The Frullani integrals
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We consider integrals of the form

\[ I_f(a, b) = \int_0^\infty \frac{f(ax) - f(bx)}{x} \, dx, \]

where \( f \) is a continuous function (real or complex) on \([0, \infty)\) and \( a, b > 0 \). The basic theorem is as follows. I do not know many books or articles where proofs are given: two references (for which I am grateful to Nick Lord) are [Fer, p. 134–135] and [Tr].

**FRU1 THEOREM.** Consider the following conditions:

(C1) \( f(x) \to c_0 \) as \( x \to 0^+ \);
(C2) \( f(x) \to c_\infty \) as \( x \to \infty \);
(C3) there exists \( K \) such that \( |F(x)| \leq K \) for all \( x > 0 \), where \( F(x) = \int_0^x f(t) \, dt \).

Under conditions (C1) and (C2), we have

\[ I_f(a, b) = (c_0 - c_\infty)(\log b - \log a). \tag{1} \]

Under conditions (C1) and (C3), we have

\[ I_f(a, b) = c_0(\log b - \log a). \tag{2} \]

**Proof.** We may assume that \( b > a \): interchanging \( a \) and \( b \) then gives the case \( b < a \).

Let \( 0 < \delta < X \). The substitution \( ax = y \) gives

\[ \int_\delta^X \frac{f(ax)}{x} \, dx = \int_{a\delta}^{aX} \frac{f(y)}{y} \, dy. \]

So

\[ \int_\delta^X \frac{f(ax) - f(bx)}{x} \, dx = \int_{a\delta}^{aX} \frac{f(y)}{y} \, dy - \int_{b\delta}^{bX} \frac{f(y)}{y} \, dy = I(\delta) - I(X), \]

where, for any \( r > 0 \),

\[ I(r) = \int_{ar}^{br} \frac{f(y)}{y} \, dy. \]

Let \( \varepsilon > 0 \) be given. Under condition (C1), there exists \( \delta_0 > 0 \) such that if \( 0 < y \leq b\delta_0 \), then \( |f(y) - c_0| \leq \varepsilon \). Then for \( \delta \leq \delta_0 \), we have \( I(\delta) = c_0(\log b - \log a) + r_1(\delta) \), where

\[ r_1(\delta) = \int_{a\delta}^{b\delta} \frac{f(y) - c_0}{y} \, dy, \]
hence \(|r_1(\delta)| \leq \varepsilon (\log b - \log a)\). So \(I(\delta) \to c_0(\log b - \log a)\) as \(\delta \to 0^+\).

In exactly the same way, condition (C2) implies that \(I(X) \to c_\infty(\log b - \log a)\) as \(X \to \infty\).

Under condition (C3), integration by parts gives

\[
I(X) = \left[ \frac{F(y)}{y} \right]_a^b X + \int_a^b \frac{F(y)}{y^2} \, dy,
\]

hence

\[
|I(X)| \leq \frac{2K}{aX} + K \int_a^b \frac{1}{y^2} \, dy < \frac{3K}{aX},
\]

so \(I(X) \to 0\) as \(X \to \infty\). (Of course, this also shows that \(\int_1^\infty \frac{f(x)}{x} \, dx\) converges; this statement could be taken as the hypothesis instead of (C3)). \(\square\)

We record a number of particular examples which are transparently cases of (1) or (2), without repeatedly writing out the integral expressions:

- \(f(x) = e^{-x}\), \(I_f(a, b) = \log b - \log a\)
- \(f(x) = e^{-x^2}\), \(I_f(a, b) = \log b - \log a\)
- \(f(x) = \frac{1}{1 + x^2}\), \(I_f(a, b) = \log b - \log a\)
- \(f(x) = \cos x\), \(I_f(a, b) = \log b - \log a\)
- \(f(x) = e^{-x} \cos x\), \(I_f(a, b) = \log b - \log a\)
- \(f(x) = \tan^{-1} x\), \(I_f(a, b) = \frac{\pi}{2} (\log a - \log b)\)
- \(f(x) = \tanh x\), \(I_f(a, b) = \log a - \log b\)

Among these examples, \(\cos x\) is the only one satisfying (C3), but not (C2). Of course, the first three examples satisfy both.

