Hilbert's inequality and related results

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1. Introduction

A more precise title would be "Hilbert's discrete inequality for $p = 2$ and some related results". The "related results" are norm evaluations for various other matrices of loosely Hilbert type, and some applications. A very brief survey of some generalizations, such as $p \neq 2$ and the continuous case, is attempted in section 12.

Our basic theme is the result known variously as "Hilbert's inequality" or "Hilbert's double series theorem". According to [HLP], Hilbert simply included the result in his lectures, and it was actually published by Weyl in his inaugural dissertation in 1908. There are actually three Hilbert inequalities, not just one. They can be stated in various equivalent ways. For the moment, we will just give the formulation in terms of bilinear forms, which is the one most frequently quoted. We need some notation.

For vectors $x = (x_1, x_2, \ldots, x_n)$, with x_j real or complex, we define the (Euclidean) *norm* by $\|x\| = (\sum_{j=1}^n |x_j|^2)^{1/2}$. Vectors are, of course, just finite sequences. We extend this to infinite sequences by defining ℓ_2 to be the set of all (real or complex) sequences $x = (x_j)$ with $\sum_{j=1}^\infty |x_j|^2$ convergent, with the norm defined in the same way.

Given a matrix $A = (a_{j,k})$ and sequences x, y (finite of infinite), the *bilinear form* $A(x, y)$ is defined by

$$A(x,y) = \sum_j \sum_k a_{j,k} x_j y_k. \tag{1}$$

Of course, in the infinite case, this only makes sense if we know that the double series converges.

To facilitate the statement of Hilbert's inequalities, we also introduce the following notation. It is not standard, but it will be used consistently in these notes. For $r \in \mathbb{Z}$, write

$$c_r = \begin{cases} 1/r & \text{if } r \neq 0 \\ 0 & \text{if } r = 0. \end{cases}$$

We define the *Hilbert matrices* as follows:

$$H_1 = (c_{j+k})_{j,k \geq 1}, \qquad H_{-1} = (c_{j-k})_{j,k \geq 1}, \qquad H_0 = (c_{j+k-1})_{j,k \geq 1}.$$

Further, we denote by $H_1^{(n)}$ the finite matrix $(c_{j+k})_{j,k=1}^n$, and similarly for the others.

The statement is:

Theorem 1.1. *Let H be any of H_1, H_0, H_{-1}. Given x, y in ℓ_2, the double series defining $H(x, y)$ is convergent, and*

$$|H(x,y)| \leq \pi \|x\|.\|y\|. \tag{2}$$

Further, π is the best constant in each case. Hilbert actually obtained the constant π for H_1 and H_0, but 2π for H_{-1}; this was improved to π by Schur [Sch].

A further variation is to restate the results for infinite two-sided sequences $(x_j)_{j \in \mathbb{Z}}$: in this form, it becomes clear that the statements for c_{j-k} and c_{j+k} are equivalent.

The reader may recognise already that the statement is equivalent to saying that the norms of the matrices in question are not greater than π. For readers needing it, we explain this equivalence in section 2; readers who do not need it could move on to section 3.

We will present several completely different proofs of Hilbert's inequalities, but we will widen the scope of these notes to present norm estimations for other matrices of this type, such as $[1/(j + k + \alpha)]$ and (c_{j-k}^2). The two main methods apply readily to wider classes of matrices, each having its natural area of applications. Rather more special methods apply to $[\operatorname{cosec} \frac{\pi}{n}(j - k)]$ and $[1/(\lambda_j - \lambda_k)]$. The estimation of the last type is a result of Montgomery and Vaughan [MontV], which has extensive applications in analytic number theory.

2. Matrix norms; bilinear and quadratic forms

The language of inner products and norms provides a neat and efficient way to present results like Hilbert's inequality, as well as delivering an important reformulation. Here we give a brief summary of these notions and the results assumed.

These results essentially belong to linear algebra (as presented, for example, in [Hal]), augmented by natural extensions to infinite matrices and sequences. We will not require any serious results from the theory of Hilbert or Banach spaces. Some readers will be able to pass over this section very quickly.

Let X be a linear space (alias vector space) over the real or complex field. An *inner product* on X is a mapping $(x, y) \to \langle x, y\rangle$ from $X \times X$ to the scalar field satisfying the following axioms: for all $x, y, z \in X$ and scalars λ:

(IP1) $\langle y, x\rangle = \overline{\langle x, y\rangle}$,

(IP2) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$,

(IP3) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,

(IP4) $\langle x, x\rangle \geq 0$ for all x, and is only 0 when $x = 0_X$.

A linear space equipped with an inner product is called an *inner-product space.* Note that the axioms imply that $\langle x, \lambda y\rangle = \overline{\lambda}\langle x, y\rangle$ and $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

The derived *norm* is defined by $\|x\| = \langle x, x\rangle^{1/2}$. The *Cauchy-Schwarz inequality* states that $|\langle x, y\rangle| \leq \|x\|.\|y\|$ (sketch proof: assume $\|y\| = 1$; apply $\langle x - \lambda y, x - \lambda y\rangle \geq 0$, with $\lambda = \langle x, y\rangle$). It follows that $\|x + y\| \leq \|x\| + \|y\|$. Also, the "parallelogram law" holds: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$. It is elementary that $\|x\| = \sup\{|\langle x, y\rangle| : \|y\| = 1\}$ (take $y = x/\|x\|$).

Elements x, y are *orthogonal* if $\langle x, y\rangle = 0$. Elements $z_1, z_2, \ldots, z_n$ are an *orthonormal basis* of an n-dimensional inner-product space X if they are mutually orthogonal and have norm 1. Then for $x \in X$ we have $x = \sum_{j=1}^{n}\langle x, z_j\rangle z_j$ and $\|x\|^2 = \sum_{j=1}^{n} |\langle x, z_j\rangle|^2$.

We denote by ℓ_2^n the space $\mathbb{R}^n$ or $\mathbb{C}^n$ with inner product defined by $\langle x, y\rangle = \sum_{j=1}^{n} x_j\overline{y}_j$; this generates the Euclidean norm $\|x\| = (\sum_{j=1}^{n} |x_j|^2)^{1/2}$. The unit vectors e_j $(1 \leq j \leq n)$ form the natural orthonormal basis of ℓ_2^n. The natural infinite-dimensional extension, denoted by ℓ_2, is the space of infinite sequences (real or complex) $x = (x_j)_{j\geq1}$ with $\sum_{j=1}^{\infty} |x_j|^2$ convergent: the n-dimensional Cauchy-Schwarz inequality applies to show that if x and y are in ℓ_2, then the series $\sum_{j=1}^{\infty} x_j\overline{y}_j$ is (absolutely) convergent, so can be used to define $\langle x, y\rangle$; it follows that $x + y \in \ell_2$. If we need to distinguish real and complex ℓ_2, we write $\ell_2^{\mathbb{R}}$ and $\ell_2^{\mathbb{C}}$

respectively. We can extend this further to the space $\ell_2(\mathbb{Z})$ of all infinite *two-sided* sequences $x = (x_j)_{j\in\mathbb{Z}}$ with $\sum_{j=-\infty}^{\infty} |x_j|^2$ convergent.

The continuous analogue is the inner product $\langle f, g\rangle = \int_I f\overline{g}$, defined on a suitable space of functions (for example, Riemann integrable ones) on a real interval I. We spell out specifically the Cauchy-Schwarz inequality arising from this inner product:

$$\left|\int_a^b fg\right| \le \left(\int_a^b |f|^2\right)^{1/2} \left(\int_a^b |g|^2\right)^{1/2}. \tag{3}$$

A linear operator A between inner product spaces is said to be *bounded* if there is a constant M such that $\|Ax\| \le M\|x\|$ for all x (we are using the same notation $\|.\|$ for the norm in both spaces). This happens automatically in finite dimensions. The least such constant M is the *norm* (more specifically, *operator norm*) of A, denoted by $\|A\|$. So we have $\|Ax\| \le \|A\|.\|x\|$, and in fact one verifies easily (by scalar multiples) that $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$. Also, $\|AB\| \le \|A\|.\|B\|$.

A linear operator from ℓ_2^n to ℓ_2^m is represented by an $m\times n$ matrix $(a_{j,k})$ (which we also denote by A) according to: $(Ax)(j) = \sum_{k=1}^n a_{j,k}x_k$ for each j, in which we are using the notation $(Ax)(j)$ for component j of the vector Ax. Note that $a_{j,k} = (Ae_k)(j)$. A linear operator on ℓ_2 or $\ell_2(\mathbb{Z})$ is represented by an infinite matrix in the same way; however, there is no simple condition on the matrix that ensures that A actually maps ℓ_2 into ℓ_2; we return to this point below.

Written out explicitly, for $A = (a_{j,k})$, if $y_j = \sum_k a_{j,k}x_k$ for each j, then

$$\sum_j |y_j|^2 \le \|A\|^2 \sum_k |x_k|^2. \tag{4}$$

Clearly, $\|A\| \ge \sup_{j,k} |a_{j,k}|$. Some further elementary facts about norms of matrices are summarised in the next result.

Proposition 2.1. *(i) If some rows or columns are removed, or replaced by 0, then the norm of a matrix is reduced (i.e. not increased).*

(ii) *If* $a_{j,k} \ge 0$ *and* $|b_{j,k}| \le a_{j,k}$ *for all* j, k, *then* $\|B\| \le \|A\|$.

(iii) *A real matrix has the same norm as an operator on* $\mathbb{R}^n$ *and on* $\mathbb{C}^n$ *(or on* $\ell_2^{\mathbb{R}}$ *and* $\ell_2^{\mathbb{C}}$*).*

Brief proof. (i) If row j is replaced by 0, then $(Ax)(j)$ is replaced by 0. If column k is replaced by 0, the effect is to specialise to vectors x with $x_k = 0$.

(ii) Let $y_j = \sum_k a_{j,k}|x_k|$ and $z_j = \sum_j b_{j,k}x_k$. Then $|z_j| \le |y_j|$ for each j.

(iii) Let M be the norm of A as an operator on $\mathbb{R}^n$. An element z of $\mathbb{C}^n$ can be expressed in the obvious way as $x+iy$, where $x, y \in \mathbb{R}^n$ and $\|z\|^2 = \|x\|^2 + \|y\|^2$. Then $Az = Ax + iAy$, so $\|Az\|^2 = \|Ax\|^2 + \|Ay\|^2 \le M^2(\|x\|^2 + \|y\|^2) = M^2\|z\|^2$. $\square$

Warning. By (ii), the norm of $(|a_{j,k}|)$ is not less than the norm of $(a_{j,k})$. In general, the two are *not* equal!

Bilinear forms. Recall from (1) that we defined the bilinear form $A(x, y)$ by

$$A(x, y) = \sum_j \sum_k a_{j,k} x_j y_k,$$

(where j and k are in the appropriate range). Meanwhile, we have

$$\langle Ay, x\rangle = \sum_j \sum_k a_{j,k} y_k \overline{x}_j,$$

so the relationship is $A(x, y) = \langle Ay, \overline{x}\rangle$, in which the vector $\overline{x}$ is defined in the obvious way. In the infinite case, we are, of course, assuming that A maps ℓ_2 into ℓ_2.

For bounded A, by the Cauchy-Schwarz inequality,

$$|\langle Ax, y\rangle| \le \|Ax\|.\|y\| \le \|A\|.\|x\|.\|y\|.$$

Further, since $\|Ax\| = \sup\{|\langle Ax, y\rangle| : \|y\| = 1\}$, it is clear that $\|A\|$ is the best constant M in the statement $|\langle Ax, y\rangle| \le M\|x\|.\|y\|$ for all x, y. Equivalently,

$$\|A\| = \sup\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1\}.$$

If A is real, this holds with x, y restricted to real vectors. In the complex case, multiplication by a suitable complex scalar shows that the stated supremum is the same as $\sup\{\text{Re } \langle Ax, y\rangle : \|x\| = \|y\| = 1\}$. Of course, we can replace $\langle Ax, y\rangle$ by $A(x, y)$ in these statements.

So Hilbert's inequalities simply equate to the statement that $\|H_1\| = \|H_0\| = \|H_{-1}\| = \pi$. In the ensuing notes, we shall present all results of this type as statements on norms of matrices.

We return to the question of infinite sequences and matrices. The basic result is:

Proposition 2.2. *Let A be the infinite matrix $(a_{j,k})$ and $A^{(n)}$ the finite matrix $(a_{j,k})_{j,k=1}^n$. Then A is a bounded operator on ℓ_2 if and only if there is a constant M such that $\|A^{(n)}\| \le M$ for all n, and then $\|A\| = \sup_{n\ge1} \|A^{(n)}\|$.*

Proof. If A is bounded, then by Proposition 2.1(i), $\|A^{(n)}\| \le \|A\|$ for all n. Conversely, suppose that $\|A^{(n)}\| \le M$ for all n. Fix j. For any n and and $x \in \ell_2^n$ with $\|x\| = 1$, we have $|\sum_{k=1}^{n} a_{j,k}x_k| \le M$. This implies that $\sum_{k=1}^{n} |a_{j,k}|^2 \le M^2$. This holds for all n, so $\sum_{k=1}^{\infty} |a_{j,k}|^2 \le M^2$. It now follows that $\sum_{k=1}^{\infty} a_{j,k}x_k$ is convergent for any $x \in \ell_2$: denote the sum by y_j, and let $y_{j,n} = \sum_{k=1}^{n} a_{j,k}x_k$, so that $y_{j,n} \to y_j$ as $n \to \infty$. For any $m \le n$, we have

$$\sum_{j=1}^{m} |y_{j,n}|^2 \le \sum_{j=1}^{n} |y_{j,n}|^2 \le M^2 \sum_{k=1}^{n} |x_k|^2.$$

Taking the limit as $n \to \infty$, we have

$$\sum_{j=1}^{m} |y_j|^2 \le M^2 \sum_{k=1}^{\infty} |x_k|^2.$$

This is true for each m, so $\sum_{j=1}^{\infty} |y_j|^2 \le M^2 \sum_{k=1}^{\infty} |x_k|^2$. In other words, $\|Ax\| \le M\|x\|$. □

Note. Consider the following pair of conditions on rows and columns:

(R) $\sum_{k=1}^{\infty} |a_{j,k}|^2 \le M^2$ for each j;

(C) $\sum_{j=1}^{\infty} |a_{j,k}|^2 \le M^2$ for each k.

We saw in the proof that the statement $\|A\| \le M$ implies (R). It also implies (C), since $\sum_{j=1}^{\infty} |a_{j,k}|^2 = \|Ae_k\|^2$. However, conditions (R) and (C) are by no means sufficient for A to map ℓ_2 into ℓ_2: we will see an example in section 5.

Convergence of sequences. Let $x^{(n)}$ be a sequence of elements of an inner-product space X. We say that $x^{(n)}$ converges to the limit x if $\|x^{(n)} - x\| \to 0$ as $n \to \infty$. A basic example is: if $x \in \ell_2$ and $x^{(n)}$ is the sequence obtained by truncating to 0 after n terms, then $x^{(n)} \to x$. It is easily shown, using the Cauchy-Schwarz inequality, that if $x^{(n)} \to x$ and $y^{(n)} \to y$, then $\langle x^{(n)}, y^{(n)} \rangle \to \langle x, y \rangle$. Further, if A is a bounded linear operator on X, then $Ax^{(n)} \to Ax$ and $\langle Ax^{(n)}, y^{(n)} \rangle \to \langle Ax, y \rangle$.

Adjoints. For a bounded operator A on ℓ_2^n or ℓ_2, the *adjoint* A^* is the operator given by $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all x, y. If A is given by the matrix $(a_{j,k})$, then A^* is given by the conjugate transpose matrix $a^*_{j,k} = \overline{a_{k,j}}$). This is verified by simple algebra in the finite case. In the infinite case, it amounts to a reversal of the order of summation in the double series; it can be deduced from the finite case using $\langle Ax, y \rangle = \lim_{n\to\infty} \langle Ax^{(n)}, y^{(n)} \rangle$, where $x^{(n)}$ and $y^{(n)}$ are the truncations of x and y.

If $A^* = A$, then A is *self-adjoint* (alias *hermitian*). Adjoints satisfy $A^{**} = A$, $(\lambda A)^* = \overline{\lambda} A^*$ and $(AB)^* = B^*A^*$ (so A^*A is self-adjoint). Also, $\|A^*\| = \|A\|$. Note that if A is given by a real, symmetric matrix, then $A^* = A$, and if A is given by a real, skew-symmetric matrix, then $A^* = -A$, hence iA is self-adjoint.

