An inequality for the gamma function conjectured by D. Kershaw

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1. Introduction

It has long been known that $\Gamma(x) + \Gamma(1/x) \geq 2$; a quick proof is by convexity of $\Gamma(x)$ and $\Gamma(1/x)$. Gautschi [3] generalized this statement by showing that the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is not less than 1, which of course implies also that $\Gamma(x)\Gamma(1/x) \geq 1$. Since then, inequalities concerning $\Gamma(1/x)$ have been investigated in many further articles. Alzer [1] extended Gautschi's result to the power means $M_r[\Gamma(x), \Gamma(1/x)]$. Kershaw and Laforgia [5] showed that $[\Gamma(1 + 1/x)]^x$ is decreasing, while $x[\Gamma(1 + 1/x)]^x$ is increasing. Giordano and Laforgia [4] extended Gautschi's product inequality by proving that

$$\tfrac{1}{2}\Gamma\left(1 + x + \frac{1}{x}\right) \leq \Gamma(x)\Gamma\left(\frac{1}{x}\right) \leq \Gamma\left(1 + x + \frac{1}{x}\right).$$

In connection with these results, Donald Kershaw, in private communication, formulated the following conjecture: for all $x > 0$,

$$\Gamma(x) + \Gamma\left(\frac{1}{x}\right) \leq \Gamma\left(1 + x + \frac{1}{x}\right), \tag{1}$$

with equality only at $x = 1$. In a recent article [2], Alzer obtained the following result in this direction (alongside a further generalization of Gautschi's result on harmonic means): $\Gamma(x) + \Gamma(1/x) \leq b\Gamma(x + 1/x)$, where $b \approx 2.098$. Since $x + 1/x \geq 2$, this implies a version of (1) with an intervening factor $b/2$ (which, of course, fails to reproduce equality at 1). The methods of [2] rely on a considerable number of specific values of the gamma function and higher derivatives.

Here we will prove that Kershaw's conjecture is true, without any extra factor.

Since (1) is unchanged when x is replaced by $1/x$, it is enough to prove it for $x > 1$. We use an assortment of different methods on different parts of the domain. The inequality (indeed, a rather stronger one) is obtained quite easily for all $x \geq 2$. For $\frac{5}{4} \leq x \leq 2$, we use convexity of $\Gamma(x)$ and $\Gamma(1/x)$ to derive linear bounds for the two sides of (1), which then only need to be compared at the end points; we do this on two shorter intervals. Only a few specific values of $\Gamma(x)$ are needed, to no great degree of accuracy.

The most interesting part of the problem is for x close to 1. Both sides of (1) have derivative 0 at 1, so there is no longer any chance of deducing the result from linear upper and lower bounds. However, after substituting $1+x$ for x, a lower bound for the right-hand side of the form $2+c(x^2-x^3)$ can still (as before) be derived from the tangent to $\Gamma(x)$ at 3. To estimate the left-hand side, we now use the power series for $\Gamma(1+x)$. We apply the exact values of the first three coefficients, together with a bound for the remaining ones, to establish (1) for $1 < x \leq \frac{5}{4}$.

2. Proof of (1) for $x \geq 2$

Note that $\Gamma(1/x) < x$ for $x > 1$. For $x \geq 2$, we prove (1) with $\Gamma(1/x)$ replaced by x.

Case $x \geq 3$. Since $\Gamma(x) > \Gamma(3) = 2$ for $x > 3$, we have

$$\Gamma\left(1+x+\frac{1}{x}\right) - \Gamma(x) > \Gamma(1+x) - \Gamma(x) = (x-1)\Gamma(x) \geq 2(x-1) > x.$$

LEMMA 1. *Let $a > 0$, and let $P_a(x) = \Gamma(1+x+a) - \Gamma(x) - x$. Then $P_a(x)$ is increasing for $x \geq 2$.*

Proof. Since $\Gamma'(x)$ is increasing for all $x > 0$, so is $\Gamma(x+a) - \Gamma(x)$. Also, $\Gamma(x)$ is increasing for $x \geq 2$. The statement follows, since

$$\begin{aligned} P_a(x) &= (x+a)\Gamma(x+a) - \Gamma(x) - x \\ &= x(\Gamma(x+a) - \Gamma(x)) + a\Gamma(x+a) + x\Gamma(x) - \Gamma(x) - x \\ &= x(\Gamma(x+a) - \Gamma(x)) + a\Gamma(x+a) + (\Gamma(x) - 1)(x-1) - 1. \qquad \square \end{aligned}$$

