(t) - f(t+a)$ for $a > 0$. Also, if $f$ and $g$ are completely monotonic, then so is $fg$. Numerous further examples of completely monotonic functions are given in [AB] and references listed there. There is, in fact, a well-developed general theory of such functions (e.g. [Wid, chap. 4]), but we do not need it for our purposes.

We denote by ($\mathrm{CM}_k$) condition (CM) restricted to the first $k$ derivatives, with $f^{(k)}$ assumed continuous.

To start the investigation of these integrals, we write

$$I_f(x,y) = \int_x^y f(t)e^{it}\,dt$$

and integrate by parts, temporarily assuming only ($\mathrm{CM}_1$):

$$\begin{aligned} I_f(x,y) &= \Big[-if(t)e^{it}\Big]_x^y + i\int_x^y f'(t)e^{it}\,dt \\ &= if(x)e^{ix} - if(y)e^{iy} + iI_{f'}(x,y). \end{aligned} \tag{1}$$

Since $f'(t) \le 0$, we have $|f'(t)e^{it}| \le -f'(t)$, hence

$$|I_{f'}(x,y)| \le \int_x^y (-f'(t))\,dt = f(x) - f(y), \tag{2}$$

which converges to $f(x)$ as $y \to \infty$. Hence the integral defining $I_{f'}(x)$ is convergent, with absolute value not greater than $f(x)$ (distinguish between $I_{f'}(x)$ and $I'_f(x)$!).

Now taking the limit as $y \to \infty$, we obtain the basic result on convergence, together with a long list of identities and inequalities, which we summarise as follows:

PROPOSITION 1. *If $f$ satisfies ($CM_1$), then the integrals defining $I_f(x)$ and $I_{f'}(x)$ are convergent for all $x > 0$, and the following statements apply:*

$$I_f(x) = if(x)e^{ix} + iI_{f'}(x), \tag{3}$$

$$C_f(x) = -f(x)\sin x - S_{f'}(x), \tag{4}$$

$$S_f(x) = f(x)\cos x + C_{f'}(x), \tag{5}$$

$$|I_{f'}(x)| \le f(x), \tag{6}$$

$$|I_f(x)| \le 2f(x), \tag{7}$$

$$-f(x)(1+\sin x) \le C_f(x) \le f(x)(1-\sin x), \tag{8}$$

$$f(x)(\cos x - 1) \le S_f(x) \le f(x)(\cos x + 1). \tag{9}$$

*Note.* Using (1) and (2), one can easily derive corresponding statements for $I_f(x,y)$, $C_f(x,y)$ and $S_f(x,y)$, without any assumptions about $f(t)$ outside the interval $[x,y]$. For example, $|I_f(x,y)| \le f(x)+f(y)+f(x)-f(y) = 2f(x)$. By writing out the real and imaginary parts in (1), one can show that (8) and (9) still apply for all $y > x$. If $y = x + 2k\pi$, then $e^{iy} = e^{ix}$, and we see that $|I_f(x,y)| \le 2[f(x) - f(y)]$.

Our standing assumption from now on will be that $f$ satisfies condition (CM), though it will be clear that particular results only require ($\mathrm{CM}_k$) for suitable $k$. Then (7) applies to $-f'$ to give $|I_{f'}(x)| \le -2f'(x)$. Typically, (and in particular for $f(x) = x^{-p}$), $|f'(x)|$ will be of a smaller order of magnitude than $f(x)$ for large $x$, so this will be a better estimation of $|I_{f'}(x)|$ than (6). It then makes sense to restate (3) in the form $I_f(x) = if(x)e^{ix} + r_f(x)$, where $|r_f(x)| \le -2f'(x)$ (and similarly for (4) and (5)).

So under the condition $f'(x)/f(x) \to 0$ as $x \to \infty$, we have $|I_f(x)| \sim f(x)$ as $x \to \infty$. This suggests that the natural comparison in (7) is really with $f(x)$ rather than $2f(x)$. It certainly shows that if $C$ is the best constant in the inequality $|I_f(x)| \le Cf(x)$, then $C \ge 1$. We will see later that the factor 2 can indeed be removed in (7).