Quadratic forms and positivity. For $A = (a_{j,k})$, we have

$$\langle Ax, x\rangle = \sum_j \sum_k a_{j,k}\overline{x}_j x_k = A(\overline{x}, x).$$

Such expressions are called *quadratic forms.* For a self-adjoint operator, $\langle Ax, x\rangle = \langle x, Ax\rangle$, so $\langle Ax, x\rangle$ is real for all x. We write $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all x. In linear algebra, such operators are usually described as "positive semi-definite", but we shall follow the culture of operator theory and just call them "positive". We also write $A \geq B$ if $A - B \geq 0$.

For *any* operator A, we have $\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2$, hence $A^*A \geq 0$, which is analogous to the fact that $\overline{z}z$ is real and non-negative for complex numbers z. Also, this shows that $\|A^*A\| = \|A\|^2$.

For bounded, self-adjoint A, we have $|\langle Ax, x\rangle| \leq \|A\|.\|x\|^2$, so inequalities of the form $m\|x\|^2 \leq \langle Ax, x\rangle \leq M\|x\|^2$ will hold (equivalently, $mI \leq A \leq MI$, where I is the identity operator). If A is also real and these inequalities hold for real vectors x, then they also hold for complex x, since if $x = y + iz$, then $\langle Ax, x\rangle = \langle Ay, y\rangle + \langle Az, z\rangle$.

In fact, the norm of a self-adoint operator is determined by its quadratic form. We include the proof, although it is very well known.

Proposition 2.3. *If $A^* = A$, then*

$$\|A\| = \sup\{|\langle Ax, x\rangle| : \|x\| = 1\}.$$

So quadratic forms, instead of bilinear ones, are sufficient to determine $\|A\|$.

Proof. Denote the stated supremum by M. Then $|\langle Ax, x\rangle| \leq M\|x\|^2$ for all x. To show that $\|A\| \leq M$, it is sufficient to show that Re $\langle Ax, y\rangle \leq M$ for all x, y with $\|x\| = \|y\| = 1$. Now

$$\begin{aligned}
\langle A(x+y),\, x+y\rangle - \langle A(x-y),\, x-y\rangle &= 2\langle Ax, y\rangle + 2\langle Ay, x\rangle \\
&= 2\langle Ax, y\rangle + 2\langle y, Ax\rangle \qquad \text{since } A^* = A \\
&= 4\mathrm{Re}\ \langle Ax, y\rangle.
\end{aligned}$$

Hence

$$4\mathrm{Re}\ \langle Ax, y\rangle \leq M\,(\,\|x+y\|^2 + \|x-y\|^2) = M\,(2\|x\|^2 + 2\|y\|^2) = 4M. \qquad \square$$

For real, symmetric matrices, as already remarked, it is sufficient to restrict to real vectors x in Proposition 2.3. Now suppose that B is a real, *skew-symmetric* matrix. Then

$(iB)^* = iB$, so the statement applies to iB, hence to B itself. However, iB is a complex matrix, and it is essential to consider *complex* vectors x: indeed, it is trivial that $\langle Bx, x\rangle = 0$ for all real vectors x! For complex vectors x, the (j,k) and (k,j) terms combine to give $a_{j,k}(x_j\overline{x}_k - x_k\overline{x}_j)$, which is purely imaginary, so $\langle Bx, x\rangle$ is purely imaginary. This also shows that $\langle B\overline{x}, \overline{x}\rangle = -\langle Bx, x\rangle$, so the set $\{\langle Bx, x\rangle : \|x\| = 1\}$ is symmetrical about 0. In this sense, skew-symmetric quadratic forms are more "symmetrical" than symmetric forms!

Eigenvalues and the spectral theorem. All eigenvalues of a self-adjoint operator A are real, since if $Ax = \lambda x$, then

$$\lambda\langle x, x\rangle = \langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\lambda}\langle x, x\rangle.$$

If A is real, then of course real-valued eigenvectors can be chosen. A fundamental property of self-adjoint operators on ℓ_2^n is the theorem on diagonalisation (alias the *spectral theorem*), which can be stated as follows:

Theorem 2.4. *If A is a self-adjoint operator on ℓ_2^n, with eigenvalues λ_j $(1 \le j \le n)$, then there is an orthonormal basis of ℓ_2^n consisting of corresponding eigenvectors z_j, so that for all $x \in \ell_2^n$,*

$$Ax = \sum_{j=1}^{n} \lambda_j \langle x, z_j\rangle z_j, \qquad \|Ax\|^2 = \sum_{j=1}^{n} \lambda_j^2 |\langle x, z_j\rangle|^2, \qquad \langle Ax, x\rangle = \sum_{j=1}^{n} \lambda_j |\langle x, z_j\rangle|^2.$$

This displays the true nature of such operators with dazzling clarity. It follows that $\|A\| = \max_{1\le j\le n} |\lambda_j|$, and if the eigenvalues are listed in descending order $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, then

$$\lambda_n \|x\|^2 \le \langle Ax, x\rangle \le \lambda_1 \|x\|^2,$$

with the bounds attained at z_n and z_1. Of course, this implies Proposition 3.3, at the cost of a harder proof.

If $B^* = -B$, then $B = iA$, where $A^* = A$. With A represented as in Theorem 2.4, the corresponding statements for B are clear. In particular, the eigenvalues of B are $i\lambda_j$ $(1 \le j \le n)$.

Actually, none of our proofs of Hilbert's inequality will depend on Theorem 2.4, but we will use it in examples and in the proof of the Montgomery-Vaughan theorem.

Example 2.1. Let

$$B = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

with a, b, c real. The eigenvalue equation is $\lambda^3 + (a^2+b^2+c^2)\lambda = 0$, so the eigenvalues are $\pm iM$ and 0, where $M = (a^2+b^2+c^2)^{1/2}$, and $\|B\| = M$.

We conclude this section with a remark that will not be used later (so the reader is at liberty to ignore it). In both the linear and bilinear formulations of matrix norms, the scalars can be replaced by vectors (in the sense of elements of an inner-product space), as follows: *If A is an $m \times n$ matrix, and H is an inner-product space, then:*

(i) If $x_1, \ldots, x_n \in H$ and $y_j = \sum_{k=1}^n a_{j,k} x_k$ for $1 \le j \le m$, then

$$\sum_{j=1}^m \|y_j\|^2 \le \|A\|^2 \sum_{k=1}^n \|x_k\|^2.$$

(ii) For elements x_j, y_k of H, we have

$$\left| \sum_{j=1}^m \sum_{k=1}^n a_{j,k} \langle y_k, x_j \rangle \right| \le \|A\| \left(\sum_{j=1}^m \|x_j\|^2 \right)^{1/2} \left(\sum_{k=1}^n \|y_k\|^2 \right)^{1/2}.$$

Statement (i) is easily proved by expressing the x_j in terms of an orthonormal basis, and (ii) is deduced by writing the sum as $\sum_{j=1}^m \langle z_j, x_j \rangle$. A further development (see [Jam2]) is that the vectors x_j, y_k can be replaced by *operators on a Hilbert space*, with terms like $\|x_j\|^2$ replaced by $X_j^* X_j$, and the partial ordering defined in the usual way for operators.

3. Relationships between the Hilbert matrices.

The notation H_1, H_0, H_{-1}, $H_1^{(n)}$ was introduced in section 1. Here we note some obvious facts about these matrices. The matrices H_1 and H_0 are symmetric, while H_{-1} is skew-symmetric. In H_1 and H_0, the (j,k) term is a function of $j+k$ (so these are *Hankel* matrices), while in H_{-1} it is a function of $j-k$ (so H_{-1} is a *Toeplitz* matrix). The inclusion of H_0 might seem like an unnecessary complication, but it arises naturally in some applications, as we shall see. Note that it can be relabelled as $(c_{j+k+1})_{j,k \ge 0}$.

We write out $H_1^{(3)}$, $H_0^{(3)}$ and $H_{-1}^{(3)}$ explicitly.

$$H_1^{(3)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix}, \quad H_0^{(3)} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}, \quad H_{-1}^{(3)} = \begin{pmatrix} 0 & -1 & -\frac{1}{2} \\ 1 & 0 & -1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}.$$

We now also introduce the doubly infinite matrices

$$\tilde{H}_1 = (c_{j+k})_{j,k \in \mathbb{Z}}, \qquad \tilde{H}_{-1} = (c_{j-k})_{j,k \in \mathbb{Z}}.$$

Further, we denote by $\tilde{H}_1^{(n)}$ the finite matrix $(c_{j+k})_{j,k=-n}^{n}$, and define $\tilde{H}_{-1}^{(n)}$ similarly.

Hilbert's inequalities state that the norms of H_1, H_0 and H_{-1} are not greater than π. Later, we will show that in each case the norm actually equals π. Ahead of our various proofs of these statements, we record some simple relationships between these matrices and their norms.

By Proposition 2.2, $\|H_{-1}\| = \sup_{n\geq 1} \|H_{-1}^{(n)}\|$ $(= \pi)$. However, this does *not* mean that $\|H_{-1}^{(n)}\| = \pi$ for each n ! Indeed, $H_{-1}^{(2)}(x_1, x_2) = (-x_2, x_1)$, so $\|H_{-1}^{(2)}\| = 1$. In fact, the determination of $\|H_1^{(n)}\|$ and $\|H_{-1}^{(n)}\|$ for general n is a much harder problem, and the exact values are not known. Some estimations will be described in later sections.

Next, $\tilde{H}_{-1}^{(n)}$ is exactly the same matrix as $H_{-1}^{(2n+1)}$, with the rows and columns labelled from $-n$ to n instead of 1 to $2n+1$. By Proposition 2.2, this is enough to show that $\|\tilde{H}_{-1}\| = \|H_{-1}\|$.

Clearly, $\tilde{H}_1 x = \tilde{H}_{-1} y$, where $y_j = x_{-j}$ for $j \in \mathbb{Z}$, so $\|\tilde{H}_1\| = \|\tilde{H}_{-1}\|$. The move to doubly infinite sequences very neatly reveals these two operators as essentially the same. Of course, $\|H_1\| \leq \|\tilde{H}_1\|$.

Also, H_1 is obtained from H_0 by removing the first row, so $(H_1 x)(j) = (H_0 x)(j+1)$. Hence $\|H_0 x\| \geq \|H_1 x\|$ for all x, and $\|H_0\| \geq \|H_1\|$. This does not quite apply to the finite truncations, but Proposition 2.1(i) shows that $\|H_0^{(n)}\| \geq \|H_1^{(n)}\|$.

We now show that $\|H_0\| \leq \|H_{-1}\|$. In fact, the following stronger statement holds:

Proposition 3.1. *Let $\|\tilde{H}_{-1}\| = M$. Then for x in ℓ_2, we have*

$$\|H_{-1}x\|^2 + \|H_0 x\|^2 \leq M^2 \|x\|^2.$$

Hence, in particular, $\|H_0\| \leq M$. Similarly for the finite truncations.

Proof. Extend x to a two-sided sequence x^* by putting $x_j^* = 0$ for all $j \leq 0$. For $j \geq 1$, we have, obviously, $(\tilde{H}_{-1}x^*)(j) = (H_{-1}x)(j)$. For $j \geq 0$ we have

$$(\tilde{H}_{-1}x^*)(-j) = \sum_{k\geq 1} c_{-j-k} x_k = -\sum_{k\geq 1} c_{j+k} x_k = -(H_0 x)(j+1).$$

Hence

$$\|H_{-1}x\|^2 + \|H_0 x\|^2 = \|\tilde{H}_{-1}x^*\|^2 \leq M^2 \|x\|^2.$$

In the finite case, the same holds with equality replaced by $\leq$, since the term $(\tilde{H}_{-1}^{(n)})(-n)$ is left out. $\square$

To summarise, we have established the following:

Proposition 3.2. *We have*

$$\|\tilde{H}_1\| = \|\tilde{H}_{-1}\| = \|H_{-1}\| \geq \|H_0\| \geq \|H_1\|,$$

also

$$\|\tilde{H}_{-1}^{(n)}\| = \|H_{-1}^{(2n+1)}\| \geq \|H_0^{(n)}\| \geq \|H_1^{(n)}\|.$$

Hence a proof that $\|H_{-1}\| \leq \pi$ implies the same for H_0 and H_1, and a proof that $\|H_1\| \geq \pi$ implies the same for H_0 and H_{-1}.

We conclude this section with some further elementary observations. It is trivial that $\|H_0\| \geq \pi/\sqrt{6}$: if e_1 is the sequence with 1 in place 1 and 0 elsewhere, then $(H_0 e_1)(j) = \frac{1}{j}$, so $\|H_0 e_1\|^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} = \pi^2/6$. Similarly, $\|H_{-1}\| \geq \pi/\sqrt{3}$.

By Example 2.1, we have $\|H_{-1}^{(3)}\| = \frac{3}{2}$. The reader might like to check that $\|H_0^{(2)}\| = \frac{1}{6}(4 + \sqrt{13})$.

4. Method 1: integrals involving trigonometric polynomials

The idea of this elegant method is to exhibit the bilinear forms in question explicitly as integrals, the value of which can be estimated. Hilbert's original proof of his inequality was along these lines. Here we give a slightly modified version due to Toeplitz.

Following the custom of analytic number theory, we shall use the notation $e(x)$ for $e^{2\pi i x}$. Note that $e(x)e(y) = e(x+y)$, $e(n) = 1$ for $n \in \mathbb{Z}$ and $e(\frac{1}{2}) = -1$. Also, $\int_0^1 e(rt)\, dt = 0$ for non-zero integers r.

A function of the form

$$X(t) = \sum_{j=1}^{n} x_j e(jt)$$

is called a *trigonometric polynomial.* Observe that

$$|X(t)|^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} x_j \overline{x}_k e[(j-k)t] = \sum_{j=1}^{n} |x_j|^2 + \sum_{j=1}^{n} \sum_{k \neq j} x_j \overline{x}_k e[(j-k)t]$$

(in which $\sum_{k \neq j}$ means that k runs from 1 to n, excluding j). Hence

$$\int_0^1 |X(t)|^2\, dt = \sum_{j=1}^{n} |x_j|^2 = \|x\|^2. \tag{5}$$

Proof of Theorem 1.1. We will prove simultaneously that the norms of H_1, H_{-1} and H_0 are not greater than π, although we have seen that the statement for H_{-1} actually implies the others. By Proposition 2.2, it is sufficient to consider the n-dimensional truncations.

Given (complex) vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, let $X(t) = \sum_{j=1}^n x_j e(jt)$ and $Y(t) = \sum_{k=1}^n y_k e(kt)$. Then

$$X(t)Y(t) = \sum_{j=1}^n \sum_{k=1}^n x_j y_k e[(j+k)t],$$

$$X(t)\overline{Y(t)} = \sum_{j=1}^n \sum_{k=1}^n x_j \overline{y}_k e[(j-k)t].$$

Consider the integrals

$$I_1 = \int_0^1 (t - \tfrac{1}{2}) X(t)Y(t)\, dt, \qquad I_{-1} = \int_0^1 (t - \tfrac{1}{2}) X(t)\overline{Y(t)}\, dt.$$

Note that $\int_0^1 (t - \frac{1}{2})\, dt = 0$ (the term $\frac{1}{2}$ was chosen to ensure this), while for integers $r \neq 0$, integration by parts gives

$$\int_0^1 (t - \tfrac{1}{2}) e(rt)\, dt = \frac{\frac{1}{2} - (-\frac{1}{2})}{2\pi i r} - \frac{1}{2\pi i r}\int_0^1 e(rt)\, dt = \frac{1}{2\pi i r}.$$

Hence

$$I_1 = \frac{1}{2\pi i}\sum_{j=1}^n \sum_{k=1}^n c_{j+k} x_j y_k = \frac{1}{2\pi i} H_1^{(n)}(x, y),$$

$$I_{-1} = \frac{1}{2\pi i}\sum_{j=1}^n \sum_{k=1}^n c_{j-k} x_j \overline{y}_k = \frac{1}{2\pi i} H_{-1}^{(n)}(x, \overline{y}).$$

Now $|t - \frac{1}{2}| \leq \frac{1}{2}$ on $[0, 1]$, so by (5) and the Cauchy-Schwarz inequality for integrals, we have

$$|I_1| \leq \tfrac{1}{2}\int_0^1 |X(t)Y(t)|\, dt \leq \tfrac{1}{2}\|x\|.\|y\|,$$

hence $|H_1^{(n)}(x, y) \leq \pi\|x\|.\|y\|$, as required. Similarly for H_{-1} (of course, we can then replace $\overline{y}$ by y). For H_0, we just replace $Y(t)$ by $e(-t)Y(t) = \sum_{k=1}^n y_k e[(k-1)t]$ in I_1. □

Notes. (1) The proof applies without change if j ranges from 1 to m and k from 1 to n, where $m \neq n$.

