INTEGRATING THE ERROR IN THE DIVISOR PROBLEM

Notes by Tim Jameson

Let B_k be the k^{th} Bernoulli polynomial. In particular $B_1(x) = x - \frac{1}{2}$ and $B_2(x) = x^2 - x + \frac{1}{6}$. Write $\lfloor x \rfloor$ for the integer part of x, $\{x\}$ for the fractional part, and $\tilde{B}_k(x) = B_k(\{x\})$.

We will require two terms in the Euler-Maclaurin expansion of

$$S = \sum_{n \le x} \frac{1}{n},$$

obtained as follows. For all $x > 0$ we have

$$\begin{aligned} S &= \frac{\lfloor x \rfloor}{x} + \int_1^x \frac{\lfloor t \rfloor}{t^2} dt \\ &= \log x + 1 - \frac{\{x\}}{x} - \int_1^x \frac{\{t\}}{t^2} dt \\ &= \log x + \gamma - \frac{\{x\}}{x} + \int_x^\infty \frac{\{t\}}{t^2} dt \\ &= \log x + \gamma - \frac{\tilde{B}_1(x)}{x} + \int_x^\infty \frac{\tilde{B}_1(t)}{t^2} dt, \end{aligned}$$

where

$$\gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2} dt$$

is Euler's constant. Now for $k \ge 2$ we have

$$\begin{aligned} \int_{\lfloor x \rfloor + 1}^\infty \tilde{B}_{k-1}(t) f^{(k-1)}(t) dt &= \sum_{n=\lfloor x \rfloor + 1}^\infty \int_0^1 B_{k-1}(u) f^{(k-1)}(n+u) du \\ &= \sum_{n=\lfloor x \rfloor + 1}^\infty \left(\left[\frac{B_k(u)}{k} f^{(k-1)}(n+u) \right]_0^1 - \int_0^1 \frac{B_k(u)}{k} f^{(k)}(n+u) du \right) \\ &= -\frac{B_k(0)}{k} f^{(k-1)}(\lfloor x \rfloor + 1) - \int_{\lfloor x \rfloor + 1}^\infty \frac{\tilde{B}_k(t)}{k} f^{(k)}(t) dt \end{aligned}$$

(using $B_k(0) = B_k(1)$) and

$$\begin{aligned} \int_x^{\lfloor x \rfloor + 1} \tilde{B}_{k-1}(t) f^{(k-1)}(t) dt &= \int_{\{x\}}^1 B_{k-1}(u) f^{(k-1)}(\lfloor x \rfloor + u) du \\ &= \left[\frac{B_k(u)}{k} f^{(k-1)}(\lfloor x \rfloor + u) \right]_{\{x\}}^1 - \int_{\{x\}}^1 \frac{B_k(u)}{k} f^{(k)}(\lfloor x \rfloor + u) du \\ &= \left[\frac{\tilde{B}_k(t)}{k} f^{(k-1)}(t) \right]_x^{\lfloor x \rfloor + 1} - \int_x^{\lfloor x \rfloor + 1} \frac{\tilde{B}_k(t)}{k} f^{(k)}(t) dt. \end{aligned}$$

Adding these two results together gives

$$\int_x^\infty \tilde{B}_{k-1}(t) f^{(k-1)}(t)dt = -\frac{\tilde{B}_k(x)}{k} f^{(k-1)}(x) - \int_x^\infty \frac{\tilde{B}_k(t)}{k} f^{(k)}(t)dt.$$

The case $f(t) = -1/t$, $k = 2$ says

$$\begin{aligned}\int_x^\infty \frac{\tilde{B}_1(t)}{t^2} dt &= -\frac{\tilde{B}_2(x)}{2x^2} + \int_x^\infty \frac{\tilde{B}_2(t)}{t^3} dt \\ &= -\frac{\tilde{B}_2(x)}{2x^2} + O\left(\frac{1}{x^3}\right),\end{aligned}$$

where the error term may be obtained by a geometrical argument, or by integrating by parts again ($k = 3$ in the above) and using the trivial estimate

$$\int_x^\infty \frac{\tilde{B}_3(t)}{t^4} dt \ll \frac{1}{x^3}.$$

We have now proved

$$S = \log x + \gamma - \frac{\tilde{B}_1(x)}{x} - \frac{\tilde{B}_2(x)}{2x^2} + O\left(\frac{1}{x^3}\right),$$

valid for all $x > 0$.