Case $2\frac{1}{3} \leq x \leq 3$. Then $\Gamma(1+x+\frac{1}{x}) \geq \Gamma(1+x+\frac{1}{3})$. Our inequality follows by Lemma 1 provided that $P_{1/3}(2\frac{1}{3}) > 0$. We verify this:

$$P_{1/3}(2\tfrac{1}{3}) = \tfrac{8}{3}\Gamma(\tfrac{8}{3}) - \Gamma(\tfrac{7}{3}) - \tfrac{7}{3} \approx 4.012 - 1.191 - 2.333 = 0.488 > 0.$$

Case $2 \leq x \leq 2\frac{1}{3}$. We now have $\frac{1}{x} \geq \frac{3}{7}$ on the interval, so we verify

$$P_{3/7}(2) = \tfrac{17}{7}\Gamma(\tfrac{17}{7}) - 1 - 2 \approx 3.074 - 3 > 0.$$

3. Proof of (1) for $\frac{5}{4} \leq x \leq 2$

LEMMA 2. *$\Gamma(1/x)$ is a convex function of x for $x > 0$.*

Proof. Let $0 < x_1 < x_2$, and write $y_j = 1/x_j$ $(j = 1, 2)$. Choose a, b so that $\Gamma(1+y_j) = ay_j + b$ for $j = 1, 2$. Since Γ is convex, $\Gamma(1+y) \leq ay + b$ for $y_2 \leq y \leq y_1$. Also, since $\Gamma(1+y) = y\Gamma(y)$, we have $\Gamma(y) \le a + b/y$ for $y_2 \le y \le y_1$, with equality at y_1 and y_2. So $\Gamma(1/x) \le a + bx$ for $x_1 \le x \le x_2$, with equality at x_1 and x_2, so that $a + bx$ is the linear function agreeing with $\Gamma(1/x)$ at these points. □

(In general, if a function f is convex and increasing, then $f(1/x)$ is convex, but this is not true for decreasing f.)

Note. Let $G(x) = \Gamma(x) + \Gamma(1/x)$. It is shown in [2, Lemma 2] that $G(x)$ is decreasing on $(0, 1]$. This follows at once from our Lemma 2, since $G(x)$ is convex and $G'(1) = 0$. Of course, the inequality $G(x) \ge 2$ (for all x) follows.

LEMMA 3. *For all* $x > 0$,

$$\Gamma\left(1 + x + \frac{1}{x}\right) \ge 2 + (3 - 2\gamma)\frac{(x-1)^2}{x}.$$

Proof. By convexity of the gamma function,

$$\Gamma(3+y) \ge \Gamma(3) + y\Gamma'(3) = 2 + (3 - 2\gamma)y$$

for all $y > 0$. The statement follows, since

$$1 + x + \frac{1}{x} = 3 + \frac{(x-1)^2}{x}. \qquad \square$$

Proof of (1) for $\frac{5}{4} \le x \le 2$. We consider the intervals $[\frac{5}{4}, \frac{3}{2}]$ and $[\frac{3}{2}, 2]$ separately. Write $\Gamma(x) + \Gamma(1/x) = G(x)$. By Lemma 2, $G(x)$ is convex. Using Lemma 3, we define a *linear* function $F(x)$ that is a lower bound for $\Gamma(1 + x + 1/x)$ on the interval in question. The statement then follows on verification that $F(x) > G(x)$ at the end points.

Let $h(x) = (x-1)^2/x$. Then $h'(x) = 1 - 1/x^2$. Hence $h(x)$ is convex, and $h(x) \ge h(x_0) + (x - x_0)h'(x_0)$ for any $x, x_0 > 0$. For the interval $[\frac{5}{4}, \frac{3}{2}]$, take $x_0 = \frac{5}{4}$. We find that $h(x) \ge h_1(x)$ on the interval, where

$$h_1(x) = \tfrac{1}{20} + \tfrac{9}{25}\left(x - \tfrac{5}{4}\right).$$

Our linear lower bound is $F_1(x) = 2 + (3 - 2\gamma)h_1(x)$. Note that $h_1(\frac{3}{2}) = \frac{7}{50}$. The values are

$$F_1(\tfrac{5}{4}) \approx 2.092, \qquad G(\tfrac{5}{4}) = \Gamma(\tfrac{5}{4}) + \Gamma(\tfrac{4}{5}) \approx 0.906 + 1.164 = 2.070.$$

$$F_1(\tfrac{3}{2}) \approx 2.258, \qquad G(\tfrac{3}{2}) = \Gamma(\tfrac{3}{2}) + \Gamma(\tfrac{2}{3}) \approx 0.886 + 1.354 = 2.240.$$