At the same time, (6) will typically give a better estimation of $|I_{f'}(x)|$ than (7) for $x$ close to 0.

Repetition of the process that gave (3) leads at once to the following identities.

PROPOSITION 2. *If $f$ is completely monotonic, then for all $x > 0$,*

$$I_f(x) = if(x)e^{ix} - f'(x)e^{ix} - I_{f''}(x), \tag{10}$$

$$C_f(x) = -f(x)\sin x - f'(x)\cos x - C_{f''}(x), \tag{11}$$

$$S_f(x) = f(x)\cos x - f'(x)\sin x - S_{f''}(x). \tag{12}$$

*Further,*

$$I_f(x) = i\,[f(x) - f''(x)]\,e^{ix} - \left[f'(x) - f^{(3)}(x)\right] e^{ix} + I_{f^{(4)}}(x), \tag{13}$$

$$C_f(x) = -\,[f(x) - f''(x)]\sin x - \left[f'(x) - f^{(3)}(x)\right]\cos x + C_{f^{(4)}}(x), \tag{14}$$

$$S_f(x) = [f(x) - f''(x)]\cos x - \left[f'(x) - f^{(3)}(x)\right]\sin x + S_{f^{(4)}}(x). \tag{15}$$

*Proof.* Applying (3) to $-f'(t)$ and substituting back into (3), we obtain (10). Now apply (10) to $f''(t)$ and substitute, obtaining:

$$I_f(x) = if(x)e^{ix} - f'(x)e^{ix} - if''(x)e^{ix} + f^{(3)}(x)e^{ix} + I_{f^{(4)}}(x).$$

which equates to (13). The other statements are derived by taking real and imaginary parts. □

The reader is urged not to be daunted by this proliferation of formulae! It should be clear that they simply record the consequences of repeating a very simple integration by parts, with versions for each of the three integrals.

Bounds for the remainder terms convert these identities into inequalities. As before, we have alternative bounds. For example, $|I_{f^{(4)}}(x)|$ is bounded both by $-f^{(3)}(x)$ and by $2f^{(4)}(x)$.

As an illustration, we restate (13) explicitly for the case $f(t) = 1/t$:

$$I_1(x) = \left(\frac{1}{x} - \frac{2}{x^3}\right) ie^{ix} + \left(\frac{1}{x^2} - \frac{6}{x^4}\right) e^{ix} + 24I_5(x),$$

in which $24I_5(x)$ is bounded both by $6/x^4$ and $48/x^5$. It is quite easy to deduce that $I_1(x) < 1/x$ for all $x > 4$ (see [Jam1]). For $S_1(x)$, the method is developed further in [JLM] to derive the stronger inequality $|S_1(x)| \le \frac{\pi}{2} - \tan^{-1} x$ (note that this is exact at 0).

Of course, the process can be continued: successive derivatives of $f(x)$ appear in the expressions multiplying $e^{ix}$ and $ie^{ix}$. The outcome is an asymptotic expansion for $I_f(x)$. However, this does not simply deliver ever-closer approximations, because typically for a fixed $x$, the derivatives $f^{(n)}(x)$ will ultimately grow large in magnitude.

*The functions $S_f(x)$ and $C_f(x)$*

(The reader could omit, or defer, this subsection). We now present a number of identities and inequalities for $S_f(x)$ and $C_f(x)$, particularly relating to the points $n\pi$ and $(n - \frac{1}{2})\pi$ (which we denote by $u_n$ to shorten the formulae).

Typically, inequalities relating to $S_f(n\pi)$ reverse according to whether $n$ is even or odd: this is handled most efficiently by stating them in terms of $(-1)^n S_f(n\pi)$.

Since $S_f'(x) = -f(x)\sin x$ and $\sin t \ge 0$ on intervals $[2n\pi, (2n+1)\pi]$, $S_f(x)$ is decreasing on such intervals, and similarly it is increasing on intervals $[(2n-1)\pi, 2n\pi]$. Hence it has maxima at the points $2n\pi$ and minima at $(2n+1)\pi$. Similarly, $C_f(x)$ has maxima at $u_{2n}$ and minima at $u_{2n+1}$.

