# Hilbert's inequality and related results

Notes by G.J.O. Jameson

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## 1. Introduction

A more precise title would be "Hilbert's discrete inequality for $p = 2$ and some related results". The "related results" are norm evaluations for various other matrices of loosely Hilbert type, and some applications. A very brief survey of some generalizations, such as $p \neq 2$ and the continuous case, is attempted in section 10.

Our basic theme is the result known variously as "Hilbert's inequality" or "Hilbert's double series theorem". According to [HLP], Hilbert simply included the result in his lectures, and it was actually published by Weyl in his inaugural dissertation in 1908. The inequality can be stated in various equivalent ways. For the moment, we will just give the formulation in terms of bilinear forms, which is the one most frequently quoted. It is most effectively presented in terms of infinite two-sided sequences $x = (x_j)_{j\in\mathbb{Z}}$. Denote by $\ell_2(\mathbb{Z})$ the set of all such sequences with $\sum_{j=-\infty}^{\infty} |x_j|^2$ convergent (we write this sum as $\sum_{j\in\mathbb{Z}} |x_j|^2$). To simplify the statement, we also introduce the following notation (which we will use consistently): for $r \in \mathbb{Z}$, write

$$c_r = \begin{cases} 1/r & \text{if } r \neq 0 \\ 0 & \text{if } r = 0. \end{cases}$$

The statement is:

**Theorem**. *Given (real or complex) sequences* $x = (x_j)$ *and* $y = (y_k)$*, let*

$$u(x, y) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j-k} x_j y_k.$$

*Then*

$$|u(x, y)|^2 \leq \pi^2 \sum_{j \in \mathbb{Z}} |x_j|^2 \sum_{k \in \mathbb{Z}} |y_k|^2.$$

Two immediate comments:

(1) The substitution $y_k = z_{-k}$ transforms $u(x, y)$ to $\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j+k} x_j z_k$, so the theorem applies equally with $c_{j-k}$ replaced by $c_{j+k}$. Indeed, the advantage of considering two-sided sequences as above is that it reveals these two cases as being the same.

(2) The statement obviously still applies when $j$ and $k$ are restricted to a smaller range (simply take the other $x_j$ and $y_k$ to be 0). In particular, by restricting to the positive integers, we obtain two statements (for $c_{j-k}$ and $c_{j+k}$ respectively, no longer obviously equivalent) applying to normal "one-sided" sequences $(x_j)_{j \geq 1}$; this was Hilbert's original version. Also, we shall see that the result for infinite two-sided sequences, as stated above, is really equivalent to the statement for finite sequences of the form $(x_{-n}, \ldots, x_{n-1}, x_n)$, for all $n$ .

The reader may recognize already that the statement is equivalent to saying that the norm of the matrix $(c_{j-k})$ is not greater than $\pi$ (for readers needing it, we explain this equivalence in section 2). We shall widen the scope of these notes to present norm estimations for other matrices of this type, such as $[1/(j + k + \alpha)]$, $(c_{j-k}^2)$ and $[1/(\lambda_j \pm \lambda_k)]$. The last type has extensive applications in analytic number theory.

We will present three completely different approaches to the proof of results of this sort, two of which subdivide further into a number of variants. It will be seen that for any particular problem, any one of these methods may turn out to be the most effective.

## 2. Review of results assumed

These notes presuppose a basic knowledge of linear algebra and inner-product, or Hilbert, spaces. Here we summarize the main concepts and results used. Details can be found in any textbook on the subject, such as [BP]. Though inner-product spaces undeniably provide the most satisfactory setting for these results, what we really use is the concrete version for "vectors" (i.e. sequences, finite or infinite) and matrices.

Let $H$ be an inner-product space, over the real or complex field. We denote the inner product of $x$ and $y$ by $\langle x, y\rangle$. The norm is derived from the inner product by putting $\|x\| = \langle x, x\rangle^{1/2}$. The *Cauchy-Schwarz inequality* states that $|\langle x, y\rangle| \leq \|x\|.\|y\|$. Given two such spaces, we shall freely use the same notation $\| \ \|$ for the norms in both spaces. An inner-product space is called a *Hilbert space* if it is complete with respect to the norm.

The most basic example of an inner-product space is $H = \ell_2^n$, the space of vectors $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ or $\mathbb{C}^n$, with the inner product defined by $\langle x, y\rangle = \sum_{j=1}^n x_j \overline{y_j}$. The natural infinite-dimensional extension, denoted by $\ell_2$, is the space of infinite sequences $x = (x_k)_{k\geq 1}$ with $\sum_{k=1}^\infty |x_k|^2$ convergent. We denote by $\ell_2(\mathbb{Z})$ the space of infinite *two-sided* sequences $x = (x_k)_{k\in\mathbb{Z}}$ with $\sum_{k=-\infty}^\infty |x_k|^2$ convergent. These spaces are all Hilbert spaces.

A linear operator from $\ell_2^n$ to $\ell_2^m$ is represented by an $m \times n$ matrix $(a_{j,k})$ (which we also denote by $A$) according to: $y = Ax$, where $y_j = \sum_{k=1}^n a_{j,k}x_k$ for each $j$. A linear operator on $\ell_2$ or $\ell_2(\mathbb{Z})$ is represented by an infinite matrix in the same way.

If $H_1$, $H_2$ are Hilbert spaces, and $A$ is a (continuous) linear operator from $H_1$ to $H_2$, then there is an *adjoint* operator $A^* : H_2 \to H_1$, given by $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y$. If $H_1 = H_2$, we may have $A^* = A$: then $A$ is "self-adjoint". Adjoints satisfy $A^{**} = A$, $(\lambda A)^* = \overline{\lambda} A^*$ and $(AB)^* = B^*A^*$ (so $A^*A$ is self-adjoint). In the matrix case, $A^*$ corresponds to the conjugate transpose matrix $(\overline{a_{k,j}})$. Hence if $A$ is given by a real, symmetric matrix, then $A^* = A$, and if $A$ is given by a real, skew-symmetric matrix, then $A^* = -A$.

*Positivity.* For a self-adjoint operator, $\langle Ax, x\rangle$ is real for all $x$, and we write $A \geq 0$ if $\langle Ax, x\rangle \geq 0$ for all $x$. In linear algebra, such operators are usually described as "positive semi-definite", but we shall follow the culture of operator theory and just call them "positive". For $A = (a_{j,k})$, the statement $A \geq 0$ says explicitly

$$\sum_{j=1}^n \sum_{k=1}^n a_{j,k} \overline{x_j} x_k \geq 0$$

for scalars $x_j$ (such expressions are called *quadratic forms*). For a real matrix, it is sufficient if this holds for real scalars. For *any* operator $A$, we have $\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2$, hence $A^*A \geq 0$, which is analogous to the fact that $\overline{z}z$ is real and non-negative for complex numbers $z$. We write $A \leq B$ (for self-adjoint $A$ and $B$) if $B - A \geq 0$. In particular, the inequality $m \leq \langle Ax, x\rangle \leq M$ for $\|x\| = 1$ is equivalent to $mI \leq A \leq MI$.

If $A$ is a linear operator between Hilbert spaces, its *norm* (more specifically, *operator norm*), denoted by $\|A\|$, is the least constant $M$ such that $\|Ax\| \leq M\|x\|$ for all $x$; the operator is continuous (alias bounded) if $M$ is finite. So we have $\|Ax\| \leq \|A\|.\|x\|$, and hence, by the Cauchy-Schwarz inequality, $|\langle Ay, x\rangle| \le \|A\|.\|x\|.\|y\|$. In fact,

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\} = \sup\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1\}. \tag{1}$$

Multiplication by a suitable complex scalar shows that this is also the same as $\sup\{\mathrm{Re}\ \langle Ax, y\rangle : \|x\| = \|y\| = 1\}$. Operator norm satisfies: $\|A^*\| = \|A\|$, $\|AB\| \le \|A\|.\|B\|$ and $\|A^*A\| = \|A\|^2$ (the last equality follows again from the identity $\langle A^*Ax, x\rangle = \|Ax\|^2$).

In these notes, the norm of a matrix means the norm of the corresponding operator, as just defined. For $A = (a_{j,k})$, the inequalities just mentioned say explicitly:

(2) if $x_1, x_2, \ldots$ are scalars and $y_j = \sum_k a_{j,k} x_k$ for each $j$, then

$$\sum_j |y_j|^2 \le \|A\|^2 \sum_k |x_k|^2;$$

(3) for scalars $x_j$, $y_k$,

$$\left| \sum_j \sum_k a_{j,k} \overline{x}_j y_k \right| \le \|A\| \left( \sum_j |x_j|^2 \right)^{1/2} \left( \sum_k |y_k|^2 \right)^{1/2}.$$

Here the sums will be finite or infinite according to context. In (3), the statement would be the same if we wrote just $x_j$. We summarize some elementary facts about norms of matrices in the next result.

**Proposition 2.1**. *(i) If some rows or columns are removed, the norm of a matrix is reduced (i.e. not increased).*

*(ii) If the order of the rows (or columns) is permuted, the norm is unchanged.*

*(iii) If $a_{j,k} \ge 0$ and $|b_{j,k}| \le a_{j,k}$ for all $j, k$, then $\|B\| \le \|A\|$.*

*(iv) A real matrix has the same norm as an operator on $\mathbb{R}^n$ and on $\mathbb{C}^n$.*

*Brief proof.* We use the formulation (2) in each case. For (i): removing row $j$ (or column $k$) is equivalent to taking $y_j$ (or $x_k$) to be 0.

(ii) This is equivalent to permuting the order of the $y_j$'s or $x_k$'s.

(iii) Let $y_j = \sum_k a_{j,k}|x_k|$ and $z_j = \sum_j b_{j,k} x_k$. Then $|z_j| \le |y_j|$ for each $j$.

(iv) Let $M$ be the norm of $A$ as an operator on $\mathbb{R}^n$. An element $z$ of $\mathbb{C}^n$ can be expressed in the obvious way as $x + iy$, where $x, y \in \mathbb{R}^n$ and $\|z\|^2 = \|x\|^2 + \|y\|^2$. Then $Az = Ax + iAy$, so $\|Az\|^2 = \|Ax\|^2 + \|Ay\|^2 \le M^2(\|x\|^2 + \|y\|^2) = M^2\|z\|^2$. □

*Warning.* By (iii), the norm of $(|a_{j,k}|)$ is not less than the norm of $(a_{j,k})$. In general, the two are *not* equal!

The determination of the norms of particular matrices is a thoroughly non-trivial exercise. The topic of these notes is precisely this problem for matrices of certain types. We mention, without proof, that there is an explicit formula for the norm of a real $2 \times 2$ matrix: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\|A\| = \tfrac{1}{2}[(a+d)^2 + (b-c)^2]^{1/2} + \tfrac{1}{2}[(a-d)^2 + (b+c)^2]^{1/2}.$$

*Determination of norms by quadratic forms.* The next result is very well known, but we include the proof, because it will be used very frequently in the following sections.

**Proposition 2.2**. *If $A^* = A$, then*

$$\|A\| = \sup\{|\langle Ax, x\rangle| : \|x\| = 1\}. \tag{4}$$

*So quadratic forms, instead of bilinear ones, are sufficient to determine $\|A\|$.*

*Proof.* Denote the stated supremum by $M$. Then $|\langle Ax, x\rangle| \le M\|x\|^2$ for all $x$. Take $x$, $y$ with $\|x\| = \|y\| = 1$. Then

$$\begin{aligned} \langle A(x+y),\, x+y\rangle - \langle A(x-y),\, x-y\rangle &= 2\langle Ax, y\rangle + 2\langle Ay, x\rangle \\ &= 2\langle Ax, y\rangle + 2\langle y, Ax\rangle \qquad \text{since } A^* = A \\ &= 4\mathrm{Re}\ \langle Ax, y\rangle. \end{aligned}$$

Hence

$$\begin{aligned} 4\mathrm{Re}\ \langle Ax, y\rangle &\le M\,(\,\|x+y\|^2 + \|x-y\|^2) \\ &= M\,(\,2\|x\|^2 + 2\|y\|^2) \\ &= 4M. \end{aligned}$$

By the remark following (1) above, the statement follows. □

A slightly subtle point needs to be mentioned here. If $A$ is a real, symmetric matrix, then its norm is determined by regarding it as an operator on $\mathbb{R}^n$, so it is enough to consider *real* vectors $x = (x_1, x_2, \ldots, x_n)$ in Proposition 2.1. Now suppose that $A$ is a real, *skew-symmetric* matrix. Then $(iA)^* = iA$, so (4) applies to $iA$, hence to $A$ itself. However, $iA$ is a complex matrix, and it is essential to consider *complex* vectors $x$: indeed, it is trivial that $\langle Ax, x\rangle = 0$ for all real vectors $x$! For complex vectors $x$, the $(j,k)$ and $(k,j)$ terms combine to give $a_{j,k}(x_j\overline{x}_k - x_k\overline{x}_j)$, which is purely imaginary, so $\langle Ax, x\rangle$ is purely imaginary. This also shows that $\langle A\overline{x}, \overline{x}\rangle = -\langle Ax, x\rangle$, so the set $\{\langle Ax, x\rangle : \|x\| = 1\}$ is symmetrical about 0. In this sense, skew-symmetric quadratic forms are more "symmetrical" than symmetric forms!

A deeper fact is: if $H$ is finite-dimensional, and $A$ is self-adjoint, with eigenvalues $\lambda_j$ ($1 \le j \le n$, necessarily real), then $\|A\| = \max|\lambda_j|$. The same applies if $A^* = -A$, since $iA$ is self-adjoint, but the eigenvalues are purely imaginary.