(2) A minor variation is to take $X(t) = \sum_{j=-n}^n x_j e^{ijt}$, and to consider integrals like $\int_0^{2\pi} (t - \pi) X(t)Y(t)\, dt$.

(3) Hilbert's original proof used separate terms $\cos jt$ and $\sin jt$ instead of e^{ijt}, resulting in the constant 2π instead of π for H_{-1}. See [HLP, section 9.6].

Since H_1 is symmetric and H_{-1} is skew-symmetric, *quadratic* forms, instead of bilinear ones, are sufficient to determine their norms, with real scalars in the case of H_1 (Proposition 2.3). We now describe a variant of the method taking advantage of this fact. This time, the integral is estimated by a simple inequality for the integrand instead of the Cauchy-Schwarz inequality.

Variant proof of Theorem 1.1. With $X(t)$ as before, let

$$J_1 = \int_0^1 (t - \tfrac{1}{2}) X(t)^2 \, dt, \qquad J_{-1} = \int_0^1 (t - \tfrac{1}{2}) |X(t)|^2 \, dt.$$

In the case of J_1, restrict to real scalars x_j. We then have

$$J_1 = \frac{1}{2\pi i} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{j+k} x_j x_k = \frac{1}{2\pi i} \langle H_1^{(n)} x, x \rangle,$$

$$J_{-1} = \frac{1}{2\pi i} \sum_{j=1}^{n} \sum_{k=1}^{n} c_{j-k} x_j \overline{x}_k = \frac{1}{2\pi i} \langle H_{-1}^{(n)} x, x \rangle.$$

Since $|t - \frac{1}{2}| \le \frac{1}{2}$ on $[0, 1]$, we have

$$|J_1| \le \tfrac{1}{2} \int_0^1 |X(t)|^2 \, dt = \tfrac{1}{2} \|x\|^2,$$

hence $|H_1^{(n)}(x)| \le \pi \|x\|^2$, and similarly for H_{-1}. □

A further variation is to consider the double integral $\int_0^1 \int_0^u |X(t)|^2 dt\, du$.

This has proved Hilbert's inequalities in elegant style. However, it is clear that the method applies more generally. We only have to replace $t - \frac{1}{2}$ by a general function $\phi(t)$. We obtain:

Theorem 4.1. *Let ϕ be an integrable function with $|\phi(t)| \le M$ for $0 \le t \le 1$. For integers r, let*

$$\int_0^1 \phi(t) e(rt) \, dt = d_r.$$

Let D be the matrix $(d_{j-k})_{j,k \ge 1}$ and $\tilde{D}$ the matrix $(d_{j-k})_{j,k \in \mathbb{Z}}$. Then $\|D\|$ and $\|\tilde{D}\|$ are not greater than M. The same applies to (d_{j+k}).

Further, if $\phi(t)$ is real and $M_1 \le \phi(t) \le M_2$ for $0 \le t \le 1$, then D is self-adjoint and for all $x = (x_j)$ in ℓ_2,

$$M_1 \|x\|^2 \le \langle Dx, x \rangle \le M_2 \|x\|^2.$$

Proof. Let $D^{(n)} = (d_{j,k})_{j,k=1}^{n}$. Given scalars x_j, y_k, define $X(t)$, $Y(t)$ as before and let

$$I = \int_0^1 \phi(t) X(t) \overline{Y(t)} \, dt.$$

Then

$$I = \sum_{j=1}^{n} \sum_{k=1}^{n} d_{j-k} x_j \overline{y}_k = D^{(n)}(x, \overline{y}).$$

But by the Cauchy-Schwarz inequality for integrals, $|I| \leq M\|x\|.\|y\|$. The statement for (d_{j+k}) is obtained by considering the integral of $\phi(t)X(t)Y(t)$. In the same way as for the Hilbert matrices, we have $\|\tilde{D}\| = \|D\|$.

If $\phi(t)$ is real with $M)_1 \leq \phi(t) \leq M_2\phi(t)$, then $d_{-r} = \overline{d}_r$, so D is self-adjoint. Let $J = \int_0^1 \phi(t)|X(t)|^2 \, dt$. Then

$$J = \sum_{j=1}^{n} \sum_{k=1}^{n} d_{j-k} x_j \overline{x}_k = \langle D^{(n)} x, x \rangle.$$

Clearly, $M_1\|x\|^2 \leq J \leq M_2\|x\|^2$. □

The interval $[0, 1]$ can be replaced by any interval $[a, a+1]$ of length 1, because we still have $\int_a^{a+1} e(rt) \, dt = 0$ for non-zero integers r. This leads to the following further variant:

Proposition 4.2. *In Theorem 4.1, the same norm estimations apply with d_r replaced by $(-1)^r d_r$.*

Proof. Since $e(\frac{1}{2}r) = e^{\pi i r} = (-1)^r$, we have

$$\int_{-1/2}^{1/2} \phi(t + \tfrac{1}{2}) e(rt) \, dt = (-1)^r \int_0^1 \phi(u) e(ru) \, du = (-1)^r d_r.$$

The result follows on applying the Theorem to $\phi(t + \frac{1}{2})$ on $[-\frac{1}{2}, \frac{1}{2}]$. □

Note. Our d_r is the Fourier coefficient $\hat{\phi}(-r)$ of ϕ on $[0, 1]$, but the proof did not require any results from Fourier analysis.

The choice $\phi(t) = t - \frac{1}{2}$ gave $D = \frac{1}{2\pi i} H_{-1}$. By feeding in different choices of $\phi(t)$, we will obtain norm estimations for many more Toeplitz and Hankel matrices (this is an easier way forward than starting from the matrix and trying to identify the appropriate $\phi(t)$). We record some examples.

Proposition 4.3. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Write $a_{j,k} = 1/(j - k + \alpha)$. Let A_α be the matrix $(a_{j,k})_{j,k \in \mathbb{Z}}$. Then $\|A_\alpha\| \leq \pi|\operatorname{cosec} \alpha\pi|$. In particular, $\|A_{1/2}\| \leq \pi$.*

Proof. Take $\phi(t) = e(\alpha t)$. Then $M = 1$ and

$$d_r = \int_0^1 e(\alpha t) e(rt) \, dt = \frac{1}{2\pi i(r + \alpha)} (e(\alpha) - 1) = \frac{e^{\pi i \alpha} \sin \alpha\pi}{\pi(r + \alpha)}.$$

So $D = (1/\pi) e^{\pi i \alpha} (\sin \alpha\pi) \, A_\alpha$, and the statement follows. □

Remarks. (1) In this case, it is easy to show that the stated constant is the best possible (which we have not yet done in the case of Hilbert's inequality), if we assume the well-known identity $\sum_{j\in\mathbb{Z}} 1/(j+\alpha)^2 = \pi^2\text{cosec}^2\alpha\pi$, Let e_0 be the two-sided sequence with 1 in place 0 and 0 elsewhere. Then $(A_\alpha e_0)(j) = 1/(j+\alpha)$, so $\|A_\alpha e_0\| = \pi|\text{cosec }\alpha\pi|$.

(2) The given statement says the same if $1/(j-k+\alpha)$ is replaced by $a'_{j,k} = 1/(j+k+\alpha)$. However, if we restrict to *positive* j, k, and $\alpha > 0$, we already know a stronger estimate: $0 < a'_{j,k} < 1/(j+k)$, so the matrix $(a'_{j,k})$ has norm not greater than $\|H_1\| = \pi$.

Proposition 4.4. *Let*

$$c_r^* = \begin{cases} 1/r & \textit{if } r \textit{ is odd,} \\ 0 & \textit{if } r \textit{ is even.} \end{cases}$$

Then the matrix $(c_{j-k}^*)_{j,k\in\mathbb{Z}}$ *has norm not greater than* $\pi/2$.

Proof. Let

$$\phi(t) = \begin{cases} -1 & \text{for } 0 \le t < \frac{1}{2}, \\ \;\;1 & \text{for } \frac{1}{2} \le t < 1. \end{cases}$$

Then $M = 1$, and for $r \neq 0$,

$$d_r = \int_{1/2}^{1} e(rt)\,dt - \int_0^{1/2} e(rt)\,dt = \frac{2 - 2e(\frac{r}{2})}{2\pi i r} = \frac{2}{\pi i} c_r^*.$$

Also, $d_0 = 0$. So the matrix (c_{j-k}^*) equates to $(\pi i/2)D$. □

Proposition 4.5. *Let T be the matrix $(c_{j-k}^2)_{j,k\ge 1}$ or $(c_{j-k}^2)_{j,k\in\mathbb{Z}}$. Then $\|T\| \le \pi^2/3$. Further, for any x in ℓ_2,*

$$-\frac{\pi^2}{6}\|x\|^2 \le \langle Tx, x\rangle \le \frac{\pi^2}{3}\|x\|^2.$$

Proof. Take $\phi(t) = t^2 - t + \frac{1}{6}$. The term $\frac{1}{6}$ is included to ensure that $\int_0^1 \phi(t)\,dt = 0$: in fact, $\phi(t)$ is the *Bernoulli polynomial* $B_2(t)$). It is easily checked that $-\frac{1}{12} \le \phi(t) \le \frac{1}{6}$ on $[0,1]$, so, in the notation of Theorem 4.1, we have $-\frac{1}{12}\|x\|^2 \le \langle Dx, x\rangle \le \frac{1}{6}\|x\|^2$. For $r \neq 0$, integration by parts gives

$$\begin{aligned} d_r &= 0 - \frac{1}{2\pi i r}\int_0^1 (2t-1)e(rt)\,dt \\ &= -\frac{1}{\pi i r}\int_0^1 t e(rt)\,dt \\ &= -\frac{1}{(\pi i r)(2\pi i r)} \\ &= \frac{1}{2\pi^2 r^2} \end{aligned}$$

so $T = 2\pi^2 D$ and the statement follows. □

This process can be extended to (c_{j-k}^n), using the fact that the Bernoulli polynomials $B_n(t)$ has Fourier coefficients $-n!/(2\pi i r)^n$. In particular, the reader might like to show that the norm of (c_{j-k}^3) is not greater than $\pi^3/(9\sqrt{3})$. using $B_3(t) = t(t - \frac{1}{2})(t-1)$. However, for even n, the norms of these matrices are found much more easily by Method 2.

To finish this section, we sketch how the method can be developed to deliver lower bounds, and hence the exact norm in Theorem 4.1. However, for this we do need to assume some Fourier analysis, and some familiarity with the Hilbert space $L_2[0,1]$; readers are at liberty to leave it out. More exactly, we assume that the functions $\psi_n(t) = e(nt)$ $(n \in \mathbb{Z})$ are an orthonormal basis of $L_2[0,1]$, so that a function $X \in L_2[0,1]$ is the sum of its Fourier series $\sum_{j\in\mathbb{Z}} x_j e(jt)$ in the sense of L_2-convergence. This implies that, as in the finite case,

$$|X(t)|^2 = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} x_j \overline{x}_k e[(j-k)t],$$

again in the sense of L_2-convergence, and (Parseval's identity)

$$\int_0^1 |X(t)|^2 \, dt = \sum_{j\in\mathbb{Z}} |x_j|^2 = \|x\|^2.$$

Proposition 4.6. *Let ϕ be an integrable real-valued function on $[0,1]$, with essential supremum M. Define $\tilde{D}$ as in Theorem 4.1. Then $\|\tilde{D}\| = M$.*

Sketch of proof. The meaning of "essential supremum" is that for any $\varepsilon > 0$, we have $|\phi(t)| > M - \varepsilon$ on a set of positive measure. It follows that there is a function $X(t)$ such that $\int_0^1 |X(t)|^2 \, dt > 0$ and

$$\left| \int_0^1 \phi(t)|X(t)|^2 \, dt \right| > (M - \varepsilon) \int_0^1 |X(t)|^2 \, dt.$$

Let $X(t)$ have Fourier series $\sum_{n\in\mathbb{Z}} x_j e(jt)$, as above. Then

$$\int_0^1 \phi(t)|X(t)|^2 \, dt = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} d_{j-k} x_j \overline{x}_k = \langle \tilde{D}x, x \rangle.$$

So $|\langle \tilde{D}x, x\rangle| > (M-\varepsilon)\|x\|^2$, hence $\|\tilde{D}\| > M - \varepsilon$. $\square$

For $\phi(t) = t - \frac{1}{2}$, generating $\tilde{H}_{-1}$, we can take $X(t) = t^n$ for suitably large n, since then $\int_0^1 X(t)^2 \, dt = 1/(2n+1)$ and

$$\begin{aligned} \int_0^1 (t - \tfrac{1}{2}) X(t)^2 \, dt &= \int_0^1 (t^{2n+1} - \tfrac{1}{2} t^{2n}) \, dt \\ &= \frac{1}{2n+2} - \frac{1}{2(2n+1)} \\ &= \frac{n}{2(n+1)(2n+1)}. \end{aligned}$$

However, as we have stated it, this method does not give equality of norm for H_1 and H_0, and it also fails to give stronger estimations for finite-dimensional truncations. We shall give a more direct proof for all the Hilbert matrices in section 6.

A more sophisticated presentation of the result is as follows. Corresponding to ϕ, a *multiplication operator* on $L_2[0,1]$ is defined by $(M_\phi f)(t) = \phi(t)f(t)$. This operator has norm M. Now $\phi = \sum_{r\in\mathbb{Z}} d_r\psi_{-r}$ (the Fourier series), so $\phi\psi_j = \sum_{r\in\mathbb{Z}} d_r\psi_{j-r}$, hence $\langle \phi\psi_j, \psi_k\rangle = d_{j-k}$, so $\tilde{D}$ is the matrix of M_ϕ with respect to the basis (ψ_j). For a systematic treatment of this and further developments, see [Young, chapters 13 and 15].

5. Method 2: row and column sums

This is a very general method for the estimation of norms of matrices with *non-negative* entries. The basic result, due to Schur, is as follows.

Theorem 5.1. *Let $A = (a_{j,k})$ be a matrix (finite or infinite) such that $a_{j,k} \geq 0$ for all j, k and*

$$\sum_k a_{j,k} \leq M_1 \quad \text{for all } j \qquad (\text{all row sums} \leq M_1),$$

$$\sum_j a_{j,k} \leq M_2 \quad \text{for all } k \qquad (\text{all column sums} \leq M_2).$$

Then A maps ℓ_2 into ℓ_2 and $\|A\| \leq (M_1M_2)^{1/2}$.

This is a pleasant result, which should be compared with conditions (R) and (C) from section 2. However, it does not apply to the Hilbert matrices H_1 and H_0, because the row sums are divergent. With only slightly more work, one can prove the following enhanced version incorporating a weighting sequence (w_j).

Theorem 5.2. *Let $A = (a_{j,k})$ be a matrix (finite or infinite) with $a_{j,k} \geq 0$ for all j, k. Suppose that, for some strictly positive sequence (w_j), we have*

$$\sum_k a_{j,k}w_k \leq M_1\, w_j \quad \text{for all } j,$$

$$\sum_j a_{j,k}w_j \leq M_2\, w_k \quad \text{for all } k.$$

Then $\|A\| \leq (M_1M_2)^{1/2}$.

Proof. Choose non-negative, real vectors $x = (x_j)$ and $y = (y_k)$ (in ℓ_2 in the infinite case), and let $A(x,y) = \sum_j \sum_k x_j a_{j,k} y_k$. Then $A(x,y) = \sum_j \sum_k r_{j,k} s_{j,k}$, where

$$r_{j,k} = a_{j,k}^{1/2} x_j \left(\frac{w_k}{w_j}\right)^{1/2}, \qquad s_{j,k} = a_{j,k}^{1/2} y_k \left(\frac{w_j}{w_k}\right)^{1/2}.$$

By the Cauchy-Schwarz inequality (applied to the double sum), $A(x,y) \le (RS)^{1/2}$, where

$$R = \sum_j \sum_k r_{j,k}^2 = \sum_j x_j^2 w_j^{-1} \sum_k a_{j,k} w_k \le M_1 \sum_j x_j^2,$$

$$S = \sum_k \sum_j s_{j,k}^2 = \sum_k y_k^2 w_k^{-1} \sum_j a_{j,k} w_j \le M_2 \sum_k y_k^2.$$

So $A(x,y) \le (M_1 M_2)^{1/2} \|x\|.\|y\|$. □

Note. If A is symmetric, then the two hypotheses say the same (with $M_2 = M_1 = M$, say) and the conclusion is $\|A\| \le M$. In 6.1, this case can be proved instantly by substituting the inequality $2x_j x_k \le x_j^2 + x_k^2$ in the quadratic form $\sum_j \sum_k a_{j,k} x_j x_k$.