For the interval $[\frac{3}{2}, 2]$, take $x_0 = \frac{3}{2}$, giving

$$h_2(x) = \tfrac{1}{6} + \tfrac{5}{9}\left(x - \tfrac{3}{2}\right),$$

with corresponding $F_2(x)$. Clearly, $F_2(\frac{3}{2}) > F_1(\frac{3}{2})$. Also, $h_2(2) = \frac{4}{9}$, and we find

$$F_2(2) \approx 2.820, \qquad G(2) = \Gamma(2) + \Gamma(\tfrac{1}{2}) \approx 2.772.$$

4. The power series for $\Gamma(1+x)$

We write the power series for $\Gamma(1+x)$ in the form $\sum_{n=0}^{\infty}(-1)^n a_n x^n$, since (as we now show) the coefficients alternate in sign. Note that $a_0 = \Gamma(1) = 1$. Now

$$\Gamma^{(n)}(x) = \int_0^\infty t^{x-1}e^{-t}(\log t)^n \, dt,$$

hence

$$a_n = \frac{(-1)^n}{n!}\Gamma^{(n)}(1) = \frac{1}{n!}\int_0^\infty e^{-t}(-\log t)^n \, dt.$$

The following bound is not optimal, but it is adequate for our purposes.

LEMMA 4. *With this notation, we have* $0 < a_n \le m$ *for* $n \ge 4$*, where* $m \le \frac{13}{12}$.

Proof. We have

$$\int_0^1 e^{-t}(-\log t)^n \, dt < \int_0^1 (-\log t)^n \, dt = \int_0^\infty u^n e^{-u} \, du = n!.$$

At the same time, this integral is greater than $e^{-1}n!$. Also, since $\log t < t^{1/2}$ for $t > 1$,

$$\int_1^\infty e^{-t}(\log t)^n dt < \int_1^\infty e^{-t}t^{n/2} \, dt < \Gamma\left(\frac{n}{2}+1\right) < \Gamma(n-1) = \frac{n!}{n(n-1)} \le \frac{n!}{12}$$

for $n \ge 4$. The statement follows. □

Note. Using the series expansion for e^{-t}, one finds that

$$\frac{1}{n!}\int_0^1 e^{-t}(-\log t)^n \, dt = 1 - \frac{1}{2!2^n} + \frac{1}{3!3^n} - \cdots.$$

One can deduce that $\lim_{n\to\infty} a_n = 1$ and $a_n < 1$ for all n. The authors are grateful to Pascal Sebah for these observations, and for the calculated values of a_n for $n \le 20$.

Meanwhile, explicit values for the first few coefficients can be derived more pleasantly as follows. We use the power series (convergent for $|x| < 1$)

$$\frac{\Gamma'(1+x)}{\Gamma(1+x)} = \sum_{n=0}^{\infty}(-1)^{n+1}c_n x^n,$$

where $c_0 = \gamma$ and $c_n = \zeta(n+1)$ for $n \ge 1$ [6, p. 12]. Now equating coefficients in the identity

$$\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)a_{n+1}x^n = \left(\sum_{n=0}^{\infty}(-1)^{n+1}c_n x^n\right)\left(\sum_{n=0}^{\infty}(-1)^n a_n x^n\right)$$

we see that

$$(n+1)a_{n+1} = c_n a_0 + c_{n-1}a_1 + \cdots + c_0 a_n$$

for all $n \geq 1$. In particular, $a_1 = c_0 = \gamma$,

$$a_2 = \tfrac{1}{2}(c_1 a_0 + c_0 a_1) = \tfrac{1}{2}(\zeta(2) + \gamma^2) \approx 0.9891,$$

$$a_3 = \tfrac{1}{3}(c_2 a_0 + c_1 a_1 + c_0 a_2) = \tfrac{1}{6}(2\zeta(3) + 3\zeta(2)\gamma + \gamma^3) \approx 0.9075.$$

5. Proof of (1) for $1 \leq x \leq \frac{5}{4}$

We now substitute $1+x$ for x, so that (1) becomes

$$\Gamma(1+x) + \Gamma\left(\frac{1}{1+x}\right) \leq \Gamma\left(2 + x + \frac{1}{1+x}\right) \tag{2}$$

We have to prove (2) for $0 \leq x \leq \frac{1}{4}$. We continue to use Lemma 3. In the new notation, this says

$$\Gamma\left(2 + x + \frac{1}{1+x}\right) \geq 2 + (3 - 2\gamma)\frac{x^2}{1+x}.$$

For $0 < x < 1$, we have $1/(1+x) > 1 - x$, hence

$$\Gamma\left(2 + x + \frac{1}{1+x}\right) \geq 2 + (3 - 2\gamma)(x^2 - x^3). \tag{3}$$

LEMMA 5. *For* $0 \leq x \leq \frac{1}{4}$,

$$\Gamma(1+x) \leq 1 - \gamma x + a_2 x^2 + b_3 x^3, \tag{4}$$

where $b_3 = -a_3 + \frac{4}{15}m \approx -0.619$.