## 3. Equivalent forms of the statement

With the concepts and results of section 2 in place, we are ready to give a more thorough statement of Hilbert's inequality in its various equivalent forms. Recall our notation

$$c_r = \begin{cases} 1/r & \text{if } r \neq 0 \\ 0 & \text{if } r = 0. \end{cases}$$

**Theorem 3.1**. *Let $C_n$ be the matrix $(c_{j-k})$ $(-n \le j, k \le n)$ and $C$ the infinite matrix $(c_{j-k})_{j,k\in\mathbb{Z}}$. Then $\|C_n\|$ (for all $n$) and $\|C\|$ are not greater than $\pi$. Hence:*

*(i) If $y_j = \sum_{k=-n}^{n} c_{j-k}x_k$ for each $j$, then $\sum_{j=-n}^{n} |y_j|^2 \le \pi^2 \sum_{k=-n}^{n} |x_k|^2$.*

*(ii) Given scalars $x_j$ $(-n \le j \le n)$ and $y_k$ $(-n \le k \le n)$, let*

$$u(x, y) = \sum_{j=-n}^{n} \sum_{k=-n}^{n} c_{j-k}x_j y_k.$$

*Then*

$$|u(x, y)|^2 \le \pi^2 \sum_{j=-n}^{n} |x_j|^2 \sum_{k=-n}^{n} |y_k|^2.$$

*These statements extend to infinite two-sided sequences $(x_j)$ and $(y_k)$ in $\ell_2(\mathbb{Z})$. Also, the same statements apply with $c_{j-k}$ replaced by $c_{j+k}$.*

In subsequent sections, we shall present several ways of proving the basic statement for the finite matrices $C_n$. For the moment, we will assume this and concentrate on equivalent versions and immediate consequences.

The equivalence of the statements for $(c_{j+k})$ is evident from the substitution $x_k = z_{-k}$ in (i) and $y_k = z_{-k}$ in (ii).

Also, similar statements apply when $j$ and $k$ are restricted to a smaller range, such as the positive integers (which gives normal "one-sided" sequences).

Next, we clarify the relationship between the finite and infinite cases. This part of the argument is much the same however the finite case is proved.

**Proposition 3.2**. *Let $C_n$ and $C$ be as in Theorem 3.1. Then $\|C\| = \sup_{n\geq 1} \|C_n\|$. So the statement that $\|C\| \leq \pi$ is equivalent to $\|C_n\| \leq \pi$ for all $n$.*

*Proof.* We use version (i) of Theorem 3.1. Of course, $\|C\| \geq \|C_n\|$ for all $n$. Let $\sup_{n\geq 1} \|C_n\| = M$ (we will show later that $M = \pi$). Let $x = (x_k)$ be in $\ell_2(\mathbb{Z})$. Convergence of $\sum_{j\in\mathbb{Z}} c_{j-k}x_k$ (for each $j$) is assured by the Cauchy-Schwarz inequality. Let this define $y_j$, and let $y_{j,n} = \sum_{k=-n}^{n} c_{j-k}x_k$, so that $y_{j,n} \to y_j$ as $n \to \infty$. Since $\|C_n\| \leq M$, for any $m < n$, we have

$$\sum_{j=-m}^{m} |y_{j,n}|^2 \leq \sum_{j=-n}^{n} |y_{j,n}|^2 \leq M^2 \sum_{k=-n}^{n} |x_k|^2.$$

Taking the limit as $n \to \infty$, we have

$$\sum_{j=-m}^{m} |y_j|^2 \leq M^2 \sum_{k=-\infty}^{\infty} |x_k|^2.$$

This is true for each $m$, so we can now take the limit as $m \to \infty$ to obtain

$$\sum_{j=-\infty}^{\infty} |y_j|^2 \leq M^2 \sum_{k=-\infty}^{\infty} |x_k|^2. \qquad \square$$

Let us take a closer look at the corresponding statements for one-sided matrices and sequences. We introduce the following notation for the matrices concerned:

$$A = (c_{j+k})_{j,k\geq 1}, \qquad A^{\#} = (c_{j+k-1})_{j,k\geq 1}, \qquad B = (c_{j-k})_{j,k\geq 1}.$$

We also write $A_n$ for the finite matrix $(c_{j+k})$ $(1 \leq j, k \leq n)$, and similarly for the others. Observe that $C_n$ is exactly the same matrix as $B_{2n+1}$, with the rows and columns labelled differently.

The matrices $A$ and $A^{\#}$ are symmetric, while $B$ is skew-symmetric. In $A$ and $A^{\#}$, the $(j, k)$ term is a function of $j + k$ (so these are *Hankel* matrices), while in $B$ it is a function of $j - k$ (so $B$ is a *Toeplitz* matrix). We introduce $A^{\#}$ because it arises naturally in some contexts. Note that its $(1, 1)$ entry is 1 (while in $A$ it is $\frac{1}{2}$). In fact, $A$ is obtained from $A^{\#}$ by removing the first row; if $(Ax)(j)$ denotes term $j$ of the sequence $Ax$, we have $(Ax)(j) = (A^{\#}x)(j + 1)$. Hence $\|A^{\#}x\| \geq \|Ax\|$ for all $x$, and $\|A^{\#}\| \geq \|A\|$.

We have already remarked that Hilbert's inequality implies that $\|A\| \leq \pi$ and $\|B\| \leq \pi$. It also implies that $\|A^{\#}\| \leq \pi$. Actually, it gives the following slightly stronger statement:

**Corollary 3.3**. *For $x$ in $\ell_2$, we have*

$$\|A^{\#}x\|^2 + \|Bx\|^2 \leq \pi^2 \|x\|^2.$$

*Proof.* Extend $x$ to a two-sided sequence $x^*$ by putting $x_j^* = 0$ for all $j \le 0$. For $j \ge 1$, we have, obviously, $(Cx^*)(j) = (Bx)(j)$. For $j \ge 0$ we have

$$(Cx^*)(-j) = \sum_{k\ge1} c_{-j-k}x_k = -\sum_{k\ge1} c_{j+k}x_k = -(A^{\#}x)(j+1).$$

Hence $\|A^{\#}x\|^2 + \|Bx\|^2 = \|Cx^*\|^2 \le \pi^2\|x\|^2$. □

*Best constants and exact norms.* We shall see in due course that $\pi$ is the best constant in Theorem 3.1, so that $\|C\| = \pi$. By Proposition 3.2, this is equivalent to the statement that $\sup_{n\ge1}\|C_n\| = \pi$. Since $C_n = B_{2n+1}$, it then follows that $\sup_{n\ge1}\|B_n\| = \|B\| = \pi$.

This does *not* mean that $\|B_n\|$ and $\|C_n\|$ equal $\pi$ for each $n$! Indeed, $B_2(x_1, x_2) = (-x_2, x_1)$, so $\|B_2\| = 1$. In general, the exact determination of the norms of these finite matrices is a much harder problem. We shall give some partial results in the following sections.

The reasoning in Corollary 3.3 (without the shift to $A^{\#}$) shows that $\|C_n\| \ge \|A_n\|$. So to show that $\|C\| = \pi$, it will be enough to show that $\sup_{n\ge1}\|A_n\|$ (equivalently, $\|A\|$) equals $\pi$. When lower estimates, instead of upper ones, are being considered, the implications among these various matrices run in the opposite direction.

## 4. Method 1: expressing the quadratic or bilinear form as an integral

One equivalent formulation of our problem is to establish bounds for certain quadratic or bilinear forms. This can sometimes be achieved in highly elegant style if the form can be expressed as the integral of a suitable function. Hilbert's original proof of his inequality was of this type. Here we give a modified version due to Toeplitz.

**Proof of Hilbert's inequality**. Given complex scalars $x_j$, write

$$S(x) = \sum_{j=-n}^{n}\sum_{k=-n}^{n} c_{j-k}x_j\overline{x_k}.$$

By (4), since the matrix $(c_{j-k})$ is real and skew-symmetric, it is enough to prove that $|S(x)| \le \pi\sum_{j=-n}^{n}|x_j|^2$ for any choice of the $x_j$. Recall that $S(x)$ is of the form $iR$, where $R$ is real.

Let $f(t)$ be the *trigonometric polynomial* $\sum_{j=-n}^{n} x_j e^{2\pi ijt}$. Then

$$|f(t)|^2 = \sum_{j=-n}^{n}\sum_{k=-n}^{n} x_j\overline{x_k}e^{2\pi i(j-k)t} = \sum_{j=-n}^{n}|x_j|^2 + \sum_{j=1}\sum_{k\ne j} x_j\overline{x_k}e^{2\pi i(j-k)t}$$

(in which $\sum_{k\neq j}$ means that $k$ runs from 1 to $n$, excluding $j$). Now $\int_0^1 e^{2\pi irt}\,dt = 0$ for integers $r \neq 0$, so

$$\int_0^1 |f(t)|^2\,dt = \sum_{j=-n}^{n} |x_j|^2.$$

Consider the integral

$$I = \int_0^1 (t - \tfrac{1}{2})|f(t)|^2\,dt.$$

Note that $\int_0^1 (t-\frac{1}{2})\,dt = 0$, while for integers $r \neq 0$, integration by parts gives

$$\int_0^1 (t-\tfrac{1}{2})e^{2\pi irt}\,dt = \frac{\frac{1}{2} - (-\frac{1}{2})}{2\pi ir} - \frac{1}{2\pi ir}\int_0^1 e^{2\pi irt}\,dt = \frac{1}{2\pi ir}.$$

Hence

$$I = \frac{1}{2\pi i}\sum_{j=1}^{n}\sum_{k\neq j} \frac{x_j\overline{x_k}}{j-k} = \frac{S(x)}{2\pi i}.$$

The quadratic form $S(x)$ has been expressed as the integral $2\pi iI$. But $|t-\frac{1}{2}| \leq \frac{1}{2}$ on $[0,1]$, so

$$|I| \leq \tfrac{1}{2}\int_0^1 |f(t)|^2\,dt = \tfrac{1}{2}\sum_{j=-n}^{n}|x_j|^2.$$

Hence $|S(x)| \leq \pi\sum_{j=-n}^{n}|x_j|^2$, as required. □

*First remark.* Clearly, the same applies if $j-k$ is replaced by $q_j - q_k$ for any sequence $(q_j)$ of distinct integers.

Next, we describe a variant of the proof that is closer to Hilbert's original (see [HLP, p. 235]). It deals with bilinear forms rather than quadratic ones, proving the original (ii) directly instead of relying on (4). To do this, it uses the Cauchy-Schwarz inequality for integrals instead of simple inequalities for the integrand.

*Variant proof of Theorem 3.1.* Take scalars $x_j$, $y_k$, and write $\sum_{j=-n}^{n}|x_j|^2 = \|x\|^2$. Define $u(x,y) = \sum_{j=-n}^{n}\sum_{k=-n}^{n} c_{j-k}x_jy_k$. Let

$$f(t) = \sum_{j=-n}^{n} x_je^{2\pi ijt}, \qquad g(t) = \sum_{k=-n}^{n} y_ke^{-2\pi ikt}.$$

Then $\int_0^1 |f(t)|^2\,dt = \|x\|^2$, and similarly for $g(t)$. Now consider

$$I = \int_0^1 (t-\tfrac{1}{2})f(t)g(t)\,dt.$$

As in the first proof, we see that

$$I = \frac{1}{2\pi i}\sum_{j=1}^{n}\sum_{k\neq j}\frac{x_jy_k}{j-k} = \frac{u(x,y)}{2\pi i}.$$

But by the fact that $|t - \frac{1}{2}| \le \frac{1}{2}$ on $[0,1]$ and the Cauchy-Schwarz inequality for integrals,

$$|I| \le \tfrac{1}{2} \int_0^1 |f(t)g(t)|\, dt \le \tfrac{1}{2} \|x\|.\|y\|.$$

Hence $|u(x,y)| \le \pi \|x\|.\|y\|$, as required. $\square$

*Further notes on the proofs.* (1) A minor variation in both proofs is to take $f(t) = \sum_{j=-n}^{n} x_j e^{ijt}$, and to consider integrals on $[0, 2\pi]$, using

$$\frac{1}{2\pi} \int_0^{2\pi} (t - \pi) e^{irt}\, dt = \frac{1}{ir}.$$

(2) Another variation is to consider the double integral $\int_0^1 \int_0^u |f(t)|^2 dt\, du$.

(3) Hilbert's original proof used separate terms $\cos jt$ and $\sin jt$ instead of $e^{ijt}$, and consequently gave the result with constant $2\pi$ instead of $\pi$. See [HLP, section 9.6]. Schur [Sch] improved the constant to $\pi$.

*Best constants.* This method can also be used to show that $\pi$ is the best constant in Theorem 3.1, but we need to assume some Fourier analysis, more precisely, the fact that a continuous function is the sum of its Fourier series in the sense of $L_2$-convergence.

**Theorem 4.1**. *In Theorem 3.1, the constant $\pi$ is optimal. In other words, $\|C\| = \sup_{n\ge1} \|C_n\| = \pi$.*

*Proof.* Fix $N$ (not related to the previous $n$!) and let $f(t) = t^N$. Let $f(t)$ have Fourier series $\sum_{n\in\mathbb{Z}} x_j e^{2\pi ijt}$. Then $f(t)$ is the $L_2$-sum of this series, so, by Parseval's identity,

$$\sum_{j\in\mathbb{Z}} |x_j|^2 = \int_0^1 |f(t)|^2\, dt = \int_0^1 t^{2N}\, dt = \frac{1}{2N+1}.$$

Let $S(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j-k} x_j \overline{x_k}$ and $I = \int_0^1 (t - \frac{1}{2})|f(t)|^2\, dt$. The identity $S(x) = 2\pi i I$ still applies. But

$$I = \int_0^1 (t^{2N+1} - \tfrac{1}{2} t^{2N})\, dt = \frac{1}{2N+2} - \frac{1}{2(2N+1)} = \frac{N}{2(N+1)(2N+1)}.$$

Hence

$$\frac{|S(x)|}{\|x\|^2} = \pi \frac{N}{N+1}.$$

Since $N$ can be chosen arbitrarily large, this proves that $\|C\| = \pi$. By Proposition 3.2, $\|C\| = \sup_{n\ge1} \|C_n\|$. $\square$

In the notation of section 3, $C_n = B_{2n+1}$, so it also follows that $\sup_{n\ge1} \|B_n\| = \|B\| = \pi$. However, we cannot deduce that $\|A\| = \pi$; we will show this by a different method in section

6, not depending on Fourier analysis. Of course, this will give a second proof of Theorem 4.1.