In particular, $S_f(x)$ is decreasing on $(0, \pi]$, so it either tends to infinity or to a finite limit (which is then $S_f(0)$) as $x \to 0^+$. Similarly for $C_f(x)$.

Assuming no more than that $f(t)$ is decreasing, we can prove:

PROPOSITION 3. *Suppose that $f(t)$ is non-negative and decreasing, and that $S_f(x)$ exists for all $x > 0$. Write $S_f(n\pi) = S_n$. Then for $n \geq 0$,*

$$(-1)^n(S_n - S_{n+2}) \geq 0 \tag{16}$$

$$(-1)^n S_n \geq 0. \tag{17}$$

*For all $x \geq 2n\pi$, we have $S_{2n+1} \leq S_f(x) \leq S_{2n}$. The least value of $S_f(x)$ is $S_1$. If $S_0$ is finite, it is the greatest value.*

*Proof.* By substituting $t = u + \pi$ on $[(n+1)\pi, (n+2)\pi]$ and recombining, we see that

$$S_n - S_{n+2} = \int_{n\pi}^{(n+2)\pi} f(t) \sin t \, dt = \int_{n\pi}^{(n+1)\pi} [f(t) - f(t+\pi)] \sin t \, dt.$$

Since $f(t) \geq f(t+\pi)$ and $(-1)^n \sin t \geq 0$ on $[n\pi, (n+1)\pi]$, this implies (16). Hence $(-1)^n(S_n - S_{n+2k}) \geq 0$: taking the limit as $k \to \infty$, we have $(-1)^n S_n \geq 0$. Now take $x$ between $(2n+2k)\pi$ and $(2n+2k+2)\pi$. By the monotonicity properties, we have $S_f(x) \leq S_{2n+2k} \leq S_{2n}$ and $S_f(x) \geq S_{2n+2k+1} \geq S_{2n+1}$. The statements on greatest and least values follow. □

Similar statements apply to $C_f(x)$, with $n\pi$ replaced by $u_n$, except that if $C_f(0)$ is finite, it can be either greater or less than $C_f(u_2)$, the other candidate for the greatest value. (See [Jam2]).

We remark that (17) also follows from (9), applied to $n\pi$ for even and odd $n$.

Our earlier identities simplify pleasantly at the points $n\pi$ and $u_n$, because $\sin n\pi = \cos u_n = 0$. Also, $\cos n\pi = (-1)^n$ and $\sin u_n = (-1)^{n+1}$. In particular, (12) and (11) become:

$$S_f(n\pi) = (-1)^n f(n\pi) - S_{f''}(n\pi), \tag{18}$$

$$C_f(u_n) = (-1)^n f(u_n) - C_{f''}(u_n). \tag{19}$$

PROPOSITION 4. *If $f$ is completely monotonic, then for $n \geq 1$,*

$$0 \leq (-1)^n S_f(n\pi) \leq f(n\pi), \tag{20}$$

$$0 \leq (-1)^n C_f(u_n) \leq f(u_n). \tag{21}$$

*Proof.* By (17), applied to $f''$, we have $(-1)^n S_{f''}(n\pi) \geq 0$. With (18) rewritten in the form

$$(-1)^n S_f(n\pi) = f(n\pi) - (-1)^n S_{f''}(n\pi),$$

it follows that $(-1)^n S_f(n\pi) \leq f(n\pi)$. The proof of (21) is similar. □

Hence, for example, $-f[(2n+1)\pi \leq S_f(x) \leq f(2n\pi)$ for $x$ in $[2n\pi, (2n+2)\pi]$.