We note here that the row-sum criterion is reversible in an obvious way for finite matrices, giving a lower bound for the norm:

Proposition 5.3. *Let $A = (a_{j,k})_{j,k=1}^n$, with $a_{j,k}$ real. Suppose that there is a positive sequence $w = (w_1, w_2, \dots, w_n)$ such that $\sum_{k=1}^n a_{j,k} w_k \ge m w_j$ for each j. Then $\|A\| \ge m$.*

Proof. The hypothesis says that $(Aw)(j) \ge m w_j$ for each j, hence $\|Aw\| \ge m\|w\|$. □

This is pleasing, but it has rather limited applications. In particular, it does not extend to infinite matrices unless the sequence w belongs to ℓ_2.

Before applying these theorems to Hilbert's inequalities, we record some other matrices of similar type where the norms follow very easily. The matrix (c_{j-k}^2) (already seen in Proposition 4.5) is ready-made for an instant application of Theorem 5.1:

Proposition 5.4. *The matrix $(c_{j-k}^2)_{j,k\in\mathbb{Z}}$ has norm not greater than $\pi^2/3$.*

Proof. The matrix is symmetric, and every row sum is $2\sum_{r=1}^\infty (1/r^2) = \pi^2/3$. □

(However, this method has nothing to say about the *lower* bound for the quadratic form given in Proposition 4.5.)

Similarly, the norm of the matrix $(c_{j+k-1}^2)_{j,k\ge 1}$ is not greater than $\pi^2/6$. More generally, the norm of $(|c_{j+k-1}|^n)$ is not greater than $\zeta(n)$ (recall that $\zeta(p)$ means $\sum_{k=1}^\infty k^{-p}$ for $p > 1$).

Applications of Theorems 5.1 and 5.2 often require the estimation of discrete sums by integrals. The basic version of this is: if $f(t)$ is decreasing on $[m,n]$, then

$$f(m)+\cdots+f(n-1)\geq \int_m^n f(t)\,dt \geq f(m+1)+\cdots+f(n), \tag{6}$$

and similarly for the case where $n\to\infty$. A slightly more refined version applies to *convex* functions (a sufficient condition for f to be convex is $f''(x)\geq 0$). A convex function lies above its tangents, so that $\int_r^{r+1} f \geq f(r+\frac{1}{2})$, hence

$$\sum_{r=m}^{n-1} f(r+\tfrac{1}{2}) \leq \int_m^n f(t)\,dt. \tag{7}$$

Proposition 5.5. *Let A be the matrix $(a_{j,k})_{j,k=1}^{\infty}$, where*

$$a_{j,k}=\frac{(jk)^{1/2}}{(j+k)^2}.$$

Then $\|A\|\leq 1$.

Proof. By integral estimation in the form (6), we have

$$\sum_{k=1}^{\infty}\frac{1}{(j+k)^2}=\sum_{r=j+1}^{\infty}\frac{1}{r^2}\leq \int_j^{\infty}\frac{1}{t^2}\,dt=\frac{1}{j}.$$

We now apply Theorem 5.2 with $w_j=1/j^{1/2}$ to obtain

$$\sum_{k=1}^{\infty}\frac{a_{j,k}}{k^{1/2}}=j^{1/2}\sum_{k=1}^{\infty}\frac{1}{(j+k)^2}\leq j^{1/2}\frac{1}{j}=\frac{1}{j^{1/2}}=w_j. \qquad \square$$

This result can be rewritten in an interesting way. By substituting $u_j=j^{1/2}x_j,\ v_k=k^{1/2}y_k$ in the bilinear form $A(x,y)$, it equates to the statement that for non-negative u_j, v_k,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{u_jv_k}{(j+k)^2}\leq\left(\sum_{j=1}^{\infty}\frac{1}{j}u_j^2\right)^{1/2}\left(\sum_{k=1}^{\infty}\frac{1}{k}v_k^2\right)^{1/2}.$$

Now let us turn to the Hilbert matrices H_1 and H_0. Theorem 5.2 gives a quick proof that the norms are not greater than π, at the same time (unlike Method 1) delivering a stronger bound for the finite truncations, at the cost of a bit more work (readers are at liberty to leave out this part of the proof). We need the following integral.

Lemma 5.6. *For $a>0$,*

$$\int_0^{\infty}\frac{1}{(t+a)t^{1/2}}\,dt=\frac{\pi}{a^{1/2}}.$$

Proof. Denote the integral by I. Substituting $t = au$, then $u = v^2$, we have

$$I = \int_0^\infty \frac{1}{(u+1)(au)^{1/2}}\, du = \frac{1}{a^{1/2}} \int_0^\infty \frac{2}{v^2+1}\, dv = \frac{\pi}{a^{1/2}}. \qquad \square$$

Theorem 5.7. *We have*

$$\|H_1\| \le \|H_0\| \le \pi,$$

also

$$\|H_1^{(n)}\| \le \pi - \frac{2}{n^{1/2}}, \qquad \|H_0^{(n)}\| \le \pi - \frac{2}{(n+\frac{1}{2})^{1/2}}.$$

Proof. Although we know that $\|H_1\| \le \|H_0\|$, we give the proof for H_1 separately, because it is slightly simpler. We again apply Theorem 5.2 with $w_j = 1/j^{1/2}$. By Lemma 5.6 and (6),

$$\sum_{k=1}^\infty \frac{1}{(j+k)k^{1/2}} \le \int_0^\infty \frac{1}{(t+j)t^{1/2}}\, dt = \frac{\pi}{j^{1/2}}$$

for each j. (An alternative, purely geometric proof of this inequality is given in [Ole].) Theorem 5.2 now gives $\|H_1\| \le \pi$.

Now consider H_0, written as $(c_{j+k+1})_{j,k\ge 0}$. The following proof does not appear to be quite so well known. We now take $w_j = 1/(j+\frac{1}{2})^{1/2}$. The function $f(t) = 1/[(j+\frac{1}{2}+t)t^{1/2}]$ is convex, so by (7),

$$\sum_{k=0}^\infty \frac{1}{(j+k+1)(k+\frac{1}{2})^{1/2}} = \sum_{k=0}^\infty f(k+\tfrac{1}{2}) \le \int_0^\infty f(t)\, dt = \frac{\pi}{(j+\frac{1}{2})^{1/2}},$$

and Theorem 5.2 again gives $\|H_0\| \le \pi$.

Next, consider $H_1^{(n)}$. We must now consider finite sums. We have

$$\sum_{k=1}^n \frac{1}{(j+k)k^{1/2}} \le \int_0^n \frac{1}{(t+j)t^{1/2}}\, dt.$$

Now

$$\int_n^\infty \frac{1}{(t+j)t^{1/2}}\, dt \ge \int_n^\infty \frac{1}{(t+j)^{3/2}}\, dt = \frac{2}{(n+j)^{1/2}},$$

so for $2 \le j \le n$,

$$\sum_{k=1}^n \frac{1}{(j+k)k^{1/2}} \le \frac{\pi}{j^{1/2}} - \frac{2}{(n+j)^{1/2}} \le \frac{1}{j^{1/2}}\left(\pi - \frac{2}{n^{1/2}}\right),$$

since $nj \ge n+j$. For $j = 1$, we have

$$\sum_{k=1}^n \frac{1}{(k+1)k^{1/2}} \le \sum_{k=1}^n \frac{1}{k^{3/2}} \le 1 + \int_1^n \frac{1}{t^{3/2}}\, dt = 3 - \frac{2}{n^{1/2}} < \pi - \frac{2}{n^{1/2}}.$$

The statement follows.

Finally, consider $H_0^{(n)} = (c_{j+k+1})_{j,k=0}^{n-1}$. The case $n = 1$ is trivial. For $n \geq 2$, one shows, by minor modifications of the reasoning above, that

$$\sum_{k=0}^{n-1} \frac{1}{(j+k+1)(k+\frac{1}{2})^{1/2}} \leq \frac{1}{(j+\frac{1}{2})^{1/2}} \left(\pi - \frac{2}{(n+\frac{1}{2})^{1/2}} \right).$$

We omit the details. □

Since H_{-1} has negative entries, Method 2 does not apply to it directly. Recall, however, that $\|A^*A\| = \|A\|^2$ for any matrix A. It turns out that the method can be applied to $\tilde{H}_{-1}^* \tilde{H}_{-1}$. We include this, though the work is undeniably longer and less elegant than Method 1. We work with the doubly infinite matrix $\tilde{H}_{-1}$.

Proof that $\|\tilde{H}_{-1}\| \leq \pi$. We have $\tilde{H}_{-1}^* \tilde{H}_{-1} = (d_{j,k})$, where

$$d_{j,k} = \sum_{p \in \mathbb{Z}} c_{p-j} c_{p-k}$$

Clearly, $d_{k,j} = d_{j,k}$. Also,

$$d_{j,j} = \sum_{p \in \mathbb{Z},\, p \neq j} \frac{1}{(p-j)^2} = \frac{\pi^2}{3}. \tag{8}$$

Now let $k = j + r$, where $r > 0$. Then $d_{j,k} = S_1 + S_2 + S_3$, where

$$\begin{aligned} S_1 &= \sum_{p>k} \frac{1}{(p-j)(p-k)} = \sum_{q \geq 1} \frac{1}{q(q+r)} \qquad (\text{put } q = p-k) \\ &= \frac{1}{r} \sum_{q \geq 1} \left(\frac{1}{q} - \frac{1}{q+r} \right) \\ &= \frac{1}{r} \left(1 + \frac{1}{2} + \cdots + \frac{1}{r} \right), \end{aligned}$$

$$S_2 = \sum_{p<j} \frac{1}{(j-p)(k-p)} = \sum_{q \geq 1} \frac{1}{q(q+r)} = S_1 \qquad (\text{put } q = j-p),$$

$$\begin{aligned} S_3 &= \sum_{p=j+1}^{k-1} \frac{1}{(p-j)(p-k)} = -\sum_{q=1}^{r-1} \frac{1}{q(r-q)} \qquad (\text{put } q = p-j) \\ &= -\frac{1}{r} \sum_{q=1}^{r-1} \left(\frac{1}{q} + \frac{1}{r-q} \right) \\ &= -\frac{2}{r} \left(1 + \frac{1}{2} + \cdots + \frac{1}{r-1} \right). \end{aligned}$$

The cancellation of positive and negative terms leaves us with

$$d_{j,k} = \frac{2}{r^2} = \frac{2}{(k-j)^2}. \tag{9}$$

By (8) and (9), we have for each j

$$\sum_{k\in\mathbb{Z}} d_{j,k} = \frac{\pi^2}{3} + \frac{2\pi^2}{3} = \pi^2,$$

and the statement follows, by Theorem 5.1. □

Note that for the cancellation to occur, it was essential to consider the doubly infinite matrix. (Specifically, one can easily check that if $B = H_1^{(3)}$, then element $(1, 3)$ of B^*B is negative.) For the one-sided case, the proof can be seen to apply to $H_0^*H_0 + H_{-1}^*H_{-1}$ to recapture Proposition 3.2 (see [HLP, section 8.12]). However, it does not work for $H_0^*H_0$ and $H_{-1}^*H_{-1}$ separately, since again the cancellation is lost.

To finish this section, we consider the $n \times n$ matrix $(|c_{j-k}|)$ obtained by taking the moduli of the entries in $H_{-1}^{(n)}$. We see that the norm grows like $\log n$, serving a brisk reminder that the norm of a matrix may change drastically if the entries are replaced by their moduli!

Proposition 5.8. *Let M be the matrix $(|c_{j-k}|)_{j,k\geq 1}$ and $M^{(n)}$ the matrix $(|c_{j-k}|)_{1\leq j,k\leq n}$. Then $\log n \leq \|M^{(n)}\| \leq 2(\log n + 1)$ for $n \geq 2$, and M does not define a bounded operator on ℓ_2.*

Proof. Write $\sum_{k=1}^{n} |c_{j-k}| = r_j$ and $1 + \frac{1}{2} + \cdots + \frac{1}{n} = l_n$. By comparison with the integral of $1/t$, we have $\log(n + 1) \leq l_n \leq \log n + 1$. With a bit of thought, one sees that $r_1 = r_n = l_{n-1}$ and

$$r_j = l_{j-1} + l_{n-j} \qquad (2 \leq j \leq n - 1).$$

Clearly, $l_{n-j} > \frac{1}{j} + \cdots + \frac{1}{n-1}$, so $r_j \geq l_{n-1} \geq \log n$ for all j. By Proposition 5.3, with $w_j = 1$, we have $\|M^{(n)}\| \geq \log n$. This also shows that M is not a bounded operator on ℓ_2.

At the same time, we have $r_j \leq 2l_{n-1} < 2(\log n+1)$ for each j. Also, $M^{(n)}$ is symmetric, so $\|M^{(n)}\| \leq 2(\log n + 1)$ by Theorem 5.1. □

Note that M satisfies conditions (R) and (C) from section 2. By considering $x_j = 1/(j^{1/2} \log j)$, one can actually show that M does not map ℓ_2 into ℓ_2. See [HLP, p. 214].

It follows also that $\|M_+^{(n)}\| \geq \frac{1}{2} \log n$, where $M_+^{(n)}$ is the matrix obtained from H_{-1}^n by replacing c_{j-k} by 0 when $j < k$.

Example. The reader might care to check, rather more directly, that the matrix $[jk/(j + k)^2]_{j,k=1}^{\infty}$ does not map ℓ_2 into ℓ_2 (take $x_j = 1/j$).

6. The best constant in Hilbert's inequalities

We saw in section 4 how Method 1 can be used to show that $\|H_{-1}\| = \pi$, but only at the cost of relatively sophisticated properties of $L_2[0,1]$, and in a way that does not apply to H_1 or the finite truncations.

As we saw in section 3, any lower bound for $\|H_1\|$ will apply to the other Hilbert matrices. We now give a direct proof that $\|H_1\| = \pi$ using integral estimation (in fact, Lemma 5.6 again), at the same time obtaining a lower bound for $\|H_1^{(n)}\|$.

Theorem 6.1. *We have*

$$\|H_1\| = \|H_0\| = \|H_{-1}\| = \pi.$$

Also, there is a constant c such that $\|H_1^{(n)}\| \geq \pi - c/\log n$ for all $n > 2$.

Proof. Let $x_j = j^{-1/2}$ for $1 \leq j \leq n$. Then $\|x\|^2 = \sum_{j=1}^n \frac{1}{j} = l_n$. Let $y = H_1^{(n)} x$. By integral estimation,

$$y_j = \sum_{k=1}^n \frac{1}{(j+k)k^{1/2}} \geq \int_1^n \frac{1}{(t+j)t^{1/2}}\, dt.$$

Now

$$\int_0^1 \frac{1}{(t+j)t^{1/2}}\, dt \leq \frac{1}{j} \int_0^1 \frac{1}{t^{1/2}}\, dt = \frac{2}{j},$$

$$\int_n^\infty \frac{1}{(t+j)t^{1/2}}\, dt \leq \int_n^\infty \frac{1}{t^{3/2}}\, dt = \frac{2}{n^{1/2}}.$$

By these statements and Lemma 5.6,

$$y_j \geq \frac{\pi}{j^{1/2}} - \frac{2}{j} - \frac{2}{n^{1/2}},$$

so that

$$y_j^2 \geq \frac{\pi^2}{j} - 4\pi \left(\frac{1}{j^{3/2}} + \frac{1}{j^{1/2} n^{1/2}} \right),$$

and

$$\begin{aligned} \sum_{j=1}^n y_j^2 &\geq \pi^2 l_n - 4\pi\zeta(\tfrac{3}{2}) - \frac{4\pi}{n^{1/2}} \sum_{j=1}^n \frac{1}{j^{1/2}} \\ &\geq \pi^2 l_n - c_1, \end{aligned}$$

where $c_1 = 4\pi\zeta(\frac{3}{2}) + 8\pi$, since $\sum_{j=1}^n (1/j^{1/2}) < \int_0^n t^{-1/2}\, dt = 2n^{1/2}$. So

$$\|H_1^{(n)}\|^2 \geq \frac{\|y\|^2}{\|x\|^2} \geq \pi^2 - \frac{c_1}{l_n},$$

and hence $\|H_1^{(n)}\| \geq \pi - c/l_n$ for another constant c. □

There is visibly a gap between the upper and lower estimates for $\|H_1^{(n)}\|$ in Theorems 5.7 and 6.1. The exact value remains unknown. With a deeper analysis [Wilf, Theorem 2.2] obtains the estimation

$$\|H_1^{(n)}\| = \pi - \frac{\pi^5}{2(\log n)^2} + O\left(\frac{\log\log n}{(\log n)^3}\right).$$

7. Integrals of polynomials; positivity; another proof for H_0

Let $f(t)$ be the ordinary polynomial $\sum_{j=0}^{n} a_j t^j$ (with a_j real). Then

$$f(t)^2 = \sum_{j=0}^{n}\sum_{k=0}^{n} a_j a_k t^{j+k},$$

so

$$\int_0^1 f(t)^2\,dt = \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{a_j a_k}{j+k+1} = H_0^{(n+1)}(a,a). \tag{10}$$

In this section, we collect together several seemingly disparate results connected with this simple observation. Firstly, as a direct consequence of Hilbert's inequality for H_0, we have:

Proposition 7.1. *Let $f(t) = \sum_{j=0}^{n} a_j t^j$ (with a_j real). Then*

$$\int_0^1 f(t)^2\,dt \le \pi \sum_{j=0}^{n} a_j^2. \qquad \square$$

The same applies if $f(t) = \sum_{j=0}^{\infty} a_j t^j$, with $\sum_{j=0}^{\infty} a_j^2$ convergent. The series converges at least for $0 \le t < 1$, since (a_j) is bounded, and termwise integration, for purists, can be justified by Abel's theorem on continuity of power series.