## 5. Some applications and companion results proved by Method 1

We start with two applications of Hilbert's inequality to real functions, proved by the same style of reasoning as Method 1 itself. They actually use the matrix $A^{\#}$: note that this matrix can equally be written as $[1/(m+n+1)]_{m,n\geq 0}$.

**Proposition 5.1**. *Let $f(t)$ be either the polynomial $\sum_{n=0}^{N} a_n t^n$ (with $a_n$ real), or the power series $\sum_{n=0}^{\infty} a_n t^n$, with $\sum_{n=0}^{\infty} a_n^2$ convergent. Then*

$$\int_0^1 f(t)^2\, dt \leq \pi \sum_n a_n^2.$$

*Proof.* In the infinite case, the series converges for $0 \leq t < 1$, by the Cauchy-Schwarz inequality. In either case, $f(t)^2 = \sum_m \sum_n a_m a_n t^{m+n}$, so

$$\int_0^1 f(t)^2\, dt = \sum_m \sum_n \frac{a_m a_n}{m+n+1}.$$

By Hilbert's inequality (for $A^{\#}$), this is not greater than $\pi \sum_n a_n^2$. (Termwise integration, for purists, can be justified by uniform convergence on $[0, r]$ for any $r < 1$.) □

By contrast, recall that we have an exact formula for the integral of the square of a *trigonometric* polynomial (which we used to prove Hilbert's inequality).

The second application concerns the *moments* of a function. It uses Proposition 5.1, while finishing with an inequality in the opposite direction.

**Proposition 5.2**. *Let $f$ be an integrable real function on $[0,1]$, and let $\mu_n = \int_0^1 t^n f(t)\, dt$ for $n \geq 0$. Then*

$$\sum_{n=0}^{\infty} \mu_n^2 \leq \pi \int_0^1 f(t)^2\, dt.$$

*Proof.* It is enough to show that $\sum_{n=0}^{N} \mu_n^2$ satisfies the inequality for each $N$ (thereby avoiding any convergence problems). We have

$$\begin{aligned} \sum_{n=0}^{N} \mu_n^2 &= \sum_{n=0}^{N} \mu_n \int_0^1 t^n f(t)\, dt \\ &= \int_0^1 f(t) g(t)\, dt, \end{aligned}$$

where $g(t) = \sum_{n=0}^{N} \mu_n t^n$. By Proposition 5.1, $\int_0^1 g(t)^2\,dt \le \pi \sum_{n=0}^{N} \mu_n^2$. So, by the Cauchy-Schwarz inequality for integrals,

$$\begin{aligned}\left(\sum_{n=0}^{N} \mu_n^2\right)^2 &\le \int_0^1 f(t)^2\,dt \int_0^1 g(t)^2\,dt \\ &\le \pi\left(\sum_{n=0}^{N} \mu_n^2\right)\int_0^1 f(t)^2\,dt.\end{aligned}$$

Division by $\sum_{n=0}^{N} \mu_n^2$ gives the statement. □

Now we come to "companion results", starting with positivity of certain matrices.

**Proposition 5.3**. *Let $\lambda_j > 0$ for $1 \le j \le n$. Then the symmetric matrix $[1/(\lambda_j + \lambda_k)]$ (where $1 \le j \le n,\ \ 1 \le k \le n$) is positive.*

*Proof.* Given real scalars $x_1, x_2, \ldots, x_n$, let

$$S(x) = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{x_j x_k}{\lambda_j + \lambda_k}.$$

We have to show that $S(x) \ge 0$. Let $f(t) = \sum_{j=1}^{n} x_j t^{\lambda_j}$. Then

$$\frac{f(t)^2}{t} = \sum_{j=1}^{n}\sum_{k=1}^{n} x_j x_k t^{\lambda_j + \lambda_k - 1},$$

so

$$\int_0^1 \frac{f(t)^2}{t}\,dt = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{x_j x_k}{\lambda_j + \lambda_k} = S(x).$$

But the integrand is non-negative for $0 < t \le 1$, so $S(x) \ge 0$, as required. (Note that if any $\lambda_j + \lambda_k < 1$, the integral still converges at 0.) □

Particular cases are: (i) $\lambda_j = \frac{1}{2}$, giving $\lambda_j + \lambda_k = 1$, (ii) $\lambda_j = j$, (iii) $\lambda_j = j - \frac{1}{2}\alpha$, where $\alpha < 2$, giving $\lambda_j + \lambda_k = j + k - \alpha$.

By considering instead functions of the form $f(t) = \sum_{j=1}^{n} x_j e^{-\lambda_j t}$, we can generalize to matrices of the following type.

**Proposition 5.4**. *Let $\lambda_j > 0$ for $1 \le j \le n$, and let $r$ be a positive integer. Then the symmetric matrix $[1/(\lambda_j + \lambda_k)^r]$ is positive.*

*Proof.* We use the integral

$$\int_0^\infty t^{r-1} e^{-\alpha t}\,dt = \frac{1}{\alpha^r}\int_0^\infty u^{r-1} e^{-u}\,du = \frac{(r-1)!}{\alpha^r}.$$

Given real scalars $x_j$, let $f(t) = \sum_{j=1}^n x_j e^{-\lambda_j t}$. Then

$$0 \le \int_0^\infty t^{r-1} f(t)^2 \, dt = (r-1)! \sum_{j=1}^n \sum_{k=1}^n \frac{x_j x_k}{(\lambda_j + \lambda_k)^r}. \qquad \square$$

We now return to trigonometric polynomials, which we used in the proof of Hilbert's inequality. From now on, following the custom of analytic number theory, we shall use the notation $e(x)$ for $e^{2\pi i x}$. Note that $e(x)e(y) = e(x+y)$, $e(\frac{1}{2}) = -1$ and $e(n) = 1$ for $n \in \mathbb{Z}$.

It is worth formulating in more general terms what the proof of Hilbert's inequality really proved. It becomes beautifully simple, as follows.

**Theorem 5.5**. *Let $\phi$ be an integrable function such that $|\phi(t)| \le M$ for $0 \le t \le 1$. For integers $r$, let $\int_0^1 \phi(t) e(rt) \, dt = d_r$. Let $D_n$ be the matrix $(d_{j-k})_{-n \le j,k \le n}$ and $D$ the matrix $(d_{j-k})_{j,k \in \mathbb{Z}}$. Then $\|D_n\|$ and $\|D\|$ are not greater than $M$. The same applies to $(d_{j+k})$.*

*Further, if $d_{-r} = d_r$ for all $r$, and $M_1 \le \phi(t) \le M_2$ for $0 \le t \le 1$, then $M_1 \|x\|^2 \le \langle D_n x, x \rangle \le M_2 \|x\|^2$ for all $x = (x_j)$ in $\ell_2(\mathbb{Z})$ (and similarly for $D$).*

*Proof.* The matrix is not, in general, symmetric or skew-symmetric, so we follow the "variant proof" of section 4. Choose $x_j$, $y_k$, and let $u(x,y) = \sum_{j=-n}^n \sum_{k=-n}^n d_{j-k} x_j y_k$. Let

$$f(t) = \sum_{j=-n}^n x_j e(jt), \qquad g(t) = \sum_{k=-n}^n y_k e(-kt).$$

Then $\int_0^1 |f(t)|^2 \, dt = \|x\|^2$, and similarly for $g(t)$. Also,

$$f(t)g(t) = \sum_{j=-n}^n \sum_{k=-n}^n x_j y_k e[(j-k)t].$$

Let

$$I = \int_0^1 \phi(t) f(t) g(t) \, dt.$$

Then $I = u(x,y)$. But $|I| \le M \int_0^1 |f(t)g(t)| \, dt$, which is not greater than $M\|x\|.\|y\|$, by the Cauchy-Schwarz inequality for integrals. So $|u(x,y)| \le M\|x\|.\|y\|$. By modifying $g(t)$ to $\sum_{k=-n}^n y_k e(kt)$, we obtain the same with $d_{j-k}$ replaced by $d_{j+k}$. The extension to the infinite case is as in Proposition 3.2.

In the case when $d_{-r} = d_r$ (so the matrix is symmetric), we consider instead

$$I = \int_0^1 \phi(t) |f(t)|^2 \, dt = \sum_{j=-n}^n \sum_{k=-n}^n d_{j-k} x_j \overline{x}_k = \langle D_n x, x \rangle.$$

Clearly, $M_1 \|x\|^2 \le I \le M_2 \|x\|^2$. $\square$

*Note.* Our $d_r$ is the Fourier coefficient $\hat{\phi}(-r)$ of $\phi$ on $[0,1]$. However, the proof did not require any result about convergence of the Fourier series. Such a result *is* required if we wish to show that $M$ is the best constant (as in Theorem 4.1). We discuss this point further, and describe a rather more "abstract" approach to Theorem 5.5, in section 10.

Clearly, Theorem 5.5 will give norm estimations for many more Toeplitz and Hankel matrices when different choices of $\phi(t)$ are fed in (this is a more user-friendly approach than starting from the matrix and trying to identify the appropriate $\phi(t)$). For Hilbert's inequality, we had $\phi(t) = t - \frac{1}{2}$. We now record some other cases; we state only the infinite-dimensional versions.

**Proposition 5.6**. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Write $a_{j,k} = 1/(j-k+\alpha)$. Let $A_\alpha$ be the matrix $(a_{j,k})_{j,k\in\mathbb{Z}}$. Then $\|A_\alpha\| \le \pi|\operatorname{cosec}\pi\alpha|$. In particular, $\|A_{1/2}\| \le \pi$.*

*Proof.* Take $\phi(t) = e(\alpha t)$. Then $M = 1$ and

$$d_r = \int_0^1 e(\alpha t)e(rt)\,dt = \frac{1}{2\pi i(r+\alpha)}(e(\alpha)-1) = \frac{e^{\pi i\alpha}\sin\pi\alpha}{\pi(r+\alpha)}.$$

So $D = (1/\pi)e^{\pi i\alpha}\sin(\pi\alpha)\,A_\alpha$, and the statement follows. □

*Remarks.* (1) The stated constant is the best possible. This follows from the identity $\sum_{j\in\mathbb{Z}} 1/(j+\alpha)^2 = \pi^2\operatorname{cosec}^2\pi\alpha$, which is obtained by Parseval's identity and the Fourier series for $e(\alpha t)$ (which was found in the proof). Now let $e_0$ be the two-sided sequence with 1 in place 0 and 0 elsewhere. Then $A_\alpha e_0 = y$, where $y_j = 1/(j+\alpha)$, so $\|y\| = \pi|\operatorname{cosec}\alpha\pi|$.

(2) The given statement says the same if $1/(j-k+\alpha)$ is replaced by $a'_{j,k} = 1/(j+k+\alpha)$. However, if we restrict to *positive* $j$, $k$, and $\alpha > 0$, stronger results apply: $0 < a'_{j,k} < 1/(j+k)$, so the matrix $(a'_{j,k})$ has norm not greater than $\pi$. It is also positive, by Proposition 5.3.

**Proposition 5.7**. *Let*

$$c_r^* = \begin{cases} 1/r & \text{if } r \text{ is odd,} \\ 0 & \text{if } r \text{ is even.} \end{cases}$$

*Then the matrix $(c^*_{j-k})_{j,k\in\mathbb{Z}}$ has norm not greater than $\pi/2$.*

*Proof.* Let

$$\phi(t) = \begin{cases} -1 & \text{for } 0 \le t < \frac{1}{2}, \\ 1 & \text{for } \frac{1}{2} \le t < 1. \end{cases}$$

Then $M = 1$, and for $r \ne 0$,

$$d_r = \int_{1/2}^1 e(rt)\,dt - \int_0^{1/2} e(rt)\,dt = \frac{2 - 2e(\frac{r}{2})}{2\pi i r} = \frac{2}{\pi i}c_r^*.$$

Also, $d_0 = 0$. So $C^* = (\pi i/2)D$. □

Recall again that our notation $c_r$ means $1/r$ when $r \neq 0$ and $0$ when $r = 0$.

**Proposition 5.8**. *Let $T$ be the matrix $(c_{j-k}^2)_{j,k\in\mathbb{Z}}$. Then $\|T\| \le \pi^2/3$. Further, for any $x$ in $\ell_2(\mathbb{Z})$,*

$$-\frac{\pi^2}{6}\|x\|^2 \le \langle Tx, x\rangle \le \frac{\pi^2}{3}\|x\|^2.$$

*Proof.* Take $\phi(t) = t(1-t)$. It is easily seen that $0 \le \phi(t) \le \frac{1}{4}$ on $[0,1]$, so, in the notation of Theorem 5.5, $0 \le \langle Dx, x\rangle \le \frac{1}{4}\|x\|^2$. Also, $d_0 = \int_0^1 t(1-t)\,dt = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, and for $r \neq 0$

$$\begin{aligned} d_r = \int_0^1 t(1-t)e(rt)\,dt &= 0 - \frac{1}{2\pi i r}\int_0^1 (1-2t)e(rt)\,dt \\ &= \frac{1}{\pi i r}\int_0^1 te(rt)\,dt \\ &= \frac{1}{(\pi i r)(2\pi i r)} \\ &= -\frac{1}{2\pi^2 r^2} \end{aligned}$$

so

$$\langle Dx, x\rangle = \frac{1}{6}\|x\|^2 - \frac{1}{2\pi^2}\langle Tx, x\rangle.$$

The stated inequalities follow. □

*Exercise.* By taking $\phi(t) = t(t-\frac{1}{2})(t-1)$, show that the norm of $(c_{j-k}^3)$ is not greater than $\pi^3/(9\sqrt{3})$.