We can derive companion bounds for $S_f(u_n)$ and $C_f(n\pi)$:

PROPOSITION 5. *If $f$ is completely monotonic, then*

$$0 \leq (-1)^{n+1} S_f(u_n) \leq -f'(u_n), \tag{22}$$

$$0 \leq (-1)^n C_f(n\pi) \leq -f'(n\pi). \tag{23}$$

*Proof.* By (4), we have $C_f(n\pi) = -S_{f'}(n\pi)$. By (20), applied to $-f'$, we have $0 \leq (-1)^{n+1} S_{f'}(n\pi) \leq -f'(n\pi)$, hence (23). The proof of (22) is similar. □

Of course, we can take these results to a further stage. For $x = n\pi$, (15) becomes

$$S_f(n\pi) = (-1)^n \left[f(n\pi) - f''(n\pi)\right] + S_{f^{(4)}}(n\pi),$$

and one can deduce

$$f(n\pi) - f''(n\pi) \leq (-1)^n S_f(n\pi) \leq f(n\pi) - f''(n\pi) + f^{(4)}(n\pi).$$

*The auxiliary functions; the inequality $|I_f(x)| \leq f(x)$*

Formulae like (3) and (10) simplify pleasantly when expressed in terms of $e^{-ix} I_f(x)$ and its components, because real and imaginary parts are separated. Write $e^{-ix} I_f(x) = K_f(x)$. Note that by (E2),

$$K_f(x) = \int_0^\infty f(u+x) e^{iu}\, du. \tag{24}$$

We write $K_f(x) = V_f(x) + iU_f(x)$, so that

$$U_f(x) = S_f(x)\cos x - C_f(x)\sin x, \tag{25}$$

$$V_f(x) = C_f(x)\cos x + S_f(x)\sin x. \tag{26}$$

$U_f$ and $V_f$ are called the "auxiliary functions". (We write $U_f$, $V_f$ this way round in a gesture to the customary notation; cf. [DLMF, chapter 6] for the case $f(t) = 1/t$.)

Note that $U_f(n\pi) = (-1)^n S_f(n\pi)$ and $V_f(n\pi) = (-1)^n C_f(n\pi)$, also $U_f(u_n) = (-1)^n C_f(u_n)$ and $V_f(u_n) = (-1)^{n+1} S_f(u_n)$. We will see that our results for $S_f$ and $C_f$

at the points $n\pi$ and $u_n$ can be recaptured as special cases of similar statements for $U_f(x)$ and $V_f(x)$, now applying for all $x$.

Stated for $K_f(x)$ and its components, identity (3) becomes

$$K_f(x) = if(x) + iK_{f'}(x), \tag{27}$$

$$U_f(x) = f(x) + V_{f'}(x), \tag{28}$$

$$V_f(x) = -U_{f'}(x). \tag{29}$$

Similarly, (10) becomes

$$K_f(x) = if(x) - f'(x) - K_{f''}(x), \tag{30}$$

$$U_f(x) = f(x) - U_{f''}(x), \tag{31}$$

$$V_f(x) = -f'(x) - V_{f''}(x). \tag{32}$$

The cases $x = n\pi$ and $x = u_n$ in (31) reproduce (18) and (19).

Meanwhile, differentiation gives

$$K_f'(x) = -ie^{ix}I_f(x) - e^{-ix}e^{ix}f(x) = -iK_f(x) - f(x), \tag{33}$$

hence

$$U_f'(x) = -V_f(x), \tag{34}$$

$$V_f'(x) = U_f(x) - f(x). \tag{35}$$

Note that by (27) and (33), we have $K_f'(x) = K_{f'}(x)$: this also follows formally from (24) by differentiation under the integral sign. Also, from (34) and (35), we see that $U_f$ and $V_f$ satisfy the differential equations $U_f'' + U_f = f$ and $V_f'' + V_f = -f'$ (though we will not use this fact). In fact, they are essentially the solutions of these equations delivered by the technique of "variation of parameters".

The basic inequalities for $U_f(x)$ and $V_f(x)$ are summarised in the next result.