Proof. Since the terms of the power series alternate in sign,

$$\Gamma(1+x) \leq 1 - \gamma x + a_2 x^2 - a_3 x^3 + m(x^4 + x^6 + \cdots),$$

and for $0 \leq x \leq \frac{1}{4}$,

$$x^4 + x^6 + \cdots = \frac{x^4}{1-x^2} = x^3 \frac{x}{1-x^2} \leq \tfrac{4}{15}x^3. \qquad \square$$

LEMMA 6. *For* $0 \leq y \leq \frac{1}{5}$,

$$\Gamma(1-y) \leq 1 + \gamma y + a_2 y^2 + c_3 y^3, \tag{5}$$

where $c_3 = a_3 + \frac{1}{4}m \approx 1.178$.

Proof. We have $\Gamma(1-y) = 1 + \gamma y + \sum_{n=2}^{\infty} a_n y^n$, and for $0 \le y \le \frac{1}{5}$,

$$a_4 y^4 + a_5 y^5 + \cdots \le m \frac{y^4}{1-y} = m y^3 \frac{y}{1-y} \le \tfrac{1}{4} m y^3. \qquad \square$$

LEMMA 7. *For* $0 \le x \le \frac{1}{4}$,

$$\Gamma\left(\frac{1}{1+x}\right) \le 1 + \gamma x + d_2 x^2 + d_3 x^3, \tag{6}$$

where

$$d_2 = a_2 - \tfrac{4}{5}\gamma, \qquad d_3 = c_3 - \tfrac{36}{25} a_2 \approx -0.246.$$

Proof. Note that $1/(1+x) = 1 - x/(1+x)$. We apply Lemma 6, with $y = x/(1+x)$, using the following estimates derived from convexity of $1/(1+x)$ and $1/(1+x)^2$: for $0 \le x \le \frac{1}{4}$,

$$\frac{1}{1+x} \le 1 - \tfrac{4}{5}x, \qquad \frac{1}{(1+x)^2} \le 1 - \tfrac{36}{25}x.$$

We obtain

$$\begin{aligned} \Gamma\left(\frac{1}{1+x}\right) &\le 1 + \gamma x(1 - \tfrac{4}{5}x) + a_2 x^2 (1 - \tfrac{36}{25}x) + c_3 x^3 \\ &= 1 + \gamma x + (a_2 - \tfrac{4}{5}\gamma) x^2 + (c_3 - \tfrac{36}{25} a_2) x^3. \qquad \square \end{aligned}$$

Proof of (2) for $0 \le x \le \frac{1}{4}$. By (3), (4), (6), for $0 \le x \le \frac{1}{4}$,

$$\Gamma\left(2 + x + \frac{1}{1+x}\right) - \Gamma(1+x) - \Gamma\left(\frac{1}{1+x}\right) \ge A_2 x^2 + A_3 x^3,$$

where

$$A_2 = 3 - \tfrac{6}{5}\gamma - 2a_2 \approx 0.329,$$

$$A_3 = -(3 - 2\gamma) - b_3 - d_3 \approx -1.845 + 0.619 + 0.246 = -0.980.$$

Clearly, $A_2 x^2 + A_3 x^3 > 0$ for $0 \le x \le \frac{1}{4}$. $\square$

Remark. The power series shows clearly that we cannot replace $\Gamma[1/(1+x)]$ by $\Gamma(1-x)$ in (2), illustrating the closeness of our inequality. Indeed, for small $x > 0$, we have the slightly stronger reverse inequality $\Gamma(1+x) + \Gamma(1-x) > \Gamma(3+x^2)$, since

$$\Gamma(1+x) + \Gamma(1-x) = 2 + 2a_2 x^2 + O(x^4), \qquad \Gamma(3+x^2) = 2 + (3-2\gamma)x^2 + O(x^4),$$

and $3 - 2\gamma < 2a_2$.

References

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