This process can be extended to $(c_{j-k}^n)$ using the *Bernoulli polynomials* $B_n(t)$, because these functions have Fourier coefficients $-n!/(2\pi i k)^n$. However, at least for even $n$, the norms of these matrices are found much more easily by Method 2, which we now describe.

## 6. Method 2: row and column sums

This is a very general method for the estimation of norms of matrices with *non-negative* entries. The vigilant reader will object that the two-sided Hilbert matrix has both positive and negative entries. Please be patient and read on!

The basic result, due to Schur, is as follows.

**Theorem 6.1**. *Let $A = (a_{j,k})$ be a matrix (finite or infinite) such that $a_{j,k} \geq 0$ for all $j, k$ and*

$$\sum_k a_{j,k} \leq K_1 \quad \text{for all } j \qquad (\text{all row sums} \leq K_1),$$

$$\sum_j a_{j,k} \leq K_2 \quad \text{for all } k \qquad (\text{all column sums} \leq K_2).$$

*Then $\|A\| \leq (K_1 K_2)^{1/2}$.*

We leave out the proof of Theorem 6.1 because only slightly more work is needed to prove the following enhanced version incorporating a weighting sequence $(w_j)$.

**Theorem 6.2**. *Let $A = (a_{j,k})$ be a matrix (finite or infinite) with $a_{j,k} \geq 0$ for all $j, k$. Suppose that, for some strictly positive sequence $(w_j)$, we have*

$$\sum_k a_{j,k} w_k \leq K_1 \, w_j \quad \text{for all } j,$$

$$\sum_j a_{j,k} w_j \leq K_2 \, w_k \quad \text{for all } k.$$

*Then $\|A\| \leq (K_1 K_2)^{1/2}$.*

*Note.* If $A$ is symmetric, then the two hypotheses say the same (with $K_2 = K_1 = K$, say) and the conclusion is $\|A\| \leq K$. This case can be proved quickly using *quadratic* forms (sufficient since $A$ is symmetric) and the inequality $\frac{1}{2} x_j x_k \leq x_j^2 + x_k^2$.

*Proof.* Choose non-negative, real vectors $x = (x_j)$ and $y = (y_k)$ (in $\ell_2$ in the infinite case), and let $u(x, y) = \sum_j \sum_k x_j a_{j,k} y_k$. Then $u(x, y) = \sum_j \sum_k c_{j,k} d_{j,k}$, where

$$c_{j,k} = a_{j,k}^{1/2} x_j \left( \frac{w_k}{w_j} \right)^{1/2}, \qquad d_{j,k} = a_{j,k}^{1/2} y_k \left( \frac{w_j}{w_k} \right)^{1/2}.$$

By the Cauchy-Schwarz inequality (applied to the double sum), $u(x, y) \leq (CD)^{1/2}$, where

$$C = \sum_j \sum_k c_{j,k}^2 = \sum_j x_j^2 w_j^{-1} \sum_k a_{j,k} w_k \leq K_1 \sum_j x_j^2,$$

$$D = \sum_k \sum_j d_{j,k}^2 = \sum_k y_k^2 w_k^{-1} \sum_j a_{j,k} w_j \leq K_2 \sum_k y_k^2.$$

So $u(x, y) \leq (K_1 K_2)^{1/2} \|x\| . \|y\|$. □

The matrix $(c_{j-k}^2)$ (already seen in Proposition 5.8) is ready-made for an instant application of Theorem 6.1:

**Proposition 6.3**. *The matrix* $(c_{j-k}^2)_{j,k\in\mathbb{Z}}$ *has norm not greater than* $\pi^2/3$.

*Proof.* The matrix is symmetric, and every row sum is $2\sum_{r=1}^{\infty}(1/r^2) = \pi^2/3$. ☐

Similarly, the norm of $(|c_{j-k}|^n)$ is not greater than $2\zeta(n)$ (recall that $\zeta(p)$ means $\sum_{k=1}^{\infty} k^{-p}$ for $p > 1$). However, Method 2 has nothing to say about the *lower* bound for the quadratic form given in Proposition 5.8.

Next, consider the one-sided Hilbert matrix $A = [(1/(j+k)]_{j,k\geq 1}$. The entries are positive, but the row sums diverge, so we turn to Theorem 6.2. It gives a pleasant and quick proof of this form of Hilbert's inequality. At the same time, it enables us, rather more messily, to make a first attempt at a stronger bound for the $n$-dimensional case, a problem for which Method 1 has nothing to offer. We use integral estimation for discrete sums, together with the following lemma.

**Lemma 6.4**. *For* $a > 0$,

$$\int_0^\infty \frac{1}{(t+a)t^{1/2}}\,dt = \frac{\pi}{a^{1/2}}.$$

*Proof.* We have

$$\int_0^\infty \frac{1}{(u+1)u^{1/2}}\,du = \int_0^\infty \frac{2}{v^2+1}\,dv = \pi.$$

Now substitute $t = au$ in the given integral to obtain the stated value. ☐

**Theorem 6.5**. *Let* $A$ *be the matrix* $[1/(j+k)]_{j,k\geq 1}$ *and* $A_n$ *the matrix* $[1/(j+k)]$ *(*$1 \leq j,k \leq n$*). Then*

$$\|A\| \leq \pi, \qquad \|A_n\| \leq \pi - \frac{2}{n^{1/2}}.$$

*Proof.* Since $1/(t+j)t^{1/2}$ is a decreasing function of $t$, the Lemma gives

$$\sum_{k=1}^{\infty} \frac{1}{(j+k)k^{1/2}} \leq \int_0^\infty \frac{1}{(t+j)t^{1/2}}\,dt = \frac{\pi}{j^{1/2}}$$

for each $j$. By Theorem 6.2, with $w_j = 1/j^{1/2}$, we have $\|A\| \leq \pi$.

For $A_n$, we can amend this to

$$\sum_{k=1}^{n} \frac{1}{(j+k)k^{1/2}} \leq \int_0^n \frac{1}{(t+j)t^{1/2}}\,dt.$$

Now

$$\int_n^\infty \frac{1}{(t+j)t^{1/2}}\,dt \geq \int_n^\infty \frac{1}{(t+j)^{3/2}}\,dt = \frac{2}{(n+j)^{1/2}},$$

so

$$j^{1/2}\sum_{k=1}^{n}\frac{1}{(j+k)k^{1/2}}\le \pi-2\frac{j^{1/2}}{(n+j)^{1/2}}.$$

For $2 \le j \le n$, this is not greater than $\pi - 2/n^{1/2}$, because $j/(n+j) \ge 1/n$ (this says $nj \ge n+j$). For $j=1$, we have

$$\sum_{k=1}^{n}\frac{1}{(k+1)k^{1/2}}\le\sum_{k=1}^{n}\frac{1}{k^{3/2}}\le 1+\int_1^n\frac{1}{t^{3/2}}\,dt=3-\frac{2}{n^{1/2}}<\pi-\frac{2}{n^{1/2}}.$$

The statement follows. □

Using estimations of this sort, we now show quite directly that $\pi$ is the best constant for $A$ (which we did not achieve by Method 1), again with an estimation for $A_n$.

**Theorem 6.6**. *Let $A$ and $A_n$ be as in Theorem 6.5. Then $\|A\| = \pi$, and there is a constant $c$ such that $\|A_n\| \ge \pi - c/\log n$ for all $n > 2$.*

*Proof.* Let $x_j = j^{-1/2}$ for $1 \le j \le n$. Then $\|x\|^2 = 1+\frac{1}{2}+\cdots+\frac{1}{n}$. Denote this quantity by $l_n$. By comparison with the integral of $1/t$, we have $\log(n+1) \le l_n \le \log n + 1$. Let $y = A_n x$. Then

$$y_j=\sum_{k=1}^{n}\frac{1}{(j+k)k^{1/2}}\ge\int_1^n\frac{1}{(t+j)t^{1/2}}\,dt.$$

Now

$$\int_0^1\frac{1}{(t+j)t^{1/2}}\,dt\le\frac{1}{j}\int_0^1\frac{1}{t^{1/2}}\,dt=\frac{2}{j},$$

$$\int_n^\infty\frac{1}{(t+j)t^{1/2}}\,dt\le\int_n^\infty\frac{1}{t^{3/2}}\,dt=\frac{2}{n^{1/2}}.$$

By these statements and Lemma 6.4,

$$y_j\ge\frac{\pi}{j^{1/2}}-\frac{2}{j}-\frac{2}{n^{1/2}},$$

so that

$$y_j^2\ge\frac{\pi^2}{j}-4\pi\left(\frac{1}{j^{3/2}}+\frac{1}{j^{1/2}n^{1/2}}\right),$$

and

$$\begin{aligned}\sum_{j=1}^{n}y_j^2 &\ge \pi^2 l_n-4\pi\zeta(\tfrac{3}{2})-\frac{4\pi}{n^{1/2}}\sum_{j=1}^{n}\frac{1}{j^{1/2}}\\ &\ge \pi^2 l_n-c_1,\end{aligned}$$

where $c_1 = 4\pi\zeta(\frac{3}{2}) + 8\pi$, since $\sum_{j=1}^{n}(1/j^{1/2}) < \int_0^n t^{-1/2}\,dt = 2n^{1/2}$. So

$$\|A_n\|^2\ge\frac{\|y\|^2}{\|x\|^2}\ge\pi^2-\frac{c_1}{l_n},$$

and hence $\|A_n\| \geq \pi - c/l_n$ for another constant $c$. □

As already mentioned, it follows that $\|B\|$ and $\|C\|$ also equal $\pi$ (Theorem 4.1). Also, the same lower estimate applies to $A_n^{\#}$ and $C_n$.

Clearly, there is a gap between the upper and lower estimates for $\|A_n\|$ in Theorems 6.5 and 6.6. A more accurate evaluation is a hard problem! With a deeper analysis [Wilf, Theorem 2.2] obtains the value

$$\|A_n\| = \pi - \frac{\pi^5}{2(\log n)^2} + O\left(\frac{\log\log n}{(\log n)^3}\right).$$

Can we obtain the full strength of Hilbert's two-sided inequality by Method 2? Yes, and it is instructive to see how, though the work is undeniably longer and less elegant than Method 1. Recall that $\|Cx\|^2 = \langle C^*Cx, x\rangle$, and hence $\|C^*C\| = \|C\|^2$. We show that, although Theorem 6.1 cannot be applied to $C$, it can be applied to $C^*C$.

*Proof of Theorem 3.1 by Method 2.* Let $C$ be the matrix $(c_{j-k})_{j,k\in\mathbb{Z}}$. Then $C^*C = (d_{j,k})$, where

$$d_{j,k} = \sum_{p\in\mathbb{Z}} c_{p-j}c_{p-k}$$

Clearly, $d_{k,j} = d_{j,k}$. Also,

$$d_{j,j} = \sum_{p\in\mathbb{Z},\, p\neq j} \frac{1}{(p-j)^2} = \frac{\pi^2}{3}. \tag{5}$$

Now let $k = j + r$, where $r > 0$. Then $d_{j,k} = S_1 + S_2 + S_3$, where

$$\begin{aligned} S_1 &= \sum_{p>k} \frac{1}{(p-j)(p-k)} = \sum_{q\geq 1} \frac{1}{q(q+r)} \qquad \text{(put } q = p-k) \\ &= \frac{1}{r}\sum_{q\geq 1}\left(\frac{1}{q} - \frac{1}{q+r}\right) \\ &= \frac{1}{r}\left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right), \end{aligned}$$

$$S_2 = \sum_{p<j} \frac{1}{(j-p)(k-p)} = \sum_{q\geq 1} \frac{1}{q(q+r)} = S_1 \qquad \text{(put } q = j-p),$$

$$\begin{aligned} S_3 &= \sum_{p=j+1}^{k-1} \frac{1}{(p-j)(p-k)} = -\sum_{q=1}^{r-1} \frac{1}{q(r-q)} \qquad \text{(put } q = p-j) \\ &= -\frac{1}{r}\sum_{q=1}^{r-1}\left(\frac{1}{q} + \frac{1}{r-q}\right) \\ &= -\frac{2}{r}\left(1 + \frac{1}{2} + \cdots + \frac{1}{r-1}\right). \end{aligned}$$

The cancellation of positive and negative terms leaves us with

$$d_{j,k} = \frac{2}{r^2} = \frac{2}{(k-j)^2}. \tag{6}$$

By (5) and (6), we have for each $j$

$$\sum_{k\in\mathbb{Z}} d_{j,k} = \frac{\pi^2}{3} + \frac{2\pi^2}{3} = \pi^2,$$

and the statement follows, by Theorem 6.1. □

Note that for the cancellation to occur, it was essential to consider the doubly infinite matrix. For the one-sided case, in the notation of section 3, this proof can be seen to apply to $A^{\#*}A^{\#} + B^*B$ (see [HLP, section 8.12]). However, it does not work for $A^{\#*}A^{\#}$ and $B^*B$ separately, since the cancellation is lost.