Proposition 7.1, in turn, has the following application in terms of the *moments* of a function. By a neat twist, it delivers a *lower* bound for the integral of $f(t)^2$.

Proposition 7.2. *Let f be an integrable real function on $[0,1]$, and let $\mu_j = \int_0^1 t^j f(t)dt$ for $j \ge 0$. Then*

$$\sum_{j=0}^{\infty} \mu_j^2 \le \pi \int_0^1 f(t)^2\,dt.$$

Proof. It is enough to show that $\sum_{j=0}^{n} \mu_j^2$ satisfies the inequality for each n (thereby avoiding any convergence problems). We have

$$\sum_{j=0}^{n} \mu_j^2 \;=\; \sum_{j=0}^{n} \mu_j \int_0^1 t^j f(t)\,dt$$

$$= \int_0^1 f(t)g(t)\,dt,$$

where $g(t) = \sum_{j=0}^n \mu_j t^j$. By Proposition 7.1, $\int_0^1 g(t)^2\,dt \le \pi \sum_{j=0}^n \mu_j^2$. So, by the Cauchy-Schwarz inequality for integrals,

$$\begin{aligned}\left(\sum_{j=0}^n \mu_j^2\right)^2 &\le \int_0^1 f(t)^2\,dt \int_0^1 g(t)^2\,dt \\ &\le \pi \left(\sum_{j=0}^n \mu_j^2\right) \int_0^1 f(t)^2\,dt.\end{aligned}$$

Division by $\sum_{j=0}^n \mu_j^2$ gives the statement. □

Another conclusion is clear from (10): since $\int_0^1 f(t)^2\,dt \ge 0$, the matrix $H_0^{(n)}$ is positive. In fact, this statement extends to a much wider class of symmetric Hilbert-type matrices:

Proposition 7.3. *Let $\lambda_j > 0$ for $1 \le j \le n$, and let $r > 0$. Then the symmetric matrix $[1/(\lambda_j + \lambda_k)^r]$ is positive.*

Proof. Given real scalars $x_1, x_2, \dots, x_n$, let $f(t) = \sum_{j=1}^n x_j e^{-\lambda_j t}$, so that $f(t)^2 = \sum_{j=1}^n \sum_{k=1}^n x_j x_k e^{(\lambda_j+\lambda_k)t}$. Now

$$\int_0^\infty t^{r-1} e^{-\alpha t}\,dt = \frac{1}{\alpha^r} \int_0^\infty u^{r-1} e^{-u}\,du = \frac{\Gamma(r)}{\alpha^r},$$

so we have

$$0 \le \int_0^\infty t^{r-1} f(t)^2\,dt = \Gamma(r) \sum_{j=1}^n \sum_{k=1}^n \frac{x_j x_k}{(\lambda_j + \lambda_k)^r}. \qquad \square$$

Particular cases are: (i) $\lambda_j = j$, (ii) $\lambda_j = j - \frac{1}{2}\alpha$, where $\alpha \le 1$, giving $\lambda_j + \lambda_k = j + k - \alpha$.

Clearly, if we can prove Proposition 7.1 by another method, then this will amount to another proof that $\|H_0\| \le \pi$. Just such another method is provided by contour integration: this is the Fejér-Riesz proof [HLP, section 9.6]. We count this as our third method.

Proof that $\|H_0\| \le \pi$ by contour integration. Recall that $\|H_0^{(n+1)}\|$ is determined by quadratic forms $H_0^{(n+1)}(a, a)$ with real vectors $a = (a_j)$. Let $f(z) = \sum_{j=0}^n a_j z^j$, and let $I = \int_{-1}^1 f(x)^2\,dx$. Then $I \ge \int_0^1 f(x)^2\,dx$, which equals $H_0^{(n+1)}(a, a)$, by (10). Let C^+ and C^- be the upper and lower halves of the unit circle, both described anticlockwise. By Cauchy's integral theorem, $I = I_1 = I_2$, where

$$I_1 = -\int_{C^+} f(z)^2\,dz = -i \int_0^\pi f(e^{i\theta})^2\,d\theta,$$

$$I_2 = \int_{C^-} f(z)^2 \, dz = i \int_{-\pi}^{0} f(e^{i\theta})^2 \, d\theta.$$

Now $|I_1| \le \int_0^\pi |f(e^{i\theta})|^2 \, d\theta$ and similarly for $|I_2|$, so $2I \le \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta$. But (as in Method 1)

$$|f(e^{i\theta})|^2 = \sum_{j=0}^{n} a_j^2 + \sum_{j=0}^{n} \sum_{k \ne j} a_j a_k e^{i(j-k)\theta},$$

so $\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta = 2\pi \sum_{j=0}^{n} a_j^2$. □

It is interesting to compare this with Method 1. In both cases, we used the integral of $|f(t)|^2$ for a trigonometric polynomial f. Here we have used Cauchy's integral theorem to relate this to the integral of an ordinary polynomial, instead of considering the product with a function like $t - \frac{1}{2}$. However, the contour integration method does not readily adapt to cope with H_{-1}.

8. The matrix $[\operatorname{cosec} \frac{\pi}{n}(j-k)]$ and a fourth method for Hilbert's inequality

Let $S^{(n)}$ be the matrix $(s_{j,k})_{j,k=1}^{n}$, where

$$s_{j,k} = \begin{cases} \operatorname{cosec} \frac{\pi}{n}(j-k) & \text{for } j \ne k, \\ 0 & \text{for } j = k. \end{cases}$$

Like $H_{-1} = (c_{j-k})$, this is a skew-symmetric, Toeplitz matrix. Apart from the constant π/n, we have replaced x by $\sin x$. The matrix $S^{(4)}$ has $\sqrt{2}$, 1, $\sqrt{2}$ on the diagonals defined by $j - k = 1, 2, 3$ respectively (we save space by refraining from writing it out explicitly).

As we have seen, the exact norms of $H_1^{(n)}$ and $H_{-1}^{(n)}$ are not known for general n. Even when they are known, for small values of n, they are unpleasantly complicated numbers. In the light of these facts, it is very striking to find that $\|S^{(n)}\|$ is precisely the integer $n-1$. We shall prove this, and then show that Hilbert's inequality can be derived from it in elegant style – our fourth proof, not counting minor variants. Furthermore, this approach does something that the earlier ones failed to do: it gives an upper estimate for $\|H_{-1}^{(n)}\|$ depending on n. These results are due to K.R. Matthews [Matt].

Our evaluation of $\|S^{(n)}\|$ will be derived from its eigenvalues, as in Theorem 2.4. However, we are not relying on this theorem, because we will establish a similar result directly for another class of matrices to which $S^{(n)}$ belongs.

We say that an $n \times n$ matrix $S = (s_{j,k})$ is *skew-circulant* if $s_{j,k} = a_{k-j}$ for some sequence

(a_j) satisfying $a_{j-n} = -a_j$ for all j, so the matrix is of the form

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ -a_{n-1} & a_0 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ -a_1 & -a_2 & \cdots & a_0 \end{pmatrix}.$$

The elements a_j cycle round the rows, reappearing with a minus sign on the left of the diagonal. We prove a general result on eigenvectors of such matrices.

Recall that if $\omega = e(r/n)$, where $1 \le r \le n-1$, then $\omega^n = 1$, while $\omega \ne 1$, so, by the geometric series, $\sum_{k=1}^n \omega^k = 0$.

Proposition 8.1. *Let S be a skew-circulant $n \times n$ matrix, with $s_{j,k} = a_{k-j}$ as above. Let*

$$\rho_r = \exp \frac{(2r-1)\pi i}{n}.$$

Then S has mutually orthogonal eigenvectors x_r $(1 \le r \le n)$ defined by $x_r(k) = \rho_r^k$, forming a basis of ℓ_2^n. The corresponding eigenvalues are $\lambda_r = \sum_{k=1}^n a_k \rho_r^k$.

Proof. Choose one value of r, and write $\rho_r = \rho,\ x_r = x$. Note that $\rho^n = -1$. We have

$$(Sx)(j) = \sum_{k=1}^n a_{k-j}\rho^k = \rho^j \sum_{k=1}^n a_{k-j}\rho^{k-j}.$$

Write $g(k) = a_k \rho^k$. Clearly, $\sum_{k=j+1}^n g(k-j) = \sum_{k=1}^{n-j} g(k)$. Also, $g(k+n) = g(k)$, so $\sum_{k=1}^j g(k-j) = \sum_{k=n-j+1}^n g(k)$. Hence $\sum_{k=1}^n g(k-j) = \sum_{k=1}^n g(k)$. This sum is independent of j: denote it by λ. We have shown that $(Sx)(j) = \lambda\rho^j = \lambda x(j)$ for all j, so $S(x) = \lambda x$.

For $r \ne s$, we have $\rho_r \overline{\rho}_s = \omega$, where $\omega = e[(r-s)/n]$, so $\omega^n = 1$ and

$$\langle x_r, x_s \rangle = \sum_{k=1}^n \omega^k = 0. \qquad \square$$

Before applying this to the case we want, let us see an example of how it works in a simpler case. Let

$$S = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Let $\rho = e^{\pi i/3}$, so that $\rho^3 = -1$ and $1 - \rho + \rho^2 = 0$. The eigenvalues are

$$\lambda_1 = \rho + \rho^2 = \sqrt{3}i, \quad \lambda_2 = \rho^3 + \rho^6 = -1 + 1 = 0, \quad \lambda_3 = \rho^5 + \rho^{10} = -\rho^2 - \rho = -\sqrt{3}i,$$

(as seen in Example 2.1). Corresponding eigenvectors are $(\rho, \rho^2, 1)$, $(1, -1, 1)$ and $(\rho^2, \rho, 1)$.

Theorem 8.2. *Let $S^{(n)}$ be the matrix $(s_{j,k})_{j,k=1}^{n}$, where*

$$s_{j,k} = \begin{cases} \operatorname{cosec} \frac{\pi}{n}(j-k) & \text{for } j \neq k, \\ 0 & \text{for } j = k. \end{cases}$$

Then:

(i) *the eigenvalues of $S^{(n)}$ are $(n-2r+1)i$ $(1 \leq r \leq n)$;*

(ii) $\|S^{(n)}\| = n-1$.

Proof. The matrix $S^{(n)}$ is as in Proposition 8.1, with $a_j = -\operatorname{cosec}(\pi j/n)$ for $1 \leq j \leq n-1$ and $a_0 = 0$. First, we evaluate λ_1:

$$\lambda_1 = -\sum_{k=1}^{n-1} \operatorname{cosec} \frac{\pi k}{n} \rho_1^k = -\sum_{k=1}^{n-1} \operatorname{cosec} \frac{\pi k}{n} \exp \frac{\pi k i}{n}.$$

Substitute $n-k$ for k: since $\operatorname{cosec}(\pi - \theta) = \operatorname{cosec} \theta$ and $e^{i(\pi-\theta)} = -e^{-i\theta}$, we have

$$\lambda_1 = \sum_{k=1}^{n-1} \operatorname{cosec} \frac{\pi k}{n} \exp\left(-\frac{\pi k i}{n}\right).$$

Adding the two expressions, we see that

$$\lambda_1 = -i\sum_{k=1}^{n-1} \operatorname{cosec} \frac{\pi k}{n} \sin \frac{\pi k}{n} = -(n-1)i.$$

Now consider $\lambda_{r+1} - \lambda_r$, where $1 \leq r \leq n-1$. We have

$$\rho_{r+1}^k - \rho_r^k = \exp \frac{2rk\pi i}{n}. 2i \sin \frac{\pi k}{n} = 2i \sin \frac{\pi k}{n}\, e\left(\frac{rk}{n}\right),$$

so

$$\lambda_{r+1} - \lambda_r = -2i \sum_{k=1}^{n-1} e(rk/n) = 2i,$$

since, by the geometric series again, $\sum_{k=1}^{n-1} e(rk/n) = -1$. Hence $\lambda_{r+1} - \lambda_r = 2i$, and so $\lambda_r = (2r-n-1)i$ for $1 \leq r \leq n$ (equally, the eigenvalues can be listed as $(n-2r+1)i$ for $1 \leq r \leq n$). The largest in magnitude, determining $\|S^{(n)}\|$, are $\pm(n-1)i$. □

Without details, we mention the corresponding steps for cot instead of cosec. First, the analogue of Proposition 8.1 for a *circulant* matrix, which is proved in the same way:

Proposition 8.3. *Let T be the $n \times n$ matrix $(t_{j,k})$, where $t_{j,k} = a_{k-j}$ for some sequence (a_j) satisfying $a_{j-n} = a_j$ for all j. Write $\omega = e(1/n)$. Then mutually orthogonal eigenvectors x_r $(0 \leq r \leq n-1)$ are defined by $x_r(k) = \omega^{rk}$. The corresponding eigenvalues are $\lambda_r = \sum_{k=0}^{n-1} a_k \omega^{rk}$.*

Theorem 8.4. *Let $T^{(n)}$ be the matrix $(t_{j,k})_{j,k=1}^n$, where*

$$t_{j,k} = \begin{cases} \cot \frac{\pi}{n}(j-k) & \text{for } j \neq k, \\ 0 & \text{for } j = k. \end{cases}$$

Then

(i) *the eigenvalues of $T^{(n)}$ are 0 and $(n-2-2r)i \quad (0 \le r \le n-2)$;*

(ii) $\|T^{(n)}\| = n-2$.

Sketch of proof. It is easily seen that $\lambda_0 = 0$. One finds that

$$\begin{aligned} \lambda_1 &= -i \sum_{k=1}^{n-1} \cot \frac{\pi k}{n} \sin \frac{2\pi k}{n} \\ &= -2i \sum_{k=1}^{n-1} \cos^2 \frac{\pi k}{n} \\ &= -i \sum_{k=1}^{n-1} \left(1 + \cos \frac{2\pi k}{n}\right) \\ &= -(n-2)i, \end{aligned}$$

since $\sum_{k=1}^{n-1} \cos(2\pi k/n) = -1$. One shows further that $\lambda_{r+1} - \lambda_r = 2i$ for $1 \le r \le n-2$. (When n is even, the eigenvalue 0 occurs twice.) □

We now give the promised derivation of Hilbert's inequality from Theorem 8.2, with an estimation of $\|H_{-1}^{(n)}\|$.

Theorem 8.5. *For each $n > 1$, we have*

$$\|H_{-1}^{(n)}\| \le \pi \left(1 - \frac{1}{n}\right),$$

and hence $\|H_{-1}\| \le \pi$.