We are now in a position to derive a suitable formula for

$$D(x) = \sum_{n \le x} \tau(n) = \sum_{mn \le x} 1.$$

Thus

$$\begin{aligned} D(x) &= 2 \sum_{n \le x^{1/2}} \left\lfloor \frac{x}{n} \right\rfloor - \lfloor x^{1/2} \rfloor^2 \\ &= 2x \sum_{n \le x^{1/2}} \frac{1}{n} - 2 \sum_{n \le x^{1/2}} \tilde{B}_1\left(\frac{x}{n}\right) - \lfloor x^{1/2} \rfloor - \lfloor x^{1/2} \rfloor^2 \\ &= 2x \left(\tfrac{1}{2} \log x + \gamma - \frac{\tilde{B}_1(x^{1/2})}{x^{1/2}} - \frac{\tilde{B}_2(x^{1/2})}{2x} + O\left(\frac{1}{x^{3/2}}\right)\right) + \\ &\quad -2 \sum_{n \le x^{1/2}} \tilde{B}_1\left(\frac{x}{n}\right) - x^{1/2} + \{x^{1/2}\} - (x - 2x^{1/2}\{x^{1/2}\} + \{x^{1/2}\}^2) \\ &= x \log x + (2\gamma - 1)x + \Delta(x), \end{aligned}$$

where

$$\begin{aligned} \Delta(x) &= -2x^{1/2}\tilde{B}_1(x^{1/2}) - \tilde{B}_2(x^{1/2}) + O(x^{-1/2}) + \\ &\quad -2 \sum_{n \le x^{1/2}} \tilde{B}_1\left(\frac{x}{n}\right) - x^{1/2} + \{x^{1/2}\} + 2x^{1/2}\{x^{1/2}\} - \{x^{1/2}\}^2 \\ &= -2 \sum_{n \le x^{1/2}} \tilde{B}_1\left(\frac{x}{n}\right) - \tilde{B}_2(x^{1/2}) + \{x^{1/2}\} - \{x^{1/2}\}^2 + O(x^{-1/2}) \\ &= -2 \sum_{n \le x^{1/2}} \tilde{B}_1\left(\frac{x}{n}\right) - 2\tilde{B}_2(x^{1/2}) + \frac{1}{6} + O(x^{-1/2}). \end{aligned}$$

Now,

$$\begin{aligned}
\int_0^X \tilde{B}_2(x^{1/2})dx &= 2\int_0^{X^{1/2}} t\tilde{B}_2(t)dt \\
&= 2\sum_{n\le X^{1/2}} \int_0^1 (n+u)B_2(u)du + O(X^{1/2}) \\
&\ll X^{1/2},
\end{aligned}$$

where the "n" in the penultimate line gives nothing because $\int_0^1 B_2(u)du = 0$. We also have

$$\begin{aligned}
\int_0^X 2\sum_{n\le x^{1/2}} \tilde{B}_1\left(\frac{x}{n}\right)dx &= \sum_{n\le X^{1/2}} \int_{n^2}^X \tilde{2}B_1\left(\frac{x}{n}\right)dx \\
&= \sum_{n\le X^{1/2}} n\int_n^{X/n} \tilde{2}B_1(t)dt \\
&= \sum_{n\le X^{1/2}} n\int_0^{\{X/n\}} 2B_1(u)du \\
&= \sum_{n\le X^{1/2}} n\left(\tilde{B}_2\left(\frac{X}{n}\right) - \frac{1}{6}\right) \\
&= -\frac{X}{12} + \sum_{n\le X^{1/2}} n\tilde{B}_2\left(\frac{X}{n}\right) + O(X^{1/2}).
\end{aligned}$$

Combining our results, we see that

$$\int_0^X \Delta(x)dx = \frac{X}{4} - \sum_{n\le X^{1/2}} n\tilde{B}_2\left(\frac{X}{n}\right) + O(X^{1/2}).$$