To finish this section, we show that the norm of the $n \times n$ matrix $(|c_{j-k}|)$ grows like $\log n$, serving a brisk reminder that the norm of a matrix may change drastically if the entries are replaced by their moduli!

**Proposition 6.7**. *Let $M_n$ be the matrix $(|c_{j-k}|)_{1\le j,k\le n}$, and $M$ the matrix $(|c_{j-k}|)_{j,k\ge 1}$. Then $\log n \le \|M_n\| \le 2\log n + 1$ for $n \ge 2$, and $M$ does not define a bounded linear mapping on $\ell_2$.*

*Proof.* Let $r_j$ be the sum of row $j$. As in 6.6, write $l_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, and recall that $\log(n+1) \le l_n \le \log n + 1$. Clearly,

$$r_1 = r_n = l_{n-1},$$

$$r_j = l_{j-1} + l_{n-j} \qquad (2 \le j \le n-1).$$

So $r_j \le \log j + \log(n-j) + 2$ for $2 \le j \le n-1$. Now $j(n-j) = n^2/4 - (j - n/2)^2 \le n^2/4$, hence

$$r_j \le 2\log n - \log 4 + 2 < 2\log n + 1$$

for $2 \le j \le n-1$ (and also for $j = 1$ and $j = n$). By 6.1, $\|M_n\| \le 2\log n + 1$.

At the same time, it is clear that $r_j \ge l_{n-1} \ge \log n$ for all $j$. Let $e = (1, 1, \ldots, 1)$. Then $r_j$ is element $j$ of $M_n e$, so $\|M_n e\| \ge \log n \|e\|$, hence $\|M_n\| \ge \log n$. This also shows that $M$ is not a bounded operator on $\ell_2$. □

## 7. The matrix $[\operatorname{cosec} \frac{\pi}{n}(j-k)]$ and a third method for Hilbert's inequality

Let $S_n$ be the matrix $(s_{j,k})$ $(1 \le j,k \le n)$, where

$$s_{j,k} = \begin{cases} \operatorname{cosec} \frac{\pi}{n}(j-k) & \text{for } j \neq k, \\ 0 & \text{for } j = k. \end{cases}$$

Like $B_n = (c_{j-k})$, this is a skew-symmetric, Toeplitz matrix. Apart from the constant $\pi/n$, we have replaced $x$ by $\sin x$. The matrix $S_4$ has $\sqrt{2}$, 1, $\sqrt{2}$ on the diagonals defined by $j-k = 1,2,3$ respectively (we save space by refraining from writing it out explicitly).

As we have seen, the exact norms of $A_n$ and $B_n$ are not known for general $n$. Even when they are known, for small values of $n$, they are unpleasantly complicated numbers. In the light of these facts, it is very striking to find that $\|S_n\|$ is precisely the integer $n-1$. We shall prove this, and then show that Hilbert's inequality can be derived from it in elegant style – our third proof, not counting minor variants. Furthermore, this approach does something that the earlier ones failed to do: it gives an estimation for $\|B_n\|$ depending on $n$. These results are due to K.R. Matthews [Matt].

Our evaluation of $\|S_n\|$ will be achieved by finding the eigenvalues. First, a reminder of the following simple relationship between eigenvalues, eigenvectors and norms.

**Lemma 7.1**. *Let $S$ be an $n \times n$ matrix. Suppose that $S$ has mutually orthogonal eigenvectors $x_r$ $(1 \le r \le n)$, with corresponding eigenvalues $\lambda_r$. Then $\|S\| = \max_{1 \le r \le n} |\lambda_r|$.*

*Proof.* Let $\max_{1 \le r \le n} |\lambda_r| = M$. We may assume that $\|x_r\| = 1$ for each $r$ (if not, divide by $\|x_r\|$). Then the vectors $x_r$ form an orthonormal basis of $\ell_2^n$. Any vector $x$ in $\ell_2^n$ can be expressed as $x = \sum_{r=1}^n \alpha_r x_r$, with $\|x\|^2 = \sum_{r=1}^n |\alpha_r|^2$. Then $Sx = \sum_{r=1}^n \lambda_r \alpha_r x_r$, and

$$\|Sx\|^2 = \sum_{r=1}^n |\lambda_r^2 \alpha_r^2| \le M^2 \|x\|^2.$$

Further, $\|Sx_r\| = M\|x_r\|$ for at least one $r$. □

There is a general theorem that symmetric and skew-symmetric matrices always have an orthogonal basis consisting of eigenvectors. However, we do not need this theorem, because we will establish the property directly for another class of matrix to which $S_n$ belongs.

We say that an $n \times n$ matrix $S = (s_{j,k})$ is *skew-circulant* if $s_{j,k} = a_{k-j}$ for some sequence $(a_j)$ satisfying $a_{j-n} = -a_j$ for all $j$, so the matrix is of the form

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ -a_{n-1} & a_0 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ -a_1 & -a_2 & \cdots & a_0 \end{pmatrix}.$$

The elements $a_j$ cycle round the rows, reappearing with a minus sign on the left of the diagonal. We prove a general result on eigenvectors of such matrices.

Recall that if $\omega = e(r/n)$, where $1 \le r \le n-1$, then $\omega^n = 1$, while $\omega \ne 1$, so, by the geometric series, $\sum_{k=1}^n \omega^k = 0$.

In the following, component $k$ of a vector $x$ is denoted by $x(k)$.

**Proposition 7.2**. *Let $S$ be a skew-circulant $n \times n$ matrix, with $s_{j,k} = a_{k-j}$ as above. Let*

$$\rho_r = \exp \frac{(2r-1)\pi i}{n}.$$

*Then $S$ has mutually orthogonal eigenvectors $x_r$ $(1 \le r \le n)$ defined by $x_r(k) = \rho_r^k$. The corresponding eigenvalues are $\lambda_r = \sum_{k=1}^n a_k \rho_r^k$.*

*Proof.* Choose one value of $r$, and write $\rho_r = \rho,\ x_r = x$. Note that $\rho^n = -1$. We have

$$(Sx)(j) = \sum_{k=1}^n a_{k-j}\rho^k = \rho^j \sum_{k=1}^n a_{k-j}\rho^{k-j}.$$

Write $g(k) = a_k \rho^k$. Clearly, $\sum_{k=j+1}^n g(k-j) = \sum_{k=1}^{n-j} g(k)$. Also, $g(k+n) = g(k)$, so $\sum_{k=1}^j g(k-j) = \sum_{k=n-j+1}^n g(k)$. Hence $\sum_{k=1}^n g(k-j) = \sum_{k=1}^n g(k)$. This sum is independent of $j$: denote it by $\lambda$. We have shown that $(Sx)(j) = \lambda x(j)$ for all $j$, so $S(x) = \lambda x$.

For $r \ne s$, we have $\rho_r \overline{\rho}_s = \omega$, where $\omega = e[(r-s)/n]$, so

$$\langle x_r, x_s \rangle = \sum_{k=1}^n \omega^k = 0. \qquad \square$$

**Theorem 7.3**. *Let $S_n$ be the matrix $(s_{j,k})$ $(1 \le j,k \le n)$, where*

$$s_{j,k} = \begin{cases} \operatorname{cosec} \frac{\pi}{n}(j-k) & \text{for } j \ne k, \\ 0 & \text{for } j = k. \end{cases}$$

*Then:*

*(i) the eigenvalues of $S_n$ are $(n-2r+1)i$ $(1 \le r \le n)$;*

*(ii) $\|S_n\| = n-1$.*

*Proof.* The matrix $S_n$ is as in Proposition 7.2, with $a_j = -\operatorname{cosec}(\pi j/n)$ for $1 \le j \le n-1$ and $a_0 = 0$. First, we evaluate $\lambda_1$:

$$\lambda_1 = -\sum_{k=1}^{n-1} \operatorname{cosec} \frac{\pi k}{n}\, \rho_1^k = -\sum_{k=1}^{n-1} \operatorname{cosec} \frac{\pi k}{n}\, \exp \frac{\pi k i}{n}.$$

Substitute $n-k$ for $k$: since $\operatorname{cosec}(\pi-\theta) = \operatorname{cosec}\theta$ and $e^{i(\pi-\theta)} = -e^{-i\theta}$, we have

$$\lambda_1 = \sum_{k=1}^{n-1} \operatorname{cosec}\frac{\pi k}{n} \exp\left(-\frac{\pi k i}{n}\right).$$

Adding the two expressions, we see that

$$\lambda_1 = -i\sum_{k=1}^{n-1} \operatorname{cosec}\frac{\pi k}{n}\ \sin\frac{\pi k}{n} = -(n-1)i.$$

Now consider $\lambda_{r+1} - \lambda_r$, where $1 \le r \le n-1$. We have

$$\rho_{r+1}^k - \rho_r^k = \exp\frac{2rk\pi i}{n}.\ 2i\sin\frac{\pi k}{n} = 2i\sin\frac{\pi k}{n}\ e\left(\frac{rk}{n}\right),$$

so

$$\lambda_{r+1} - \lambda_r = -2i\sum_{k=1}^{n-1} e(rk/n) = 2i,$$

since, by the geometric series again, $\sum_{k=1}^{n-1} e(rk/n) = -1$. Hence $\lambda_{r+1} - \lambda_r = 2i$, and so $\lambda_r = (2r-n-1)i$ for $1 \le r \le n$ (equally, the eigenvalues can be listed as $(n-2r+1)i$ for $1 \le r \le n$). □

Without details, we mention the corresponding steps for cot instead of cosec. First, the analogue of Proposition 7.2 for a *circulant* matrix, which is proved in the same way:

**Proposition 7.4**. *Let $T$ be the $n \times n$ matrix $T = (t_{j,k})$, where $t_{j,k} = a_{k-j}$ for some sequence $(a_j)$ satisfying $a_{j-n} = a_j$ for all $j$. Write $\omega = e(1/n)$. Then mutually orthogonal eigenvectors $x_r$ $(0 \le r \le n-1)$ are defined by $x_r(k) = \omega^{rk}$. The corresponding eigenvalues are $\lambda_r = \sum_{k=0}^{n-1} a_k\omega^{rk}$.*

**Theorem 7.5**. *Let $T_n$ be the matrix $(t_{j,k})$ $(1 \le j,k \le n)$, where*

$$t_{j,k} = \begin{cases} \cot\frac{\pi}{n}(j-k) & \text{for } j \ne k, \\ 0 & \text{for } j = k. \end{cases}$$

*Then*

(i) *the eigenvalues of $T_n$ are 0 and $(n-2-2r)i$ $(0 \le r \le n-2)$;*

(ii) $\|T_n\| = n-2$.

*Sketch of proof.* It is easily seen that $\lambda_0 = 0$. One finds that

$$\begin{aligned} \lambda_1 &= -i\sum_{k=1}^{n-1} \cot\frac{\pi k}{n}\sin\frac{2\pi k}{n} \\ &= -2i\sum_{k=1}^{n-1} \cos^2\frac{\pi k}{n} \end{aligned}$$

$$\begin{aligned} &= -i\sum_{k=1}^{n-1}\left(1+\cos\frac{2\pi k}{n}\right) \\ &= -(n-2)i, \end{aligned}$$

since $\sum_{k=1}^{n-1}\cos(2\pi k/n) = -1$. One shows further that $\lambda_{r+1} - \lambda_r = 2i$ for $1 \le r \le n-2$. (When $n$ is even, the eigenvalue 0 occurs twice.) □

We now give the promised derivation of Hilbert's inequality from Theorem 7.3, with an estimation of $\|B_n\|$. Recall that $c_r$ means $1/r$ for $r \neq 0$ and 0 for $r = 0$.

**Theorem 7.6**. *Let $B_n = (c_{j-k})$ $(1 \le j, k \le n)$ and $B = (c_{j-k})_{j,k\ge 1}$. Then*

$$\|B_n\| \le \pi\left(1 - \frac{1}{n}\right),$$

*and hence $\|B\| \le \pi$.*

*Proof.* Take scalars $x_j$ and let $S(x) = \sum_{j=1}^n \sum_{k=1}^n c_{j-k}x_j\overline{x}_k$. Let $s_{j,k}$ be as in Theorem 7.3. For $j \neq k$, let $c_{j-k} = s_{j,k}d_{j,k}$, so that

$$d_{j,k} = \frac{1}{j-k}\sin\frac{\pi(j-k)}{n}.$$

Now

$$\int_{-1/2}^{1/2} e(\lambda t)\, dt = \frac{1}{2\pi i\lambda}(e^{\pi i\lambda} - e^{-\pi i\lambda}) = \frac{\sin\pi\lambda}{\pi\lambda}.$$

Hence

$$d_{j,k} = \frac{\pi}{n}\int_{-1/2}^{1/2} e\left(\frac{j-k}{n}t\right)\, dt.$$

For completeness (though it is not really needed), put $d_{j,j} = \pi/n$. Then

$$S(x) = \frac{\pi}{n}\int_{-1/2}^{1/2} F(t)\, dt,$$

where

$$F(t) = \sum_{j=1}^n\sum_{k=1}^n s_{j,k}\, x_j\overline{x}_k\, e\left(\frac{j-k}{n}t\right) = \sum_{j=1}^n\sum_{k=1}^n s_{j,k}y_j(t)\overline{y}_k(t),$$

where $y_j(t) = x_j\, e(jt/n)$. By Theorem 7.3,

$$|F(t)| \le (n-1)\sum_{j=1}^n |y_j(t)|^2 = (n-1)\sum_{j=1}^n |x_j|^2,$$

so

$$|S(x)| \le \frac{\pi}{n}(n-1)\sum_{j=1}^n |x_j|^2. \qquad \square$$

As with $A_n$, an exact evaluation of the norm of $B_n$ is elusive. H.L. Montgomery has stated in private correspondence that $\pi - \|B_n\|$ is $O(\log n/n)$, but it would appear that no proof has ever been published.