THEOREM 1. *If $f$ is completely monotonic, then*

$$0 \le U_f(x) \le f(x), \tag{36}$$

$$0 \le V_f(x) \le -f'(x). \tag{37}$$

*Further, $U_f(x)$, $V_f(x)$ and $f(x) - U_f(x)$ are decreasing.*

*Proof.* By (6), we have $|V_{f'}(x)| \le |I_{f'}(x)| \le f(x)$, so by (28), $U_f(x) \ge 0$. Applied to $-f'$, this gives $U_{f'}(x) \le 0$, so by (29), we have $V_f(x) \ge 0$. Now applying these statements to $f''$, we see that $U_{f''}(x)$ and $V_{f''}(x)$ are non-negative, so by (31) and (32), $U_f(x) \leq f(x)$ and $V_f(x) \leq -f'(x)$.

By (34), we now have $U_f'(x) = -V_f(x) \leq 0$ and $f'(x) - U_f'(x) = f'(x) + V_f(x) \leq 0$. By (35), we have $V_f'(x) = U_f(x) - f(x) \leq 0$. □

*Note.* As mentioned previously, weaker conditions are actually sufficient. We require ($CM_1$) for the inequality $U_f(x) \geq 0$, hence ($CM_2$) for $U_{f'}(x) \leq 0$ and $V_f(x) \geq 0$, and consequently ($CM_4$) for $V_{f''}(x) \geq 0$ and $V_f(x) \leq -f(x)$.

Note that (20) and (21) are the cases $x = n\pi$ and $x = u_n$ in (36), while (22) and (23) are similarly cases of (37). We have gained other information, for example $(-1)^n S_f(n\pi)$ and $(-1)^n C_f(n\pi)$ decrease with $n$.

We can proceed to further terms, as before. Identity (13) becomes

$$K_f(x) = i\,[f(x) - f''(x)] - \left[f'(x) - f^{(3)}(x)\right] + K_{f^{(4)}}(x),$$

$$U_f(x) = f(x) - f''(x) + U_{f^{(4)}}(x),$$
$$V_f(x) = -f'(x) - f^{(3)}(x) + V_{f^{(4)}}(x),$$

with attendant inequalities derived from $0 \leq U_{f^{(4)}}(x) \leq f^{(4)}(x)$ and $0 \leq V_{f^{(4)}}(x) \leq -f^{(5)}(x)$.

We can now show that the factor 2 in (7) can be removed, as stated earlier.

THEOREM 2. *If $f$ is completely monotonic, then $|I_f(x)|$ is decreasing and*

$$|I_f(x)| \leq f(x) \tag{38}$$

*for all $x > 0$.*

*Proof.* Let $M(x) = |I_f(x)|^2 = C_f(x)^2 + S_f(x)^2$. Then

$$\begin{aligned} M'(x) &= -2C_f(x)f(x)\cos x - 2S_f(x)f(x)\sin x \\ &= -2f(x)[C_f(x)\cos x + S_f(x)\sin x] \\ &= -2f(x)V_f(x). \end{aligned}$$

(This follows equally from (34) and (35).) By (37), we have $0 \leq V_f(x) \leq -f'(x)$. Hence $M'(x) \leq 0$, so $M(x)$ is decreasing. Also, $M'(x) \geq 2f(x)f'(x)$, so $f(x)^2 - M(x)$ is decreasing. Since it tends to 0 as $x \to \infty$, it follows that it is non-negative, so $M(x) \leq f(x)^2$, hence $|I_f(x)| \leq f(x)$, for $x > 0$. □

Hence $|C_f(x)|$ and $|S_f(x)|$ are also bounded by $f(x)$. The upper bounds in (36) and (37) are now seen to be special cases of (38) , but of course (37) was used in its proof.

We mention two corollaries.

COROLLARY 2.1. *If* $y = x + 2k\pi$*, then* $|I_f(x,y)| \le f(x) - f(y)$.

*Proof.* By (E3), $I_f(y) = I_g(x)$, where $g(t) = f(t+2k\pi)$. So $I_f(x,y) = I_f(x) - I_f(y) = I_h(x)$, where $h(t) = f(t) - f(t+2k\pi)$. The statement follows by applying Theorem 2 to $h$. □

COROLLARY 2.2. *If* $\lambda > 0$*, then*

$$\left| \int_x^\infty f(t) e^{i\lambda t}\, dt \right| \le \frac{f(x)}{\lambda}.$$

*Proof.* This is an immediate consequence of Theorem 2 and (E4). □

Theorem 2 has appeared in [Jam3]. Although these integrals are a highly classical topic, not much attention seems to have been given to inequalities of this type. I am not aware of any earlier mention of (38) in the literature, although for the case $f(t) = 1/t^p$, it is an easy consequence of a known expression for $I_p(x)$: we return to this below.