The Fourier series for $\tilde{B}_k$ (with $\tilde{B}_1(n)$ modified to mean 0 in the conditionally convergent case of $k=1$) is

$$\tilde{B}_k(x) = -k!\sum_{\substack{h=-\infty \\ h\ne 0}}^{\infty} \frac{e(hx)}{(2\pi ih)^k}.$$

For $1 \ll U < u \le 2U$ and $h>0$, the simplest van der Corput estimate gives

$$\sum_{U<n\le u} e\left(\frac{hX}{n}\right) \ll \left(\frac{hX}{U}\right)^{1/2} + \left(\frac{U^3}{hX}\right)^{1/2},$$

and by summation by parts

$$\sum_{U<n\le 2U} ne\left(\frac{hX}{n}\right) \ll (hXU)^{1/2} + \left(\frac{U^5}{hX}\right)^{1/2}.$$

Hence

$$\sum_{U<n\le 2U} n\tilde{B}_2\left(\frac{X}{n}\right) \ll \sum_{h=1}^{\infty}\frac{1}{h^2}\left((hXU)^{1/2}+\left(\frac{U^5}{hX}\right)^{1/2}\right)$$
$$\ll (XU)^{1/2}+\left(\frac{U^5}{X}\right)^{1/2}$$
$$\ll (XU)^{1/2} \quad (U \ll X^{1/2}).$$

Summing over $\ll \log X$ values of $U \ll X^{1/2}$ finally gives

$$\int_0^X \Delta(x)dx = \frac{X}{4}+O(X^{3/4}).$$

Ivić gets exactly this result using the integrated Voronoi formula. Sometimes $\Delta(x)$ is defined by $D(x)=x\log x+(2\gamma-1)x+\frac{1}{4}+\Delta(x)$ (the term $\frac{1}{4}$ is the residue of $\zeta(s)^2x^s/s$ at $s=0$) which removes the term $X/4$ from our final result.

It is left to the reader to obtain similar results for the circle problem: Let

$$r(n)=\sum_{\substack{a^2+b^2=n\\ a\ge 1,\ b\ge 0}} 1,$$

$$R(x)=\sum_{1\le n\le x} r(n)=\frac{\pi x}{4}+P(x).$$

Proceeding from the elementary formula

$$R(x)=\lfloor\sqrt{x}\rfloor+2\sum_{1\le n\le\sqrt{(x/2)}}(\lfloor\sqrt{x-n^2}\rfloor-n)+\left\lfloor\sqrt{\frac{x}{2}}\right\rfloor,$$

I get

$$P(x)=-\tilde{B}_1(\sqrt{x})-2\sum_{1\le n\le\sqrt{(x/2)}}\tilde{B}_1(\sqrt{x-n^2})-2\tilde{B}_2\left(\sqrt{\frac{x}{2}}\right)-\frac{1}{3}+O\left(\frac{1}{\sqrt{x}}\right).$$

More generally one can get formulae rather like this for the number of integer points inside any suitably smooth convex closed curve with radius of curvature bounded above and below.

Or, proceeding from

$$r(n)=\sum_{\substack{d|n\\ d\equiv 1 \bmod 2}}(-1)^{(d-1)/2}$$

I get (note: Graham and Kolesnik's version of this is incorrect)

$$P(x)=\sum_{n\le(\sqrt{x})/2}\left(-\tilde{B}_1\left(\frac{x}{4n+1}\right)+\tilde{B}_1\left(\frac{x}{4n-1}\right)-\tilde{B}_1\left(\frac{x-n}{4n}\right)+\tilde{B}_1\left(\frac{x+n}{4n}\right)\right)+O(1)$$

but I'm not sure I've ever derived a more accurate version of this. (Done with error $\ll x^{-1/2}$ on 29-3-08. See circprob.ide)

I have checked all my formulae using a computer but don't think I ever performed the integration using them. I think you'll get

$$\int_0^X P(x)dx = -\frac{X}{4} + O(X^{3/4}) \quad \text{(? –yes 29-3-08).}$$

(Note that $1 + 4R(x)$ is the number of integer points inside the full circle, including the origin.) It might be in Ivić, using the integrated Hardy formula, which corresponds to the integrated Voronoi formula.

(Done years ago at least twice. Done yet again and typed 29-9-07.)