*Parting thought.* It is not really surprising that $\|S_n\|$ and $\|T_n\|$ are "nice" while $\|A_n\|$ and $\|B_n\|$ are "nasty". The matrices $S_n$ and $T_n$ are constructed from what might be called a complete system, equally spaced values in one cycle of a periodic function. The numbers $1/(j-k)$ only form a complete system when the whole set of integers is included, and the matrices $A_n$ and $B_n$ represent unnatural truncations of this system.

## 8. The generalizations by Montgomery and Vaughan

Two powerful generalizations were established by Montgomery and Vaughan [MontV] in 1974. They deal, respectively, with matrices of the form $[1/(\lambda_j - \lambda_k)]$ (generalizing Hilbert's inequality) and $[\operatorname{cosec} \pi(\lambda_j - \lambda_k)]$ (generalizing Theorem 7.3), with the numbers $\lambda_j$ no longer equally spaced. Both results have applications in analytic number theory.

It is possible to prove either result first and deduce the other, or, with sufficiently careful formulation, to prove both together. In [MontV], the result for $[\operatorname{cosec} \pi(\lambda_j - \lambda_k)]$ is proved first. Here we will prove the result for $[1/(\lambda_j - \lambda_k)]$ first. The statement is as follows.

**Theorem 8.1**. *Suppose that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $\lambda_j - \lambda_{j-1} \geq \delta$ for each $j$. Let $G_n$ be the matrix $(g_{j,k})$, where*

$$g_{j,k} = \begin{cases} 1/(\lambda_j - \lambda_k) & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases}$$

*Then $\|G_n\| \leq \pi/\delta$.*

First, some preliminary comments. The matrix is still skew-symmetric, but no longer Toeplitz. After multiplying by a constant, we can assume that $\delta = 1$. Then $\lambda_j - \lambda_k \geq j - k$, and hence $0 < g_{j,k} \leq 1/(j-k)$ for $j > k$. One might suppose that $\|G_n\| \leq \|B_n\|$ simply because all the entries are smaller, while still the same sign. However, no such statement is true, even for matrices of this special type! A specific example is as follows.

*Example.* Let $M$ be the $4 \times 4$ skew-symmetric, Toeplitz matrix $[f(j-k)]$, where $f(1) = a$, $f(2) = b$, $f(3) = c$ and $f(-r) = -f(r)$ (again we save space by refraining from printing the matrix explicitly). One can show that $M$ has the same norm as the $2 \times 2$ matrix

$$\begin{pmatrix} c & a+b \\ b-a & a \end{pmatrix}$$

(details on request). The explicit formula for the norm of a real $2 \times 2$ matrix was given in Section 2. We find that the choice $(1, 0, 1)$ for $(a, b, c)$ gives $\|M\| = \sqrt{2}$, while the choice $(1, 0, 0)$ gives $\|M\| = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$. (For the record, this gives $\|B_4\| = \frac{1}{2}\sqrt{13}$.)

Here we give a quite simple proof based on Method 1. However, it only delivers a slightly weaker result, with an intervening constant which happens to be $2/\sqrt{3}$.

*Proof of Theorem 8.1 with an extra constant.* Since $G_n$ is skew-symmetric, it is sufficient to consider quadratic forms: given (complex) $x_j$ $(1 \le j \le n)$, let $S(x) = \sum_{j=1}^{n} \sum_{k=1}^{n} g_{j,k} x_j \overline{x}_k$.

Let $f(t) = \sum_{j=1}^{n} x_j e(\lambda_j t)$, so that

$$|f(t)|^2 = \sum_{j=1}^{n} |x_j|^2 + \sum_{j=1}^{n} \sum_{k \neq j} x_j \overline{x}_k e[(\lambda_j - \lambda_k)t]. \tag{7}$$

With $c > 0$ to be chosen later, we will use the fact that $I \ge 0$, where

$$I = \int_0^c (c - t)|f(t)|^2 \, dt.$$

(The use of $c - t$ instead of $t$ makes the following work slightly simpler.) Of course, terms no longer cancel to zero in the way they did when the $\lambda_j$ were integers. In fact, for $\lambda \neq 0$,

$$\int_0^c e(\lambda t) \, dt = \frac{e(\lambda c) - 1}{2\pi i \lambda},$$

$$\begin{aligned} \int_0^c (c - t) e(\lambda t) \, dt &= -\frac{c}{2\pi i \lambda} + \frac{1}{2\pi i \lambda} \int_0^c e(\lambda t) \, dt \\ &= -\frac{c}{2\pi i \lambda} - \frac{e(\lambda c) - 1}{4\pi^2 \lambda^2}. \end{aligned}$$

Also, $\int_0^c (c - t) \, dt = \frac{1}{2} c^2$. So

$$I = \tfrac{1}{2} c^2 \sum_{j=1}^{n} |x_j|^2 - \frac{c}{2\pi i} S(x) + R,$$

where

$$R = -\frac{1}{4\pi^2} \sum_{j=1}^{n} \sum_{k \neq j} \frac{x_j \overline{x}_k}{(\lambda_j - \lambda_k)^2} [e(\lambda_j c - \lambda_k c) - 1].$$

So

$$|R| \le \frac{2}{4\pi^2} \sum_{j=1}^{n} \sum_{k \neq j} \frac{|x_j x_k|}{(\lambda_j - \lambda_k)^2} \le \frac{1}{2\pi^2} \sum_{j=1}^{n} \sum_{k \neq j} \frac{|x_j x_k|}{(j - k)^2}.$$

By Theorem 5.2 (or 6.3),

$$|R| \le \frac{1}{2\pi^2} \frac{\pi^2}{3} \sum_{j=1}^{n} |x_j|^2 = \frac{1}{6} \sum_{j=1}^{n} |x_j|^2.$$

So the inequality $I \geq 0$ translates into

$$\frac{c}{2\pi i} S(x) \leq \left(\tfrac{1}{2}c^2 + \tfrac{1}{6}\right) \sum_{j=1}^{n} |x_j|^2.$$

This applies equally to $S(\overline{x}) = -S(x)$, so in fact

$$|S(x)| \leq \pi \left(c + \frac{1}{3c}\right) \sum_{j=1}^{n} |x_j|^2.$$

To minimize $c + 1/3c$, take $c = 1/\sqrt{3}$, giving $c + 1/3c = 2/\sqrt{3}$. (Note that if we simplified the proof by taking $c = 1$, the constant obtained would be $4/3$.) □

*Note.* The statement still holds if the numbers $\lambda_j$ are not listed in increasing order, but satisfy $|\lambda_j - \lambda_k| \geq \delta$ for $j \neq k$. This amounts to re-ordering the columns, which of course does not alter the norm of the matrix.

At the cost of rather more work, Montgomery and Vaughan obtained the result as stated, without the factor $2/\sqrt{3}$. A version of their proof can be seen in [Mont1] (and may be added to these notes eventually). Rather like our Method 2, it starts from the quadratic form $\langle G_n^* G_n x, x \rangle$, but essential use is made of the fact that a skew-hermitian operator attains its norm at an eigenvector. We will now take the liberty of assuming the result without the factor $2/\sqrt{3}$.

We mention an immediate application to functions of the form $f(t) = \sum_{j=1}^{n} x_j e(\lambda_j t)$, which were used in the proof; such functions are called *generalized Dirichlet polynomials.* We have the following estimation for integrals of $|f(t)|^2$.

**Theorem 8.2**. *Suppose that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $\lambda_j - \lambda_{j-1} \geq \delta$ for each $j$. Let $f(t) = \sum_{j=1}^{n} x_j e(\lambda_j t)$. Then, for any $T > 0$, we have*

$$\int_0^T |f(t)|^2 \, dt = \left(T + \frac{\theta}{\delta}\right) \sum_{j=1}^{n} |x_j|^2$$

*for some $\theta$ with $|\theta| \leq 1$.*

*Proof.* By (7), it is clear that $\int_0^T |f(t)|^2 \, dt = T \sum_{j=1}^{n} |x_j|^2 + r(T)$, where

$$r(T) = \frac{1}{2\pi i} \sum_{j=1}^{n} \sum_{k \neq j} x_j \overline{x}_k \frac{e[(\lambda_j - \lambda_k)T] - 1}{\lambda_j - \lambda_k}.$$

Write $x_j e(\lambda_j T) = y_j$. Then (with the notation of Theorem 8.1)

$$2\pi i r(T) = \sum_{j=1}^{n} \sum_{k \neq j} \frac{y_j \overline{y}_k - x_j \overline{x}_k}{\lambda_j - \lambda_k} = S(y) - S(x).$$

By Theorem 8.1, $|S(x)|$ and $|S(y)|$ are not greater than $(\pi/\delta)\sum_{j=1}^{n}|x_j|^2$. The statement follows. □

We can read off the following corollary for an "ordinary" Dirichlet polynomial $F(s) = \sum_{j=1}^{n} x_j/j^s$.

**Corollary 8.3**. *Let* $F(s) = \sum_{j=1}^{n} x_j/j^s$. *Then*

$$\int_0^T |F(it)|^2\,dt = (T + 2\pi\theta n)\sum_{j=1}^{n}|x_j|^2,$$

*where* $|\theta| \le 1$.

*Proof.* We have $j^{-it} = e(-\lambda_j t)$, where $\lambda_j = (\log j)/2\pi$. The smallest $\lambda_j - \lambda_{j-1}$ is

$$\lambda_n - \lambda_{n-1} = \frac{1}{2\pi}(\log n - \log(n-1)) > \frac{1}{2\pi n}. \qquad \square$$

We now come to the second theorem of Montgomery and Vaughan, applying to matrices of the form $[\operatorname{cosec}\pi(\lambda_j - \lambda_k)]$. Clearly, $\lambda_j - \lambda_k$ needs to be not only non-zero, but not an integer. The relevant measure of separation is now the distance from the nearest integer. We introduce the following notation. For any real number $\lambda$, let $\Delta(\lambda)$ be this distance:

$$\Delta(\lambda) = \inf\{|\lambda - n| : n \in \mathbb{Z}\}.$$

(The notation $\|\lambda\|$ is often used for this, but we are already using this notation for norms of vectors and matrices.)

**Theorem 8.4**. *Let* $\lambda_j$ $(1 \le j \le n)$ *be real numbers such that* $\Delta(\lambda_j - \lambda_k) \ge \delta$ *whenever* $j \ne k$, *and let* $H$ *be the matrix defined by*

$$h_{j,k} = \begin{cases} \operatorname{cosec}[\pi(\lambda_j - \lambda_k)] & \text{if } j \ne k, \\ 0 & \text{if } j = k. \end{cases}$$

*Then* $\|H\| \le 1/\delta$.

*The same conclusion applies if cosec* $[\pi(\lambda_j - \lambda_k)]$ *is replaced by* $\cot[\pi(\lambda_j - \lambda_k)]$.

*Proof.* We shall deduce the result from Theorem 8.1. We use the well-known identity

$$\pi \operatorname{cosec}\pi\lambda = \lim_{n\to\infty}\sum_{m=-n}^{n}\left(1 - \frac{|m|}{n}\right)\frac{(-1)^m}{\lambda + m}, \tag{8}$$

which can be proved by applying Fejér's convergence theorem to the Fourier series for $\cos\lambda t$.

Choose $n > 1$ and scalars $x_j$. For $1 \le j \le n$ and $1 \le r \le n$, define

$$\lambda_{j,r} = \lambda_j + r,$$

$$x_{j,r} = (-1)^r x_j.$$

If $(j,r) \neq (k,s)$, then

$$|\lambda_{j,r} - \lambda_{k,s}| = |\lambda_j - \lambda_k - (s-r)| \geq \delta,$$

(if $j = k$, then $r \neq s$, so $|r-s| \geq 1$). Let

$$S_n(x) = \sum_{(j,r)\neq(k,s)} \frac{x_{j,r}\overline{x}_{k,s}}{\lambda_{j,r} - \lambda_{k,s}}$$

(the summation is over all $j$, $k$, $r$, $s$ such that $(j,r) \neq (k,s)$). By Theorem 8.1, applied to the scalars $\lambda_{j,r}$,

$$|S_n(x)| \leq \frac{\pi}{\delta} \sum_{j=1}^{n} \sum_{r=1}^{n} |x_{j,r}|^2 = \frac{\pi n}{\delta} \sum_{j=1}^{n} |x_j|^2.$$

Now

$$S_n(x) = \sum_{(j,r)\neq(k,s)} \frac{(-1)^{r+s} x_j \overline{x}_k}{\lambda_j - \lambda_k + (r-s)}.$$

For fixed $j$, the $[(j,r),(j,s)]$ terms combine to

$$|x_j|^2 \sum_{r=1}^{n} \sum_{s\neq r} \frac{(-1)^{r+s}}{r-s} = 0,$$

so

$$S_n(x) = \sum_{j=1}^{n} \sum_{k\neq j} \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{(-1)^{r+s} x_j \overline{x}_k}{\lambda_j - \lambda_k + (r-s)}.$$

For a chosen $m$ with $-n \leq m \leq n$, there are $n - |m|$ (equal) terms with $r - s = m$ (e.g. if $m \geq 0$, these are given by $1 \leq s \leq n-m$ with $r = s+m$). Hence

$$S_n(x) = \sum_{j=1}^{n} \sum_{k\neq j} \sum_{m=-n}^{n} (n-|m|) \frac{(-1)^m x_j \overline{x}_k}{\lambda_j - \lambda_k + m}.$$

By (8),

$$\sum_{j=1}^{n} \sum_{k\neq j} \operatorname{cosec} \pi(\lambda_j - \lambda_k) x_j \overline{x}_k = \lim_{n\to\infty} \frac{1}{n} S_n(x),$$

and therefore the modulus of this expression is not greater than $(1/\delta) \sum_{j=1}^{n} |x_j|^2$.