Proof. Consider the quadratic form $H_{-1}^{(n)}(x) = \sum_{j=1}^n \sum_{k=1}^n c_{j-k} x_j \overline{x}_k$. Let $s_{j,k}$ be as in Theorem 8.2. For $j \neq k$, let $c_{j-k} = s_{j,k} d_{j,k}$, so that

$$d_{j,k} = \frac{1}{j-k} \sin \frac{\pi(j-k)}{n}.$$

Now

$$\int_{-1/2}^{1/2} e(\lambda t)\, dt = \frac{1}{2\pi i \lambda}(e^{\pi i \lambda} - e^{-\pi i \lambda}) = \frac{\sin \pi\lambda}{\pi\lambda}.$$

Hence

$$d_{j,k} = \frac{\pi}{n} \int_{-1/2}^{1/2} e\left(\frac{j-k}{n} t\right)\, dt.$$

For completeness (though it is not really needed), put $d_{j,j} = \pi/n$. Then

$$H_{-1}^{(n)}(x) = \frac{\pi}{n} \int_{-1/2}^{1/2} F(t)\, dt,$$

where

$$F(t) = \sum_{j=1}^{n} \sum_{k=1}^{n} s_{j,k}\, x_j \overline{x}_k\, e\left(\frac{j-k}{n} t\right) = \sum_{j=1}^{n} \sum_{k=1}^{n} s_{j,k} y_j(t) \overline{y}_k(t),$$

where $y_j(t) = x_j\, e(jt/n)$. By Theorem 8.2,

$$|F(t)| \le (n-1) \sum_{j=1}^{n} |y_j(t)|^2 = (n-1) \sum_{j=1}^{n} |x_j|^2,$$

so

$$|H_{-1}^{(n)}(x)| \le \frac{\pi}{n} (n-1) \sum_{j=1}^{n} |x_j|^2. \qquad \square$$

As with $H_1^{(n)}$, an exact evaluation of the norm of $H_{-1}^{(n)}$ is elusive. H.L. Montgomery has stated in private correspondence that $\pi - \|H_{-1}^{(n)}\|$ is $O(\log n/n)$, but it would appear that no proof has ever been published.

Parting thought. It is not really surprising that $\|S^{(n)}\|$ and $\|T^{(n)}\|$ are pleasant while $\|H_1^{(n)}\|$ and $\|H_{-1}^{(n)}\|$ are unpleasant. The matrices $S^{(n)}$ and $T^{(n)}$ are constructed from what might be called a complete system, equally spaced values in one cycle of a periodic function. The numbers $1/(j-k)$ only form a complete system when the whole set of integers is included, and the matrices $H_1^{(n)}$ and $H_{-1}^{(n)}$ represent unnatural truncations of this system.

9. The generalizations by Montgomery and Vaughan

Two powerful generalizations were established by Montgomery and Vaughan [MontV] in 1974. They deal, respectively, with matrices of the form $[1/(\lambda_j - \lambda_k)]$ (generalizing Hilbert's inequality) and $[\operatorname{cosec} \pi(\lambda_j - \lambda_k)]$ (generalizing Theorem 8.2), with the numbers λ_j no longer equally spaced. Both results have applications in analytic number theory.

It is possible to prove either result first and deduce the other, or, with sufficiently careful formulation, to prove both together. In [MontV], the result for $[\operatorname{cosec} \pi(\lambda_j - \lambda_k)]$ is proved first. Here, following [Mont1], we will prove the result for $[1/(\lambda_j - \lambda_k)]$ first. The statement is as follows.

Theorem 9.1. *Suppose that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $\lambda_j - \lambda_{j-1} \geq \delta$ for each j. Let $G^{(n)}$ be the matrix $(g_{j,k})_{j,k=1}^n$, where*

$$g_{j,k} = \begin{cases} 1/(\lambda_j - \lambda_k) & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases}$$

Then $\|G^{(n)}\| \leq \pi/\delta$.

First, some preliminary comments. The matrix is still skew-symmetric, but no longer Toeplitz. Of course, a proof of Theorem 9.1 will amount to a fifth proof of Hilbert's inequality.

Also, let us dispose of a tempting fallacy. One might suppose that $\|G^{(n)}\| \leq \|H_{-1}^{(n)}\|$ simply because all the entries are smaller, while still the same sign. Example 2.1 shows that this does occur in the case $n = 3$. However, it is not true in general, as the following example shows.

Example 9.1. Consider the skew-symmetric matrix

$$M_b = \begin{pmatrix} 0 & 1 & 0 & b \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -b & 0 & -1 & 0 \end{pmatrix}.$$

One finds that

$$\det(M_b - \lambda I_4) = \lambda^4 + (b^2 + 3)\lambda^2 + (b^2 + 2b + 1).$$

For $b = 1$, this becomes $\lambda^4 + 4\lambda^2 + 4$, and we deduce that $\|M_1\| = \sqrt{2}$. For $b = 0$, it becomes $\lambda^4 + 3\lambda^2 + 1$, and we find that $\|M_0\| = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$.

Proof of Theorem 9.1. After multiplying by a constant, we can assume that $\delta = 1$. Then $\lambda_j - \lambda_k \geq j - k$, and hence $0 < g_{j,k} \leq 1/(j - k)$ for $j > k$. In the following, all sums are from 1 to n, sometimes with one or two elements left out as indicated. Write G for $G^{(n)}$. Then

$$\|Gx\|^2 = \langle G^*Gx, x\rangle = \sum_j \sum_k d_{j,k} x_j \overline{x}_k,$$

where

$$d_{j,k} = \sum_p g_{p,j} g_{p,k}.$$

So $\|Gx\|^2 = S_1(x) + S_2(x)$, where

$$S_1(x) = \sum_j d_{j,j} |x_j|^2,$$

$$S_2(x) = \sum_j \sum_{k \neq j} d_{j,k} x_j \overline{x}_k.$$

Now

$$d_{j,j} = \sum_p g_{p,j}^2 \le \sum_{p\ne j} \frac{1}{(p-j)^2} \le \frac{\pi^2}{3},$$

so $S_1(x) \le \frac{\pi^2}{3} \sum_j |x_j|^2$.

We know that $\|G\|$ is attained at an eigenvector x, and that $Gx = i\mu x$, where μ is real. We will show that for such x,

$$S_2(x) = 2 \sum_j \sum_k g_{j,k}^2 x_j \overline{x}_k. \tag{11}$$

Assume this for the moment. Since $|2x_j\overline{x}_k| \le |x_j|^2 + |x_k|^2$, we then have

$$\begin{aligned} |S_2(x)| &\le \sum_j \sum_k g_{j,k}^2 (|x_j|^2 + |x_k|^2) \\ &= \sum_j d_{j,j}|x_j|^2 + \sum_k d_{k,k}|x_k|^2 \\ &\le \frac{2\pi^2}{3} \sum_j |x_j|^2, \end{aligned}$$

so that $\|Gx\|^2 \le \pi^2 \sum_j |x_j|^2$.

It remains to prove (11). For distinct j, k, p, we have

$$\frac{1}{(\lambda_p - \lambda_j)(\lambda_p - \lambda_k)} = \frac{1}{\lambda_j - \lambda_k} \left(\frac{1}{\lambda_p - \lambda_j} - \frac{1}{\lambda_p - \lambda_k} \right),$$

in other words, $g_{p,j} g_{p,k} = g_{j,k}(g_{p,j} - g_{p,k})$. So for $j \ne k$, we have $d_{j,k} = g_{j,k} t_{j,k}$, where

$$\begin{aligned} t_{j,k} &= \sum_{\substack{p\ne j \\ p\ne k}} (g_{p,j} - g_{p,k}) \\ &= \sum_p g_{p,j} - g_{k,j} - \sum_p g_{p,k} + g_{j,k} \\ &= \sum_p g_{p,j} - \sum_p g_{p,k} + 2g_{j,k}, \end{aligned}$$

since $g_{j,j} = 0$ and $g_{k,j} = -g_{j,k}$. So $S_2(x) = T_1 - T_2 + T_3$, where

$$T_1 = \sum_j \sum_{k\ne j} g_{j,k} \sum_p g_{p,j} x_j \overline{x}_k, \qquad T_2 = \sum_j \sum_{k\ne j} g_{j,k} \sum_p g_{p,k} x_j \overline{x}_k,$$

$$T_3 = 2 \sum_j g_{j,k}^2 x_j \overline{x}_k.$$

We can drop the restriction $k \ne j$, since $g_{j,j} = 0$. Since $\sum_k g_{j,k} x_k = (Gx)(j) = i\mu x_j$, we have

$$T_1 = \sum_j x_j \sum_k g_{j,k} \overline{x}_k \sum_p g_{p,j} = -i\mu \sum_j x_j \overline{x}_j \sum_p g_{p,j}.$$

Similarly, since $\sum_j g_{j,k} x_j = -\sum_j g_{k,j} x_j = -i\mu x_k$,

$$T_2 = \sum_k \overline{x}_k \sum_j g_{j,k} x_j \sum_p g_{p,k} = -i\mu \sum_k \overline{x}_k x_k \sum_p g_{p,k} = T_1.$$

Identity (11) follows. □

Note 1. The cancellation $T_1 = T_2$ is described in [Mont2] as "marvelous". For a general x, not an eigenvector, the relationship is $T_2 = -\overline{T}_1$, so equality depends on the real part being zero.

Note 2. The proof delivers a stronger estimate for the finite-dimensional truncation $G^{(n)}$. Indeed (still with $\delta = 1$), it shows that

$$\|G^{(n)}\|^2 \le 3 \sup_{1 \le p \le n} \sum_{p \ne j} \frac{1}{(p-j)^2}.$$

By integral estimation, one finds that this supremum is not greater than $\pi^2 - 12/(n+1)$ (we omit the details), and hence that

$$\|G^{(n)}\| \le \pi - \frac{6}{\pi(n+1)}.$$

For $\|H_{-1}^{(n)}\|$, this is a little weaker than the estimate $\pi(1 - \frac{1}{n})$ in Theorem 8.5.

Note 3. The statement still holds if the numbers λ_j are not listed in increasing order, but satisfy $|\lambda_j - \lambda_k| \ge \delta$ for $j \ne k$. This amounts to re-ordering the columns, which of course does not alter the norm of the matrix.

Note 4. This proof might seem a little reminiscent of Method 2 for $\tilde{H}_{-1}$. However, it is really quite different. The cancellation there depended on $(j+r) - (k+r) = j - k$, which does not work with j replaced by λ_j. Also, as we saw, it required doubly infinite sequences in an essential way.

An obvious special case of Theorem 9.1 is:

Corollary 9.2. *Let $L^{(n)}$ be the matrix $(l_{j,k})_{j,k=1}^n$, where*

$$l_{j,k} = \begin{cases} 1/(\log j - \log k) & \text{if } j \ne k, \\ 0 & \text{if } j = k. \end{cases}$$

Then $\|L^{(n)}\| \le n\pi$.

Proof. The smallest of the differences $\log j - \log k$ is $\log n - \log(n-1) > \frac{1}{n}$. □

We pause here to give a second proof by a development of Method 1; a version of it is given in [Iv]. It is somewhat simpler, but it only delivers a slightly weaker result, with an intervening constant which happens to be $2/\sqrt{3}$.

Proof of Theorem 9.1 with an extra constant. Consider the quadratic form $G(x) = \sum_{j=1}^n \sum_{k=1}^n g_{j,k} x_j \overline{x}_k$. Note that $G(x)$ is of the form $i\sigma$, where σ is real, and $G(\overline{x}) = -G(x)$.

Let $f(t) = \sum_{j=1}^n x_j e(\lambda_j t)$ (such functions are called *generalized Dirichlet polynomials*). Then

$$|f(t)|^2 = \sum_{j=1}^n |x_j|^2 + \sum_{j=1}^n \sum_{k \neq j} x_j \overline{x}_k e[(\lambda_j - \lambda_k)t]. \tag{12}$$

With $c > 0$ to be chosen later, we will use the fact that $I \geq 0$, where

$$I = \int_0^c (c-t)|f(t)|^2 \, dt.$$

(The use of $c-t$ instead of t makes the following work slightly simpler.) Where we previously had $\int_0^1 e(rt)\, dt$ for integers r, we now have, for $\lambda \neq 0$,

$$\int_0^c e(\lambda t)\, dt = \frac{e(\lambda c) - 1}{2\pi i \lambda}.$$

Also, $\int_0^c (c-t)\, dt = \frac{1}{2}c^2$, and

$$\begin{aligned} \int_0^c (c-t) e(\lambda t)\, dt &= -\frac{c}{2\pi i \lambda} + \frac{1}{2\pi i \lambda} \int_0^c e(\lambda t)\, dt \\ &= -\frac{c}{2\pi i \lambda} - \frac{e(\lambda c) - 1}{4\pi^2 \lambda^2}. \end{aligned}$$

So

$$I = \tfrac{1}{2}c^2 \sum_{j=1}^n |x_j|^2 - \frac{c}{2\pi i} G(x) + \frac{R}{4\pi^2},$$

where

$$R = \sum_{j=1}^n \sum_{k \neq j} \frac{x_j \overline{x}_k}{(\lambda_j - \lambda_k)^2} [1 - e(\lambda_j c - \lambda_k c)].$$

So

$$|R| \leq 2 \sum_{j=1}^n \sum_{k \neq j} \frac{|x_j x_k|}{(\lambda_j - \lambda_k)^2} \leq 2 \sum_{j=1}^n \sum_{k \neq j} \frac{|x_j x_k|}{(j-k)^2}.$$

By Proposition 4.5 (or 5.4),

$$|R| \leq \frac{2\pi^2}{3} \sum_{j=1}^n |x_j|^2,$$

so $R/(4\pi^2) \leq \frac{1}{6} \sum_{j=1}^n |x_j|^2$, and the inequality $I \geq 0$ translates into

$$\frac{c}{2\pi i} G(x) \leq \left(\tfrac{1}{2}c^2 + \tfrac{1}{6}\right) \sum_{j=1}^n |x_j|^2,$$

This applies equally to $G(\overline{x}) = -G(x)$, so in fact it applies to $|S(x)|$, giving

$$|G(x)| \leq \pi \left(c + \frac{1}{3c}\right) \sum_{j=1}^n |x_j|^2.$$

To minimize $c + 1/3c$, take $c = 1/\sqrt{3}$, giving $c + 1/3c = 2/\sqrt{3}$. (Note that if we simplified the proof by taking $c = 1$, the constant obtained would be $4/3$.) □

An immediate application of Theorem 9.1 is the following estimation for the integrals of generalized Dirichlet polynomials, which we used in the second proof.

Theorem 9.3. *Suppose that* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ *and* $\lambda_j - \lambda_{j-1} \geq \delta$ *for each* j. *Let* $f(t) = \sum_{j=1}^n x_j e(\lambda_j t)$. *Then, for any* $T > 0$, *we have*

$$\int_0^T |f(t)|^2 \, dt = \left(T + \frac{\theta}{\delta} \right) \sum_{j=1}^n |x_j|^2$$

for some θ *with* $|\theta| \leq 1$.

Proof. By (12), it is clear that $\int_0^T |f(t)|^2 \, dt = T \sum_{j=1}^n |x_j|^2 + r(T)$, where

$$r(T) = \frac{1}{2\pi i} \sum_{j=1}^n \sum_{k \neq j} x_j \overline{x}_k \frac{e[(\lambda_j - \lambda_k)T] - 1}{\lambda_j - \lambda_k}.$$

Write $x_j e(\lambda_j T) = y_j$. Then (with $G(x)$ as in the second proof of Theorem 9.1)

$$2\pi i r(T) = \sum_{j=1}^n \sum_{k \neq j} \frac{y_j \overline{y}_k - x_j \overline{x}_k}{\lambda_j - \lambda_k} = G(y) - G(x).$$

By Theorem 9.1, $|G(x)|$ and $|G(y)|$ are not greater than $(\pi/\delta) \sum_{j=1}^n |x_j|^2$. The statement follows. □

We can read off the following corollary for an "ordinary" Dirichlet polynomial $F(s) = \sum_{j=1}^n x_j/j^s$.

Corollary 9.4. *Let* $F(s) = \sum_{j=1}^n x_j/j^s$. *Then for some* θ *with* $|\theta| \leq 1$,

$$\int_0^T |F(it)|^2 \, dt = (T + 2\pi\theta n) \sum_{j=1}^n |x_j|^2,$$

Proof. We have $j^{-it} = e(-\lambda_j t)$, where $\lambda_j = (\log j)/2\pi$. As in Corollary 9.2, $\lambda_j - \lambda_{j-1} \geq 1/(2\pi n)$ for $1 \leq j \leq n$. □

We now come to the second theorem of Montgomery and Vaughan, applying to matrices of the form $[\operatorname{cosec} \pi(\lambda_j - \lambda_k)]$. Clearly, $\lambda_j - \lambda_k$ needs to be not only non-zero, but not an integer. The relevant measure of separation is now the distance from the nearest integer. We introduce the following notation. For any real number λ, let $\Delta(\lambda)$ be this distance:

$$\Delta(\lambda) = \inf\{|\lambda - n| : n \in \mathbb{Z}\}.$$

(The notation $\|\lambda\|$ is often used for this, but we are already using this notation for norms of vectors and matrices.)