As noted earlier, the condition really required for (37), and hence (38), is $(\mathrm{CM}_4)$,

Without any differentiability conditions, a trivial example shows that factor 2 in (7) can be attained: take $f(t)$ to be 1 for $0 \le t \le \pi$ and 0 for $t > \pi$: then $I_f(0) = 2i$. We now give an example satisfying $(\mathrm{CM}_2)$ in which $|I_f(x)|$, and indeed $S_f(x)$, is greater than $f(x)$, and (37) fails.

*Example.* Fix $b > 2\pi$, and let

$$f(t) = \begin{cases} (b-t)^2 & \text{for } 0 < t \le b, \\ 0 & \text{for } t > b. \end{cases}$$

Let $a = b - 2\pi$. Then $f''(t) = 2$ for $a \le t < b$, so $I_{f''}(a) = \int_a^b 2e^{it}\, dt = 0$, so (10) applies to give

$$I_f(a) = [if(a) - f'(a)]e^{ia} = (4\pi^2 i + 4\pi)e^{ia},$$

hence $|I_f(a)| > 4\pi^2 = f(a)$. Also, $S_f(a) = 4\pi^2 \cos a + 4\pi \sin a$, which is greater than $4\pi^2$ at least when $0 < a < \frac{\pi}{6}$. Further, one checks easily that $V_f(x) = 2(b-x) - 2\sin(b-x)$, which is greater than $-f'(x) = 2(b-x)$ when $a < x < a + \pi$. (Strictly, this example does not satisfy $(\mathrm{CM}_2)$ because $f''$ has a discontinuity, but this can be rectified by a small perturbation, or by a natural relaxation of the definition of $(\mathrm{CM}_2)$ to restrict to left derivatives at a point $b$ such that $f(t) = 0$ for all $t \ge b$.)

*An alternative method for Theorems 1 and 2*

We sketch a superficially very quick alternative proof of these theorems using a theorem of Bernstein [Wid, p. 160], which states that all completely monotonic functions can be expressed as

$$f(t) = \int_0^\infty e^{-ut}\, d\mu(u)$$

for some non-negative measure $\mu$. As an illustration, the expression for $1/t^p$' derived directly from the definition of the gamma function, is

$$\frac{1}{t^p} = \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-ut}\, du.$$

*Proof of Theorems 1 and 2.* Assuming validity of the reversal of integration, we have

$$\begin{aligned} I_f(x) &= \int_x^\infty e^{it} \int_0^\infty e^{-ut}\, d\mu(u)\, dt \\ &= \int_0^\infty \int_x^\infty e^{-(u-i)t}\, dt\, d\mu(u) \\ &= \int_0^\infty \frac{e^{-(u-i)x}}{u-i}\, d\mu(u). \end{aligned}$$

Since $|u - i| \geq 1$, we deduce

$$|I_f(x)| \leq \int_0^\infty e^{-ux}\, d\mu(u) = f(x).$$

Also, we can derive the following expressions for the auxiliary functions:

$$U_f(x) = \int_0^\infty \frac{e^{-ux}}{u^2+1}\, d\mu(u),$$

$$V_f(x) = \int_0^\infty \frac{ue^{-ux}}{u^2+1}\, d\mu(u),$$

from which it is clear that $U_f(x)$, $V_f(x)$ and $f(x) - U_f(x)$ are decreasing. □

For the case $f(t) = 1/t^p$, this is indeed an appealing method, though the reversal of integration requires justification, at least by first considering the integral on a bounded interval $[x, y]$ for $t$. The implied expression for $I_p(x)$ is surely well known.

However, for general completely monotonic functions, this method depends on heavy machinery in the form of Bernstein's theorem. Also, it does not apply to functions satisfying only ($CM_4$).

*References*


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