Finally, if (8) is modified by removing the term $(-1)^m$, the resulting expression equals $\pi \cot \pi\lambda$. The proof, simplified by removing $(-1)^m$ throughout, shows that cosec can be replaced by cot in the statement. □

We have deduced Theorem 8.4 from Theorem 8.1. Conversely, if Theorem 8.4 has been proved by another method, one can deduce Theorem 8.1 quite simply, as in the proof of Theorem 7.6.

[MontV] also proved the following variants, in which the overall spacing $\delta$ is replaced by the spacing $\delta_j$ separating $\lambda_j$ from the other points.

**Theorem 8.5**. *(i) Let $g_{j,k}$ be as in Theorem 8.1, and suppose that $|\lambda_k - \lambda_j| \geq \delta_j$ for $k \neq j$. Then*

$$\left| \sum_{j=1}^{n} \sum_{k=1}^{n} g_{j,k} x_j \overline{x}_k \right| \leq \frac{3\pi}{2} \sum_{j=1}^{n} \frac{|x_j|^2}{\delta_j}.$$

*(ii) Let $h_{j,k}$ be as in Theorem 8.4, with $\Delta(\lambda_k - \lambda_j) \geq \delta_j$ for $k \neq j$. Then*

$$\left| \sum_{j=1}^{n} \sum_{k=1}^{n} h_{j,k} x_j \overline{x}_k \right| \leq \frac{3}{2} \sum_{j=1}^{n} \frac{|x_j|^2}{\delta_j}.$$

It is not known whether the factor $\frac{3}{2}$ can be removed.

## 9. Applications to the large sieve inequality and Dirichlet polynomials

Consider again a trigonometric polynomial $f(t) = \sum_{j=1}^{n} x_j e(jt)$, as used in the first proof of Hilbert's inequality. We now address the problem of finding a bound for sums of squares of values. Such bounds are called "large value estimates", since they set a limit on the number of times large values occur. More exactly, we want a constant $C$ such that

$$\sum_{r=1}^{R} |f(t_r)|^2 \leq C \sum_{j=1}^{n} |x_j|^2. \tag{9}$$

Since $f$ has period 1, the values would simply reinforce if the points $t_r$ differed by integers, so we require the points $t_r$ to be "well separated" in the sense that $\Delta(t_r - t_s) \geq \delta$ for $r \neq s$. The constant $C$ is to depend on $n$ and $\delta$, but not $R$. Those not already soaked in the culture of analytic number theory will find it somewhat intriguing that the answer is known as the "large sieve inequality".

The problem is really another exercise in determining the norm of a matrix. Indeed, since

$$f(t_r) = \sum_{j=1}^{n} e(jt_r) x_j,$$

the constant $C$ in (9) is $\|V\|^2$, where $V$ is the matrix $[e(jt_r)]$ $(1 \leq j \leq n,\ 1 \leq r \leq R)$.

A minor remark will be useful. The modulus of $f(t)$ is unchanged if it is multiplied by $e(kt)$, putting it into the form $\sum_{j=k+1}^{k+n} y_j e(jt)$. So the problem is the same if the range of values of $j$ is translated by $k$: the role of $n$ is to state the length of this range.

Of course, *mean* values of $|f(t)|^2$ are measured by its integral, which we know very well: $\int_0^1 |f(t)|^2 dt = \sum_{j=1}^n |x_j|^2$. The following lemma shows how to convert estimates for integrals into estimates of functional values.

**Lemma 9.1**. *Let $g$ be a differentiable function (real or complex) on $[c-h, c+h]$. Then*

$$|g(c)| \leq \frac{1}{2h} \int_{c-h}^{c+h} |g(t)|\, dt + \frac{1}{2} \int_{c-h}^{c+h} |g'(t)|\, dt.$$

*Proof.* Let

$$\rho(t) = \begin{cases} t - c + h & \text{for } c - h < t < c, \\ t - c - h & \text{for } c < t < c + h. \end{cases}$$

Integration by parts on the intervals $[c - h, c]$ and $[c, c + h]$ leads easily to

$$\int_{c-h}^{c+h} \rho(t) g'(t)\, dt = 2hg(c) - \int_{c-h}^{c+h} g(t)\, dt.$$

Since $|\rho(t)| \leq h$, the statement follows. □

Using this lemma, we deduce the following provisional solution to our problem. The method is due to Gallagher [Gall].

**Proposition 9.2**. *Let $f(t) = \sum_{j=1}^n x_j e(jt)$. Let $t_r$ ($1 \leq r \leq R$) be points such that $\Delta(t_r - t_s) \geq \delta$ for $r \neq s$. Then*

$$\sum_{r=1}^R |f(t_r)|^2 \leq \left(\pi n + \frac{1}{\delta}\right) \sum_{j=1}^n |x_j|^2.$$

*Proof.* As mentioned above, we can translate the range of $j$. In particular, we can move it to an interval $J$ contained in $[-\frac{1}{2}n, \frac{1}{2}n]$, now taking $f(t)$ to be $\sum_{j \in J} x_j e(jt)$, with $J$ as stated. Also, we may assume that $t_1 < t_2 < \cdots < t_R$, with $t_r - t_{r-1} \geq \delta$, and $t_1 + 1 \geq t_R + \delta$, so that if $t_1 - \frac{1}{2}\delta = c$, then $t_R + \frac{1}{2}\delta \leq 1 + c$. So the intervals $[t_r - \frac{1}{2}\delta, t_r + \frac{1}{2}\delta]$ do not overlap, and are contained in $[c, 1 + c]$. By Lemma 9.1, applied to $f(t)^2$,

$$\sum_{r=1}^R |f(t_r)|^2 \leq \frac{1}{\delta} \int_c^{1+c} |f(t)|^2\, dt + \int_c^{1+c} |f(t) f'(t)|\, dt.$$

Now $\int_c^{1+c} |f(t)|^2 dt = \sum_{j \in J} |x_j|^2$. Also, since $f'(t) = 2\pi i \sum_{j \in J} j x_j e(jt)$ and $|j| \leq n/2$ for $j \in J$, we have

$$\int_c^{1+c} |f'(t)|^2\, dt = 4\pi^2 \sum_{j \in J} j^2 |x_j|^2 \leq \pi^2 n^2 \sum_{j \in J} |x_j|^2.$$

By the Cauchy-Schwarz inequality for integrals,

$$\int_c^{1+c} |f(t) f'(t)|\, dt \leq \pi n \sum_{n \in J} |x_j|^2.$$

The statement follows. $\square$

This result was obtained without using anything resembling Hilbert's inequality. For many purposes, it is quite good enough. However, it is not optimal. We show next that Theorem 8.4 is exactly what is needed to produce a better estimate, which in fact turns out to be optimal.

**Theorem 9.3**. *Let $t_r$ ($1 \leq r \leq R$) be points such that $\Delta(t_r - t_s) \geq \delta$ for $r \neq s$, and let $V$ be the matrix defined by $v_{r,j} = e(jt_r)$ for $1 \leq j \leq n$, $1 \leq r \leq R$. Then*

$$\|V\|^2 \leq (n-1) + \frac{1}{\delta}.$$

*Consequently, if $f(t) = \sum_{j=1}^n x_j e(jt)$, then*

$$\sum_{r=1}^R |f(t_r)|^2 \leq \left(n - 1 + \frac{1}{\delta}\right) \sum_{j=1}^n |x_j|^2.$$

*Proof.* We evaluate the norm of the transposed matrix (which, of course, is the same). Given scalars $y_r$, let

$$T(y) = \sum_{j=1}^n \left| \sum_{r=1}^R e(jt_r) y_r \right|^2 .$$

Then $\|V\|^2$ is the least constant $C$ for which we have $T(y) \leq C \sum_{r=1}^R |y_r|^2$ for all choices of $y_r$. Note first that, by the geometric series,

$$\sum_{j=1}^n e(jt) = e(t) \frac{e(nt) - 1}{e(t) - 1} = \frac{e[(n + \frac{1}{2})t] - e(\frac{1}{2}t)}{2i \sin \pi t}.$$

Hence we have

$$\begin{aligned} T(y) &= \sum_{j=1}^n \sum_{r=1}^R \sum_{s=1}^R y_r \overline{y}_s e[j(t_r - t_s)] \\ &= n \sum_{r=1}^R |y_r|^2 + \sum_{r \neq s} y_r \overline{y}_s \sum_{j=1}^n e[j(t_r - t_s)] \\ &= n \sum_{r=1}^R |y_r|^2 + \sum_{r \neq s} y_r \overline{y}_s \frac{e[(n + \frac{1}{2})(t_r - t_s)] - e[\frac{1}{2}(t_r - t_s)]}{2i \sin \pi (t_r - t_s)}, \end{aligned}$$

in which $\sum_{r \neq s}$ means summation over all pairs $(r, s)$ with $r \neq s$. Now for any $a$,

$$\sum_{r \neq s} y_r \overline{y}_s \frac{e[a(t_r - t_s)]}{2 \sin \pi (t_r - t_s)} = \sum_{r \neq s} \frac{z_r \overline{z}_s}{2 \sin \pi (t_r - t_s)},$$

where $z_r = y_r e(at_r)$. Since $|z_r| = |y_r|$, Theorem 8.4 shows that the modulus of this expression is not greater than $(1/2\delta)\sum_{r=1}^R |y_r|^2$. Apply this with $a = n + \frac{1}{2}$ and $a = \frac{1}{2}$ to obtain

$$|T(y)| \le \left(n + \frac{1}{\delta}\right)\sum_{r=1}^R |y_r|^2.$$

This completes the proof, apart from showing that $n$ can be replaced by $n-1$ (a refinement that is quite unimportant in applications). This was first shown by Selberg, but the following neat proof was given by Paul Cohen. Choose $K > 1$, and let

$$g(t) = f(Kt) = \sum_{j=1}^n x_j e(jKt),$$

which we can write as $\sum_{k=K}^{Kn} y_k e(kt)$, in which $\sum_{k=K}^{Kn} |y_k|^2 = \sum_{j=1}^n |x_j|^2$. Since $f$ has period 1,

$$K\sum_{r=1}^R |f(t_r)|^2 = \sum_{k=1}^K \sum_{r=1}^R |f(t_r + k)|^2 = \sum_{k=1}^K \sum_{r=1}^R \left| g\left(\frac{t_r + k}{K}\right)\right|^2.$$

The numbers $(t_r + k)/K$ are separated by $\delta/K$, so, by the result already proved,

$$K\sum_{r=1}^R |f(t_r)|^2 \le \left((Kn - K + 1) + \frac{K}{\delta}\right)\sum_{j=1}^n |x_j|^2.$$

Now divide by $K$ and let $K$ tend to infinity to obtain the result. □

A trivial example is enough to show that the expression in Theorem 9.3 is optimal, within the rules we have set for ourselves.

*Example.* Take $f(t) = e(0) + e(2t)$, so that $n = 3$ and $x_0 = 1$, $x_1 = 0$ and $x_2 = 1$. Let $t_1 = 0$ and $t_2 = \frac{1}{2}$, so that $\delta = \frac{1}{2}$. Then $f(t_1) = f(t_2) = 2$, so $f(t_1)^2 + f(t_2)^2 = 8$, while $\sum_{j=0}^2 x_j^2 = 2$. The ratio is 4, equal to $(n-1) + 1/\delta$.

An account of applications to number theory, and an explanation of the term "large sieve", is given in my companion website notes [Jam2]. A good historical account, with an extensive list of references, is given in [Mont1].

We now address the problem of a "large values" estimate for a generalized Dirichlet polynomial $f(t) = \sum_{j=1}^n x_j e(\lambda_j t)$. The integral of $|f(t)|^2$ is no longer trivial: it was estimated in Theorem 8.2. We used the first Montgomery-Vaughan theorem (Theorem 8.1) to get this far. We now imitate the proof of Proposition 9.2 to derive the required estimate. There are now two sets of numbers that need to be well separated, $(\lambda_j)$ and $(t_r)$.

**Theorem 9.4**. *Suppose that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ with $\lambda_j - \lambda_{j-1} \ge \delta$ and $|\lambda_j| \le M$ for all $j$. Let $f(t) = \sum_{j=1}^n x_j e(\lambda_j t)$. Also, let $t_1 < t_2 < \cdots < t_R$, with $t_r - t_{r-1} \ge h$ for each $r$,* *with* $t_1 \geq \frac{1}{2}h$ *and* $t_R \leq T - \frac{1}{2}h$. *Then*

$$\sum_{r=1}^{R} |f(t_r)|^2 \leq \left(2\pi M + \frac{1}{h}\right)\left(T + \frac{1}{\delta}\right)\sum_{j=1}^{n} |x_j|^2.$$

*Proof.* The intervals $[t_r - \frac{1}{2}h, t_r + \frac{1}{2}h]$ do not overlap, so by Lemma 9.1, applied to $g(t) = f(t)^2$,

$$\sum_{r=1}^{R} |f(t_r)|^2 \leq \frac{1}{h}\int_0^T |f(t)|^2\, dt + \int_0^T |f(t)f'(t)|\, dt.$$

By Theorem 8.2,

$$\int_0^T |f(t)|^2\, dt \leq \left(T + \frac{1}{\delta}\right)\sum_{j=1}^{n} |x_j|^2.$$

Also, $f'(t) = 2\pi i \sum_{j=1}^{n} \lambda_j x_j e(\lambda_j t)$, so

$$\begin{aligned}\int_0^T |f'(t)|^2\, dt &\leq 4\pi^2\left(T + \frac{1}{\delta}\right)\sum_{j=1}^{n} \lambda_j^2 |x_j|^2 \\ &\leq 4\pi^2 M^2\left(T + \frac{1}{\delta}\right)\sum_{j=1}^{n} |x_j|^2.\end{aligned}$$

By the Cauchy-Schwarz inequality for integrals,

$$\int_0^T |f(t)f'(t)|\, dt \leq 2\pi M\left(T + \frac{1}{\delta}\right)\sum_{j=1}^{n} |x_j|^2.$$

The statement follows. □

**Corollary 9.5**. *Let* $F(s) = \sum_{j=1}^{n} x_j/j^s$, *and let* $t_r$ *(*$1 \leq r \leq R$*) be as in Theorem 9.4. Then*

$$\sum_{r=1}^{R} |F(it_r)|^2 \leq \left(\log n + \frac{1}{h}\right)(T + 2\pi n)\sum_{j=1}^{n} |x_j|^2.$$

*Proof.* Take $\lambda_j = (\log j)/2\pi$ in Theorem 9.7: then $2\pi M = \log n$ and $\delta = 1/(2\pi n)$. □

*Note.* For $f(t)$ as in 9.4, the Cauchy-Schwarz inequality gives $|f(t)|^2 \leq n\sum_{j=1}^{n} |x_j|^2$, so it is trivial that $\sum_{r=1}^{R} |f(t_r)|^2 \leq Rn\sum_{j=1}^{n} |x_j|^2$. Also, the assumptions force $Rh \leq T$.