Theorem 9.5. *Let λ_j $(1 \le j \le n)$ be real numbers such that $\Delta(\lambda_j - \lambda_k) \ge \delta$ whenever $j \ne k$, and let H be the matrix defined by*

$$h_{j,k} = \begin{cases} \operatorname{cosec}\,[\pi(\lambda_j - \lambda_k)] & \text{if } j \ne k, \\ 0 & \text{if } j = k. \end{cases}$$

Then $\|H\| \le 1/\delta$.

The same conclusion applies if $\operatorname{cosec}\,[\pi(\lambda_j - \lambda_k)]$ *is replaced by* $\cot[\pi(\lambda_j - \lambda_k)]$.

Proof. We shall deduce the result from Theorem 9.1. We use the well-known identity

$$\pi \operatorname{cosec} \pi\lambda = \lim_{n\to\infty} \sum_{m=-n}^{n} \left(1 - \frac{|m|}{n}\right) \frac{(-1)^m}{\lambda + m}, \tag{13}$$

which can be proved by applying Fejér's convergence theorem to the Fourier series for $\cos \lambda t$.

Choose $n > 1$ and scalars x_j. For $1 \le j \le n$ and $1 \le r \le n$, define

$$\lambda_{j,r} = \lambda_j + r,$$

$$x_{j,r} = (-1)^r x_j.$$

If $(j, r) \ne (k, s)$, then

$$|\lambda_{j,r} - \lambda_{k,s}| = |\lambda_j - \lambda_k - (s - r)| \ge \delta,$$

(if $j = k$, then $r \ne s$, so $|r - s| \ge 1$). Let

$$S_n(x) = \sum_{(j,r)\ne(k,s)} \frac{x_{j,r}\overline{x}_{k,s}}{\lambda_{j,r} - \lambda_{k,s}}$$

(the summation is over all j, k, r, s such that $(j, r) \ne (k, s)$). By Theorem 9.1, applied to the scalars $\lambda_{j,r}$,

$$|S_n(x)| \le \frac{\pi}{\delta} \sum_{j=1}^{n} \sum_{r=1}^{n} |x_{j,r}|^2 = \frac{\pi n}{\delta} \sum_{j=1}^{n} |x_j|^2.$$

Now

$$S_n(x) = \sum_{(j,r)\ne(k,s)} \frac{(-1)^{r+s} x_j \overline{x}_k}{\lambda_j - \lambda_k + (r - s)}.$$

For fixed j, the $[(j, r), (j, s)]$ terms combine to

$$|x_j|^2 \sum_{r=1}^{n} \sum_{s\ne r} \frac{(-1)^{r+s}}{r - s} = 0,$$

so

$$S_n(x) = \sum_{j=1}^{n} \sum_{k\neq j} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{(-1)^{r+s} x_j \overline{x}_k}{\lambda_j - \lambda_k + (r-s)}.$$

For a chosen m with $-n \leq m \leq n$, there are $n - |m|$ (equal) terms with $r - s = m$ (e.g. if $m \geq 0$, these are given by $1 \leq s \leq n - m$ with $r = s + m$). Hence

$$S_n(x) = \sum_{j=1}^{n} \sum_{k\neq j} \sum_{m=-n}^{n} (n - |m|) \frac{(-1)^m x_j \overline{x}_k}{\lambda_j - \lambda_k + m}.$$

By (13),

$$\sum_{j=1}^{n} \sum_{k\neq j} \operatorname{cosec} \pi(\lambda_j - \lambda_k) x_j \overline{x}_k = \lim_{n\to\infty} \frac{1}{n} S_n(x),$$

and therefore the modulus of this expression is not greater than $(1/\delta)\sum_{j=1}^{n} |x_j|^2$.

Finally, if (13) is modified by removing the term $(-1)^m$, the resulting expression equals $\pi \cot \pi\lambda$. The proof, simplified by removing $(-1)^m$ throughout, shows that cosec can be replaced by cot in the statement. □

Alternatively, one can prove Theorem 9.5 directly by a modification of the first proof of Theorem 9.1. One can then deduce Theorem 9.1 quite simply, as in the proof of Theorem 8.5.

[MontV] also proved the following variants, in which the overall spacing δ is replaced by the spacing δ_j separating λ_j from the other points.

Theorem 9.6. *(i) Let $g_{j,k}$ be as in Theorem 9.1, and suppose that $|\lambda_k - \lambda_j| \geq \delta_j$ for $k \neq j$. Then*

$$\left| \sum_{j=1}^{n} \sum_{k=1}^{n} g_{j,k} x_j \overline{x}_k \right| \leq \frac{3\pi}{2} \sum_{j=1}^{n} \frac{|x_j|^2}{\delta_j}.$$

(ii) Let $h_{j,k}$ be as in Theorem 9.5, with $\Delta(\lambda_k - \lambda_j) \geq \delta_j$ for $k \neq j$. Then

$$\left| \sum_{j=1}^{n} \sum_{k=1}^{n} h_{j,k} x_j \overline{x}_k \right| \leq \frac{3}{2} \sum_{j=1}^{n} \frac{|x_j|^2}{\delta_j}.$$

In the case $g_{j,k} = 1/(\log j - \log k)$, the right-hand side in (i) becomes $\frac{3}{2}\pi \sum_{j=1}^{n} (j+1)|x_j|^2$.

The factor $\frac{3}{2}$ is certainly not optimal, but it is not known whether it can be replaced by 1.

10. Applications to the large sieve inequality and Dirichlet polynomials

Consider again a trigonometric polynomial $f(t) = \sum_{j=1}^{n} x_j e(jt)$, as used in the first proof of Hilbert's inequality. We now address the problem of finding a bound for sums of squares of values. Such bounds are called "large value estimates", since they set a limit on the number of times large values occur. More exactly, we want a constant C such that

$$\sum_{r=1}^{R} |f(t_r)|^2 \le C \sum_{j=1}^{n} |x_j|^2. \tag{14}$$

Since f has period 1, the values would simply reinforce if the points t_r differed by integers, so we require the points t_r to be "well separated" in the sense that $\Delta(t_r - t_s) \ge \delta$ for $r \ne s$. The constant C is to depend on n and δ, but not R. Those not already soaked in the culture of analytic number theory will find it somewhat intriguing that the answer is known as the "large sieve inequality".

The problem is really another exercise in determining the norm of a matrix. Indeed, since

$$f(t_r) = \sum_{j=1}^{n} e(jt_r)x_j,$$

the constant C in (14) is $\|V\|^2$, where V is the matrix $[e(jt_r)]$ $(1 \le j \le n,\ 1 \le r \le R)$.

A minor remark will be useful. The modulus of $f(t)$ is unchanged if it is multiplied by $e(kt)$, putting it into the form $\sum_{j=k+1}^{k+n} y_j e(jt)$. So the problem is the same if the range of values of j is translated by k: the role of n is to state the length of this range.

Of course, *mean* values of $|f(t)|^2$ are measured by its integral, which we know very well: $\int_0^1 |f(t)|^2 dt = \sum_{j=1}^{n} |x_j|^2$. The following lemma shows how to convert estimates for integrals into estimates of functional values.

Lemma 10.1. *Let g be a differentiable function (real or complex) on $[c-h, c+h]$. Then*

$$|g(c)| \le \frac{1}{2h}\int_{c-h}^{c+h} |g(t)|\,dt + \frac{1}{2}\int_{c-h}^{c+h} |g'(t)|\,dt.$$

Proof. Let

$$\rho(t) = \begin{cases} t-c+h & \text{for } c-h < t < c, \\ t-c-h & \text{for } c < t < c+h. \end{cases}$$

Integration by parts on the intervals $[c-h, c]$ and $[c, c+h]$ leads easily to

$$\int_{c-h}^{c+h} \rho(t)g'(t)\,dt = 2hg(c) - \int_{c-h}^{c+h} g(t)\,dt.$$

Since $|\rho(t)| \le h$, the statement follows. □

Using this lemma, we deduce the following provisional solution to our problem. The method is due to Gallagher [Gall].

Proposition 10.2. *Let $f(t) = \sum_{j=1}^{n} x_j e(jt)$. Let t_r $(1 \le r \le R)$ be points such that $\Delta(t_r - t_s) \ge \delta$ for $r \ne s$. Then*

$$\sum_{r=1}^{R} |f(t_r)|^2 \le \left(\pi n + \frac{1}{\delta}\right) \sum_{j=1}^{n} |x_j|^2.$$

Proof. As mentioned above, we can translate the range of j. In particular, we can move it to an interval J contained in $[-\frac{1}{2}n, \frac{1}{2}n]$, now taking $f(t)$ to be $\sum_{j\in J} x_j e(jt)$, with J as stated. Also, we may assume that $t_1 < t_2 < \cdots < t_R$, with $t_r - t_{r-1} \ge \delta$, and $t_1 + 1 \ge t_R + \delta$, so that if $t_1 - \frac{1}{2}\delta = c$, then $t_R + \frac{1}{2}\delta \le 1 + c$. So the intervals $[t_r - \frac{1}{2}\delta, t_r + \frac{1}{2}\delta]$ do not overlap, and are contained in $[c, 1+c]$. By Lemma 10.1, applied to $f(t)^2$,

$$\sum_{r=1}^{R} |f(t_r)|^2 \le \frac{1}{\delta} \int_c^{1+c} |f(t)|^2 \, dt + \int_c^{1+c} |f(t) f'(t)| \, dt.$$

Now $\int_c^{1+c} |f(t)|^2 dt = \sum_{j\in J} |x_j|^2$. Also, since $f'(t) = 2\pi i \sum_{j\in J} j x_j e(jt)$ and $|j| \le n/2$ for $j \in J$, we have

$$\int_c^{1+c} |f'(t)|^2 \, dt = 4\pi^2 \sum_{j\in J} j^2 |x_j|^2 \le \pi^2 n^2 \sum_{j\in J} |x_j|^2.$$

By the Cauchy-Schwarz inequality for integrals,

$$\int_c^{1+c} |f(t) f'(t)| \, dt \le \pi n \sum_{n\in J} |x_j|^2.$$

The statement follows. □

This result was obtained without using anything resembling Hilbert's inequality. For many purposes, it is quite good enough. However, it is not optimal. We show next that Theorem 9.5 is exactly what is needed to produce a better estimate, which in fact turns out to be optimal.

Theorem 10.3. *Let t_r $(1 \le r \le R)$ be points such that $\Delta(t_r - t_s) \ge \delta$ for $r \ne s$, and let V be the matrix defined by $v_{r,j} = e(jt_r)$ for $1 \le j \le n$, $1 \le r \le R$. Then*

$$\|V\|^2 \le (n-1) + \frac{1}{\delta}.$$

Consequently, if $f(t) = \sum_{j=1}^{n} x_j e(jt)$, then

$$\sum_{r=1}^{R} |f(t_r)|^2 \le \left(n - 1 + \frac{1}{\delta}\right) \sum_{j=1}^{n} |x_j|^2.$$

Proof. We evaluate the norm of the transposed matrix (which, of course, is the same). Given scalars y_r, let

$$T(y) = \sum_{j=1}^{n} \left| \sum_{r=1}^{R} e(jt_r) y_r \right|^2 .$$

Then $\|V\|^2$ is the least constant C for which we have $T(y) \le C \sum_{r=1}^{R} |y_r|^2$ for all choices of y_r. Note first that, by the geometric series,

$$\sum_{j=1}^{n} e(jt) = e(t) \frac{e(nt) - 1}{e(t) - 1} = \frac{e[(n + \frac{1}{2})t] - e(\frac{1}{2}t)}{2i \sin \pi t}.$$

Hence we have

$$\begin{aligned} T(y) &= \sum_{j=1}^{n} \sum_{r=1}^{R} \sum_{s=1}^{R} y_r \overline{y}_s e[j(t_r - t_s)] \\ &= n \sum_{r=1}^{R} |y_r|^2 + \sum_{r \ne s} y_r \overline{y}_s \sum_{j=1}^{n} e[j(t_r - t_s)] \\ &= n \sum_{r=1}^{R} |y_r|^2 + \sum_{r \ne s} y_r \overline{y}_s \frac{e[(n + \frac{1}{2})(t_r - t_s)] - e[\frac{1}{2}(t_r - t_s)]}{2i \sin \pi (t_r - t_s)}, \end{aligned}$$

in which $\sum_{r \ne s}$ means summation over all pairs (r, s) with $r \ne s$. Now for any a,

$$\sum_{r \ne s} y_r \overline{y}_s \frac{e[a(t_r - t_s)]}{2 \sin \pi (t_r - t_s)} = \sum_{r \ne s} \frac{z_r \overline{z}_s}{2 \sin \pi (t_r - t_s)},$$

where $z_r = y_r e(at_r)$. Since $|z_r| = |y_r|$, Theorem 9.5 shows that the modulus of this expression is not greater than $(1/2\delta) \sum_{r=1}^{R} |y_r|^2$. Apply this with $a = n + \frac{1}{2}$ and $a = \frac{1}{2}$ to obtain

$$|T(y)| \le \left(n + \frac{1}{\delta} \right) \sum_{r=1}^{R} |y_r|^2.$$

This completes the proof, apart from showing that n can be replaced by $n - 1$ (a refinement that is quite unimportant in applications). This was first shown by Selberg, but the following neat proof was given by Paul Cohen. Choose $K > 1$, and let

$$g(t) = f(Kt) = \sum_{j=1}^{n} x_j e(jKt),$$

which we can write as $\sum_{k=K}^{Kn} y_k e(kt)$, in which $\sum_{k=K}^{Kn} |y_k|^2 = \sum_{j=1}^{n} |x_j|^2$. Since f has period 1,

$$K \sum_{r=1}^{R} |f(t_r)|^2 = \sum_{k=1}^{K} \sum_{r=1}^{R} |f(t_r + k)|^2 = \sum_{k=1}^{K} \sum_{r=1}^{R} \left| g\left(\frac{t_r + k}{K} \right) \right|^2 .$$

The numbers $(t_r + k)/K$ are separated by δ/K, so, by the result already proved,

$$K \sum_{r=1}^{R} |f(t_r)|^2 \leq \left((Kn - K + 1) + \frac{K}{\delta} \right) \sum_{j=1}^{n} |x_j|^2.$$

Now divide by K and let K tend to infinity to obtain the result. □

A trivial example is enough to show that the expression in Theorem 10.3 is optimal, within the rules we have set for ourselves.

Example. Take $f(t) = e(0) + e(2t)$, so that $n = 3$ and $x_0 = 1$, $x_1 = 0$ and $x_2 = 1$. Let $t_1 = 0$ and $t_2 = \frac{1}{2}$, so that $\delta = \frac{1}{2}$. Then $f(t_1) = f(t_2) = 2$, so $f(t_1)^2 + f(t_2)^2 = 8$, while $\sum_{j=0}^{2} x_j^2 = 2$. The ratio is 4, equal to $(n-1) + 1/\delta$.

A beginner's account of applications to number theory, and an explanation of the term "large sieve", is given in my companion website notes [Jam1]. More extended accounts can be seen in [Mont1], [Mont2], [Dav], [Ten].

We now address the problem of a "large values" estimate for a generalized Dirichlet polynomial $f(t) = \sum_{j=1}^{n} x_j e(\lambda_j t)$. The integral of $|f(t)|^2$ is no longer trivial: it was estimated in Theorem 9.3. We used the first Montgomery-Vaughan theorem (Theorem 9.1) to get this far. We now imitate the proof of Proposition 10.2 to derive the required estimate. There are now two sets of numbers that need to be well separated, (λ_j) and (t_r).

Theorem 10.4. *Suppose that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ with $\lambda_j - \lambda_{j-1} \geq \delta$ and $|\lambda_j| \leq M$ for all j. Let $f(t) = \sum_{j=1}^{n} x_j e(\lambda_j t)$. Also, let $t_1 < t_2 < \cdots < t_R$, with $t_r - t_{r-1} \geq h$ for each r, with $t_1 \geq \frac{1}{2}h$ and $t_R \leq T - \frac{1}{2}h$. Then*

$$\sum_{r=1}^{R} |f(t_r)|^2 \leq \left(2\pi M + \frac{1}{h} \right) \left(T + \frac{1}{\delta} \right) \sum_{j=1}^{n} |x_j|^2.$$

Proof. The intervals $[t_r - \frac{1}{2}h, t_r + \frac{1}{2}h]$ do not overlap, so by Lemma 10.1, applied to $g(t) = f(t)^2$,

$$\sum_{r=1}^{R} |f(t_r)|^2 \leq \frac{1}{h} \int_0^T |f(t)|^2 \, dt + \int_0^T |f(t) f'(t)| \, dt.$$

By Theorem 9.3,

$$\int_0^T |f(t)|^2 \, dt \leq \left(T + \frac{1}{\delta} \right) \sum_{j=1}^{n} |x_j|^2.$$

Also, $f'(t) = 2\pi i \sum_{j=1}^{n} \lambda_j x_j e(\lambda_j t)$, so

$$\int_0^T |f'(t)|^2 \, dt \quad \leq \quad 4\pi^2 \left(T + \frac{1}{\delta} \right) \sum_{j=1}^{n} \lambda_j^2 |x_j|^2$$

$$\leq \quad 4\pi^2 M^2 \left(T + \frac{1}{\delta}\right) \sum_{j=1}^{n} |x_j|^2.$$

By the Cauchy-Schwarz inequality for integrals,

$$\int_0^T |f(t)f'(t)|\, dt \leq 2\pi M \left(T + \frac{1}{\delta}\right) \sum_{j=1}^{n} |x_j|^2.$$

The statement follows. □

Corollary 10.5. *Let $F(s) = \sum_{j=1}^{n} x_j/j^s$, and let t_r ($1 \leq r \leq R$) be as in Theorem 10.4. Then*

$$\sum_{r=1}^{R} |F(it_r)|^2 \leq \left(\log n + \frac{1}{h}\right)(T + 2\pi n) \sum_{j=1}^{n} |x_j|^2.$$

Proof. Take $\lambda_j = (\log j)/2\pi$ in Theorem 10.4: then $2\pi M = \log n$ and $\delta > 1/(2\pi n)$. □

Note. For $f(t)$ as in 10.4, the Cauchy-Schwarz inequality gives $|f(t)|^2 \leq n \sum_{j=1}^{n} |x_j|^2$, so it is trivial that $\sum_{r=1}^{R} |f(t_r)|^2 \leq Rn \sum_{j=1}^{n} |x_j|^2$. Also, the assumptions force $Rh \leq T$.

Again, there are numerous applications in analytic number theory. An unsolved problem, Montgomery's "large values conjecture", asks whether a bound of the form $CT^{\epsilon}(R+n)n \max |x_j|^2$ is correct in 10.5. For further discussion, see [Mont2], [Ten], [Gr].

11. Some further facts about the Hilbert matrices

First, we describe the relationship to the *Cesaro matrix*, which is defined by

$$A = \begin{pmatrix} 1 & 0 & 0 & . \\ \frac{1}{2} & \frac{1}{2} & 0 & . \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & . \\ . & . & . & . \end{pmatrix}$$

so that $(Ax)(j) = \frac{1}{j}(x_1 + \cdots + x_j)$, the "averaging operator". We show that this leads to a very quick method for Hilbert's inequality for H_0, but only with constant 4. Now

$$AA^* = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} & . \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & . \\ . & . & . & . & . \end{pmatrix},$$

in which the (j,k) entry is $1/\max(j,k)$. Now $\max(j,k) \leq j+k-1$ for $j,k \leq 1$, so by Proposition 2.1, $\|H_0\| \leq \|AA^*\|$.

Hardy's inequality (for the case $p = 2$) states that $\|A\| \leq 2$. It follows that $\|H_0\| \leq \|AA^*\| \leq 4$.

The original proof of Hardy's inequality for general p can be seen in [HLP, Theorem 326]. For $p = 2$, the following very neat proof, again using AA^*, is given in [Choi]:

Proposition 11.1. *We have* $\|I - A\| = 1$*, hence* $\|A\| \leq 2$*.*

Proof. It is clear that $A + A^* = AA^* + D$, where D is the diagonal matrix with entries $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$. Then $\|D\| = \|I - D\| = 1$. So

$$(I - A)(I - A^*) = I - A - A^* + AA^* = I - D,$$

hence $\|(I - A)(I - A^*)\| = \|I - A\|^2 = 1$. □

Lower bounds for $\|H_0 x\|$*; inverses.* The matrix $H_0^{(n)}$ is invertible. In fact, the inverse is known explicitly (e.g. see [Choi] or [FS]): the entries are products of certain binomial coefficients (hence integers). The reader can check directly that the inverse of $H_0^{(3)}$ is

$$\begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}.$$

However, $H_0^{(n)}$ is, in a sense, only just invertible: it is "ill-conditioned". The norm of its inverse is $1/\delta_n$, where δ_n is the largest constant such that $\|H_0^{(n)} x\| \geq \delta_n \|x\|$ for all x. It is very easily seen that $\delta_n \to 0$ as $n \to \infty$: in fact, by integral estimation

$$\|H_0^{(n)} e_n\|^2 \leq \|H_0 e_n\|^2 = \sum_{r=n}^{\infty} \frac{1}{r^2} \leq \frac{1}{n-1},$$

so $\delta_n^{-1} \geq (n-1)^{1/2}$. Simply by considering $H_0^{(n)}(e_n - e_{n-1})$, one can strengthen this to $\delta_n^{-1} \geq C n^{3/2}$ for a constant C. The norm of a matrix is not less than each of its entries, and from the formula for the inverse, one can see that the entry $(n-1, n-1)$ is asymptotically $c n^{5/2} 2^{2n}$ for another constant c. Of course, this shows that H_0 does not have a bounded inverse on ℓ_2. At the same time, it can be shown that H_0 is one-to-one on ℓ_2, so that $H_0 x \neq 0$ for all non-zero $x \in \ell_2$: see [Choi].

Meanwhile, $H_{-1}^{(n)}$ is not invertible for odd n: in fact, this is a general property of skew-symmetric matrices.

Approximation to H_0 *by* $H_0^{(n)}$. Let $A = (a_{j,k}) j, k \geq 1$ be an infinite matrix defining a bounded operator on ℓ_2. We now slightly vary the definition of $A^{(n)}$ to mean the infinite matrix in which $a_{j,k}$ is replaced by 0 whenever j or k is greater than n. A neat way to describe the relationship is $A^{(n)} = P_n A P_n$, where P_n is the projection onto the first n coordinates: $P_n x = x^{(n)}$. For each $x \in \ell_2$, $P_n x \to x$ as $n \to \infty$, and hence $A^{(n)} x \to Ax$ as $n \to \infty$. This is *not* the same as saying that $\|A - A^{(n)}\| \to 0$ as $n \to \infty$. For the case of H_0, rows 1 to n and columns $n+1$ to $2n$ of $H - H_0^{(n)}$ form an $n \times n$ matrix with all entries greater than $1/(3n)$. By Proposition 5.3, it follows that $\|H_0 - H_0^{(n)}\| \geq \frac{1}{3}$ for all n. In the language of operator theory, $H_0^{(n)}$ converges to H_0 in the strong operator topology but not the norm topology. It follows that H_0 is not a compact operator.

12. Summary of some other generalisations

Hilbert's inequality can be generalised in many different directions in addition to those we have discussed. These generalisations have generated a massive literature. The following is a very incomplete outline of some of them.

There are two obvious ways in which one can seek to extend matrix-norm inequalities of the kind we have been considering. One is the "continuous" analogue, in which sequences are replaced by functions, discrete sums by integrals and matrices by functions of two variables. The continuous analogue of H_1 is the integral operator (which we still denote by H_1) defined as follows: for a function f on $[0, \infty)$,

$$(H_1 f)(x) = \int_0^\infty \frac{f(y)}{x+y}\, dy.$$

Of course, convergence at 0, as well as at infinity, may have to be considered.

The other way is to vary the norm. For $p \geq 1$, the space ℓ_p is the set of sequences $x = (x_j)$ for which $\sum_{j\geq 1} |x_j|^p$ is convergent, with norm $\|x\|_p = (\sum_{j\geq 1} |x_j|^p)^{1/p}$. The L_p-norm of a function f is defined in the analogous way, using integrals. Results on these spaces frequently involve the conjugate index p^*, defined by $1/p + 1/p^* = 1$. For any operator T on ℓ_p (defined by a matrix) or L_p (defined by an integral), the implied problem is to determine its norm with respect to $\| \, \|_p$, which we denote by $\|T\|_p$.

These notes have been exclusively devoted to the discrete case, with $p = 2$. Extensions of the two types just described can often be considered together, by analogous methods. Where they differ, the discrete case is usually harder, because it often entails discrete sums which can be estimated by corresponding integrals, but are not exactly equal to them.

For $p \neq 2$, our Method 1 is a non-starter. However, Method 2 can be adapted, at the cost of only slightly greater complexity, to obtain for the discrete case

$$\|H_1\|_p = \|H_0\|_p = \pi \operatorname{cosec} \frac{\pi}{p} \tag{15}$$

for $1 < p \le \infty$, by way of the integral

$$\int_0^\infty \frac{1}{t^\alpha (t+1)}\, dt = \pi \operatorname{cosec} \alpha\pi.$$

(The fact that the column sums diverge shows at once that H_1 does not map ℓ_1 into ℓ_1). The method also adapts to establish the same value for the continuous case. See [HLP, chapter 9] or [Bor]. This extension of Hilbert's inequality was proved by G.H. Hardy, and is sometimes known as the Hardy-Hilbert inequality.

A generalisation [Bon], strengthening a result of Hardy and Littlewood, is as follows: let $\frac{1}{p} + \frac{1}{q} \ge 1$ and $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$, and let H_λ be the matrix $[1/(j+k)^\lambda]$. Then $|H_\lambda(x,y)| \le K(p,q)\|x\|_p\|y\|_q$ for a constant $K(p,q)$ that reproduces (15) when $q = p^*$.

Another generalisation is to the *weighted* ℓ_p space $\ell_p(w)$, where the norm is defined by $\|x\|_{p,w} = (\sum_{j\ge 1} w_j |x_j|^p)^{1/p}$ for a given weighting sequence (w_j), with a corresponding version $L_p(w)$ for the continuous case. For both cases, with the weight $w_j = 1/j^\alpha$ or $w(x) = 1/x^\alpha$ (where $0 < \alpha < 1$), it was shown in [JL] that the norm of H_1 is $\pi \operatorname{cosec} [(1-\alpha)\pi/p]$.

In several articles, e.g. [GY], Gao Mingzhe and others have formulated generalisations of the following type: $|H_1(x,y)| \le N_p(x) N_{p^*}(y)$, where

$$N_p(x) = \left(\sum_{j=1}^\infty (C - \frac{c}{j^{1/p}}) |x_j|^p \right)^{1/p},$$

in which $C = \pi \operatorname{cosec} \frac{\pi}{p}$ and c is a constant.

The corresponding questions for H_{-1} or $\tilde{H}_{-1}$ are distinctly harder. The continuous analogue is, of course, the integral of $f(y)/(y-x)$. However, even if $f(y)$ is the constant function 1 on $[0,1]$ (and 0 elsewhere), the integral will diverge on both $[0,x)$ and $(x,1]$ if $0 < x < 1$. This is overcome by using the "principal value" integral to define the *Hilbert (integral) operator* H as follows: for a function f on $\mathbb{R}$,

$$(Hf)(x) = \frac{1}{\pi} \lim_{\delta \to 0^+} \left(\int_{-\infty}^{x-\delta} + \int_{x+\delta}^\infty \right) \frac{f(y)}{y-x}\, dy.$$

By obvious substitutions, this can be rewritten as

$$(Hf)(x) = \frac{1}{\pi} \lim_{\delta \to 0^+} \int_\delta^\infty \frac{f(x+t) - f(x-t)}{t}\, dt.$$

(Note that the factor $(1/\pi)$ is included in the definition; equally, the "Hilbert matrix" is sometimes defined, in our notation, to be $\frac{1}{\pi}\tilde{H}_{-1}$.) As an exercise, the reader might like to show that if f is 1 on $[0,1]$, then $\pi(Hf)(x) = \log|1-x| - \log|x|$ for all $x \neq 0,1$. It was shown by M. Riesz in 1927 [R] that H is a bounded operator on $L_p(\mathbb{R})$ for all $p > 1$, without giving a good estimate of the constants. His method was to show that the problem is roughly equivalent to the following problem. Given a function defined by a Fourier series $f(t) = \sum_{j\in\mathbb{Z}} x_j e(jt)$, the "conjugate function" is $\tilde{f}(t) = \sum_{j>0} x_j e(jt) - \sum_{j<0} x_j e(jt)$ (strictly, this multiplied by $-i$). Show that there is a constant A_p such that $\|\tilde{f}\|_p \leq A_p\|f\|_p$ for $f \in L_p[0,1]$ (note that Parseval's identity gives at once $A_2 = 1$). See [Zyg], chapters 7 and 16.

Pichorides [Pich] developed the method to give exact values:

$$\|H\|_p = \begin{cases} \tan(\pi/2p) & \text{for } 1 < p \leq 2, \\ \cot(\pi/2p) & \text{for } p \geq 2. \end{cases}$$

Note that the least value is 1, occurring when $p = 2$, and that $\|H\|_{p^*} = \|H\|_p$.

In the discrete case, the norm is not less than these values, but it is only known that the values are exact when p is of the form 2^n or $2^n/(2^n-1)$.

Generalised and weighted versions of these results have appeared in [HMW], [ONW] and numerous subsequent articles.

REFERENCES

[Bon] F.F. Bonsall, Inequalities with non-conjugate parameters, *Quart. J. Math. Oxford* (2) **2** (1951), 135–150.

[Bor] Jonathan M. Borwein, Hilbert's inequality and Witten's zeta-function, *Amer. Math. Monthly* **115** (2008), 125–137.

[Choi] Man-Duen Choi, Tricks or treats with the Hilbert matrix, *Amer. Math. Monthly* **90** (1983), 3011–312.

[Dav] H. Davenport, *Multiplicatice Number Theory*, 2nd ed. Springer (1980).

[Gall] P.X. Gallagher, The large sieve, *Mathematika* **14** (1967), 14–20.

[GY] Gao Mingzhe and Yang Bicheng, On the extended Hilbert's inequality, *Proc. Amer. Math. Soc.* **126** (1998), 751–759.

[Gr] B.J. Green, The Kakeya problem, `www.maths.bris.ac.uk/~mabjg/papers` .

[Hal] P.R. Halmos, *Finite-Dimensional Vector Spaces*, Van Nostrand (1958).

[HLP] G.H. Hardy, J. Littlewood and G. Polya, *Inequalities*, 2nd ed., Cambridge Univ. Press (1967).

[HMW] R.A. Hunt, B. Muckenhoupt and R.L. Wheeden, Weighted norm inequalities for the conjugate function and the Hilbert transform, *Trans. Amer. Math. Soc.* **176** (1973), 227–251.

[Iv] A. Ivić, *The Riemann Zeta Function*, Wiley (1985).

[Jam1] G.J.O. Jameson, Notes on the large sieve, `www.maths.lancs.ac.uk/~jameson`

[Jam2] G.J.O. Jameson, Operator-valued extensions of matrix norm inequalities, `www.maths.lancs.ac.uk/~jameson`

[JL] G.J.O. Jameson and R. Lashkaripour, Norms of certain operators on weighted ℓ_p spaces and Lorentz sequence spaces, *J. Ineq. Pure Appl. Math.* **3**, issue 1, article 6 (2002), 1–17.

[Matt] K.R. Matthews, Hilbert's inequalities, `www.maths.uq.edu.au/~krm` .

[Mont1] H.L. Montgomery, The analytic principle of the large sieve, *Bull. Amer. Math. Soc.* **84** (1978), 547–567.

[Mont2] H.L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS no. 84, Amer. Math. Soc. (1990).

[MontV] H.L. Montgomery and R.C. Vaughan, Hilbert's inequality, *J. London Math.* (2) *Soc.* **8** (1974), 73-82.

[Ole] K. Oleszkiewicz, An elementary proof of Hilbert's inequality, *Amer. Math. Monthly* **100** (1993), 276–280.

[ONW] R. O'Neil and G. Weiss, The Hilbert Transform and rearrangement of functions, *Studia Math.* **23** (1963), 189–198.

[Pich] S.K. Pichorides, On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov, *Studia Math.* **44** (1972), 165–179.

[R] M. Riesz, Sur les fonctions conjuguées, *Math. Z.* **27** (1927), 218–244.

[Sch] I. Schur, Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen, *J. Reine Angew. Math.* **140** (1911), 1–28.

[Ten] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press (1995).

[Weyl] H. Weyl, *Singuläre Integralgleichungen mit besonderer Berücksichtigung des Fourierschen Integraltheorems*, inaugural dissertation, Göttingen (1908).

[Wilf] H.S. Wilf, *Finite sections of some classical inequalities*, Ergebnisse der Mathematik, vol. 52, Springer (1970).

[Young] N.J. Young, *An Introduction to Hilbert Space*, Cambridge Univ. Press (1988).

[Zyg] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press (1959).