Again, there are numerous applications in analytic number theory. An unsolved problem, Montgomery's "large values conjecture", asks whether a bound of the form $CT^{\epsilon}(R + n)n \max |x_j|^2$ is correct in 9.5. See [Mont2], [Ten], [Gr].

## 10. Summary of some other generalizations

Hilbert's inequality can be generalized in many different directions in addition to those we have discussed. These generalizations have generated a massive literature. The following is a very incomplete outline of some of them.

*Vector-valued extensions*

Recall that (2) and (3) are, respectively, the linear and bilinear inequalities expressing the norm of a matrix. For *any* matrix, including all the ones we have discussed, the scalars in both statements can be replaced by vectors. This is easily proved, as follows.

**Proposition 10.1**. *Let $A$ be an $m \times n$ matrix, and let $H$ be a Hilbert space. Then:*

*(i) If $x_1, \dots, x_n \in H$ and $y_j = \sum_{k=1}^n a_{j,k} x_k$ for $1 \le j \le m$, then*

$$\sum_{j=1}^m \|y_j\|^2 \le \|A\|^2 \sum_{k=1}^n \|x_k\|^2.$$

*(ii) For elements $x_j$, $y_k$ of $H$, we have*

$$\left| \sum_{j=1}^m \sum_{k=1}^n a_{j,k} \langle y_k, x_j \rangle \right| \le \|A\| \left( \sum_{j=1}^m \|x_j\|^2 \right)^{1/2} \left( \sum_{k=1}^n \|y_k\|^2 \right)^{1/2}.$$

*Proof.* (i) Choose an orthonormal basis $e_1, e_2, \dots, e_s$ of the subspace generated by the elements $x_j$, and write $x_j = \sum_{r=1}^m x_j(r) e_r$. Then $\|x_k\|^2 = \sum_{r=1}^s |x_k(r)|^2$. Also, $y_j(r) = \sum_{k=1}^n a_{j,k} x_k(r)$. So, by (2),

$$\sum_{j=1}^m |y_j(r)|^2 \le \|A\|^2 \sum_{k=1}^n |x_k(r)|^2$$

for each $r$. Summation over $r$ gives the statement.

(ii). The sum in question is $\sum_{j=1}^m \langle z_j, x_j \rangle$, where $z_j = \sum_{k=1}^n a_{j,k} y_k$. By the Cauchy-Schwarz inequality for general inner products, $|\langle z_j, x_j \rangle| \le \|z_j\|.\|x_j\|$, and now by the "concrete" Cauchy-Schwarz inequality for a sum of products,

$$\sum_{j=1}^m \|z_j\|.\|x_j\| \le \left( \sum_{j=1}^m \|z_j\|^2 \right)^{1/2} \left( \sum_{j=1}^m \|x_j\|^2 \right)^{1/2}.$$

By (i), $\sum_{j=1}^m \|z_j\|^2 \le \|A\|^2 \sum_{k=1}^n \|y_k\|^2$. The statement follows. □

A further development is that the scalars in (2) and (3) can be replaced by *operators on a Hilbert space*, with terms like $|x_j|^2$ replaced by $X_j^* X_j$, and the partial ordering defined in the usual way for operators. See [Jam1].

*Another variation of Method 1: multiplication operators on $L_2$*

In Method 1, we used quadratic or bilinear forms (which we expressed as an integral) to determine the norm of a matrix; a general statement of the conclusion was given in Theorem 5.5. We can reach the same conclusion using the linear, rather than bilinear, formulation of norms, by exhibiting our matrix as the matrix of a suitable linear operator, the norm of which is (almost) obvious. However, this approach requires a full understanding of the space $L_2[0,1]$ (note: many writers prefer the notation $L^2$): it consists of equivalence classes, under equality almost everywhere, of Lebesgue measurable functions, with norm

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2 \, dt\right)^{1/2}.$$

In particular, we need to know that $L_2[0,1]$ is complete (so a Hilbert space), and that the functions $\psi_n(t) = e(nt)$ $(n \in \mathbb{Z})$ are an orthonormal basis of it; the expression of a function in terms of this basis is, of course, its Fourier series.

Similarly, $L_\infty[0,1]$ is the space of equivalence classes of "essentially bounded" functions, with norm $\|\phi\|_\infty$ defined by the "essential supremum". Given such a function $\phi$, the corresponding *multiplication operator* $M_\phi$ on $L_2[0,1]$ is defined by: $(M_\phi f)(t) = \phi(t)f(t)$. Clearly,

$$\int_0^1 |\phi(t)f(t)|^2 \, dt \le \|\phi\|_\infty^2 \int_0^1 |f(t)^2 \, dt,$$

and hence $\|M_\phi\| \le \|\phi\|_\infty$. With a bit more care, one can establish equality here. Let $d_r = \hat{\phi}(r) = \int_0^1 \phi(t)e(-rt)\, dt$ (this was $d_{-r}$ in Theorem 5.5). Then $\phi = \sum_{r\in\mathbb{Z}} d_r\psi_r$ in the sense of $L_2$-convergence. Clearly, $\psi_j\psi_k = \psi_{j+k}$. Hence

$$\phi\psi_k = \sum_{r\in\mathbb{Z}} d_r\psi_{r+k} = \sum_{j\in\mathbb{Z}} d_{j-k}\psi_j.$$

This shows that the matrix of $M_\phi$ with respect to the orthonormal basis $(\psi_j)_{j\in\mathbb{Z}}$ is $(d_{j-k})_{j,k\in\mathbb{Z}}$. The norm of this matrix (as an operator on $\ell_2(\mathbb{Z})$) is the same as the norm of $M_\phi$ as an operator on $L_2[0,1]$, hence equal to $\|\phi\|_\infty$. This recaptures Theorem 5.5, strengthened to an equality.

For $p$ equal to 2 or $\infty$, let $H_p$ and $K_p$ be the subspaces of $L_p[0,1]$ spanned by $\{\psi_j : j \ge 0\}$ and $\{\psi_j : j < 0\}$ respectively ($H_2$ and $H_\infty$ are *Hardy* spaces). One-sided Toeplitz and Hankel matrices represent $M_\phi$ restricted to, and projected onto, either $H_2$ or $K_2$. *Nehari's theorem* states that if $d_r = \hat{\phi}(r)$, then the norm of the Hankel matrix $(d_{-j-k+1})_{j,k\ge1}$ equates to the distance from $\phi$ to $H_\infty$ in the space $L_\infty[0,1]$, which in turn equates to

$$\inf\{\|\psi\|_\infty : \hat{\psi}(-j) = d_{-j} \text{ for all } j \ge 1\}.$$

See [Young], chapters 13 and 15.

*The case $p \neq 2$ and the continuous case: the matrix $A$*

There are two further important ways in which one can seek to extend matrix-norm inequalities of the kind we have been considering. One is the "continuous" analogue, in which sequences are replaced by functions, discrete sums by integrals and matrices by functions of two variables.

The other way is to vary the norm. For $p \geq 1$, the space $\ell_p$ is the set of sequences $x = (x_j)$ for which $\sum_{j\geq 1} |x_j|^p$ is convergent, with norm $\|x\|_p = (\sum_{j\geq 1} |x_j|^p)^{1/p}$. The $L_p$-norm of a function $f$ is defined in the analogous way, using integrals. Results on these spaces frequently involve the conjugate index $p^*$, defined by $1/p + 1/p^* = 1$. For any operator $T$ on $\ell_p$ (defined by a matrix) or $L_p$ (defined by an integral), the implied problem is to determine its norm with respect to $\| \ \|_p$, which we denote by $\|T\|_p$.

These notes have been exclusively devoted to the discrete case, with $p = 2$. Extensions of the two types just described can often be considered together, by analogous methods. Where they differ, the discrete case is usually harder, because it often entails discrete sums which can be estimated by corresponding integrals, but are not exactly equal to them.

Both these generalizations are fairly straightforward for our original matrix $A = (c_{j+k})_{j,k\geq 1}$. The continuous analogue is the operator (which we still denote by $A$) defined as follows: for a function $f$ on $[0, \infty)$,

$$(Af)(x) = \int_0^\infty \frac{f(y)}{x+y}\, dy.$$

Of course, convergence at 0, as well as at infinity, has to be considered.

Because $1/(j+k)$ and $1/(x+y)$ are *positive*, our Method 2 can be adapted, at the cost of slightly greater complexity, to obtain (in both cases)

$$\|A\|_p = \pi \operatorname{cosec} \frac{\pi}{p}$$

for $1 < p \leq \infty$, by way of the integral

$$\int_0^\infty \frac{1}{t^\alpha (t+1)}\, dt = \pi \operatorname{cosec} \alpha\pi.$$

(The fact that the column sums diverge shows at once that $A$ does not map $\ell_1$ into $\ell_1$). The same value applies to the matrix $A^{\#}$, but a different method is needed (interestingly, one way is to derive it from the continuous case). Details of these results are given in [HLP, chapter

9]. This extension of Hilbert's inequality was proved by G.H. Hardy, and is sometimes known as the Hardy-Hilbert inequality.

Some estimations have been given for the norm of $A$ as an operator from $\ell_p$ to $\ell_q$, where $p < q$, (e.g. [Bon]), but exact values are not known.

A further generalization is to the *weighted* $\ell_p$ space $\ell_p(w)$, where the norm is defined by $\|x\|_{p,w} = (\sum_{j\geq 1} w_j|x_j|^p)^{1/p}$ for a given weighting sequence $(w_j)$. For the case $w_j = 1/j^\alpha$, [JL] shows that the norm of $A$ is $\pi \operatorname{cosec}[(1-\alpha)\pi/p]$.

*The general case: the Hilbert operator and matrix*

The corresponding questions for the "full" Hilbert matrix $C$ are distinctly harder. The continuous analogue is, of course, the integral of $f(y)/(y-x)$. However, even if $f(y)$ is the constant function 1 on $[0,1]$ (and 0 elsewhere), the integral will diverge on both $[0,x)$ and $(x,1]$ if $0 < x < 1$. This is overcome by using the "principal value" integral to define the *Hilbert (integral) operator* $H$ as follows: for a function $f$ on $\mathbb{R}$,

$$(Hf)(x) = \frac{1}{\pi} \lim_{\delta\to 0^+} \left( \int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty} \right) \frac{f(y)}{y-x}\, dy.$$

By obvious substitutions, this can be rewritten as

$$(Hf)(x) = \frac{1}{\pi} \lim_{\delta\to 0^+} \int_{\delta}^{\infty} \frac{f(x+t) - f(x-t)}{t}\, dt.$$

(Note that the factor $(1/\pi)$ is included in the definition; equally, the "Hilbert matrix" is usually defined, in our notation, to be $C/\pi$.) As an exercise, the reader might like to show that if $f$ is 1 on $[0,1]$, then $\pi(Hf)(x) = \log|1-x| - \log|x|$ for all $x \neq 0, 1$. It was shown by M. Riesz in 1927 [R] that $H$ is a bounded operator on $L_p(\mathbb{R})$ for all $p > 1$, without giving a good estimate of the constants. His method was to show that the problem is roughly equivalent to the following problem. Given a function defined by a Fourier series $f(t) = \sum_{j\in\mathbb{Z}} x_j e(jt)$, the "conjugate function" is $\tilde{f}(t) = \sum_{j>0} x_j e(jt) - \sum_{j<0} x_j e(jt)$ (strictly, this multiplied by $-i$). Show that there is a constant $A_p$ such that $\|\tilde{f}\|_p \leq A_p\|f\|_p$ for $f \in L_p[0,1]$ (note that Parseval's identity gives at once $A_2 = 1$). See [Zyg], chapters 7 and 16.

Pichorides [Pich] developed the method to give exact values:

$$\|H\|_p = \begin{cases} \tan(\pi/2p) & \text{for } 1 < p \leq 2, \\ \cot(\pi/2p) & \text{for } p \geq 2. \end{cases}$$

Note that the least value is 1, occurring when $p = 2$, and that $\|H\|_{p^*} = \|H\|_p$.

In the discrete case, the norm is not less than these values, but it is only known that the values are exact when $p$ is of the form $2^n$ or $2^n/(2^n - 1)$.

Among many other articles on the subject, we mention just two, both of which have generated further papers. [ONW] sets out a rather different approach to these theorems, and [HMW] presents weighted versions.

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