Next, we state a simple extension of the Theorem.

FRU2. Let \(F(x) = \sum_{j=1}^n m_j f(a_j x)\), where \(a_j > 0\) for \(1 \leq j \leq n\) and \(\sum_{j=1}^n m_j = 0\). If \(f\) satisfies (C1) and (C2), then

\[
\int_0^\infty \frac{F(x)}{x} \, dx = - (c_0 - c_\infty) \sum_{j=1}^n m_j \log a_j.
\]

(3)

If \(f\) satisfies (C1) and (C3), the same applies with \(c_\infty\) replaced by 0.

Proof. Write \(M_j = m_1 + \cdots + m_j\). By Abel summation,

\[
F(x) = \sum_{j=1}^{n-1} M_j [f(a_j x) - f(a_{j+1} x)].
\]
By (1),
\[ \int_0^\infty \frac{F(x)}{x} \, dx = -\sum_{j=1}^{n-1} M_j (\log a_j - \log a_{j+1}) = -\sum_{j=1}^n m_j \log a_j. \]

**Double integral method for (1).**

For monotonic \( f \), the following is an alternative method for (1) (but not (2)). It appears in some books, at least for the special case \( f(x) = e^{-x} \), e.g. [AAR, p. 27].

Note that for \( x, y > 0 \),
\[ \frac{1}{y} \frac{d}{dx} f(xy) = \frac{1}{x} \frac{d}{dy} f(xy) = f'(xy). \]
Hence
\[ \int_a^b f'(xy) \, dy = \left[ \frac{1}{x} f(xy) \right]_{y=a}^{y=b} = \frac{f(bx) - f(ax)}{x}. \]
So, reversing the double integral, we obtain
\[ -I_f(a,b) = \int_0^\infty \int_a^b f'(xy) \, dy \, dx = \int_a^b \int_0^\infty f'(xy) \, dx \, dy. \]
But
\[ \int_0^\infty f'(xy) \, dx = \left[ \frac{1}{y} f(xy) \right]_{x=0}^{x=\infty} = \frac{1}{y} (c_\infty - c_0). \]
So
\[ -I_f(a,b) = (c_\infty - c_0) \int_a^b \frac{1}{y} \, dy = (c_\infty - c_0)(\log b - \log a). \]

**Some derived integrals**

We give two applications of the case \( f(x) = \cos x \). First, since \( 2 \sin ax \sin bx = \cos(a - b)x - \cos(a + b)x \), we have, for \( a > b > 0 \),
\[ \int_0^\infty \frac{\sin ax \sin bx}{x} \, dx = \frac{1}{2} \log \frac{a+b}{a-b}. \]
Second, since \( \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x \), we have
\[ \int_0^\infty \frac{\sin^3 x}{x^2} \, dx = \left[ -\frac{\sin^3 x}{x} \right]_0^\infty + \int_0^\infty \frac{1}{x} \left( \frac{3}{4} \cos x - \frac{3}{4} \cos 3x \right) \, dx = \frac{3}{4} \log 3. \]
Integrals of this type are discussed more generally in [Tr].

We now describe some applications of the case \( f(x) = e^{-x} \). Written out explicitly, the statement is
\[ \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx = \log b - \log a. \]
The substitution $x = -\log y$ transforms this to
\[
\int_0^1 \frac{y^{b-1} - y^{a-1}}{\log y} \, dy = \log b - \log a.
\]
So, for example,
\[
\int_0^1 \frac{y - 1}{\log y} \, dy = \log 2.
\]

Now consider the integral
\[
J = \int_0^\infty \frac{1}{x^2} (1 - e^{-ax})(1 - e^{-bx}) \, dx.
\]
Integration by parts gives $J = J_1 + J_2$, where
\[
J_1 = \left[ -\frac{1}{x}(1 - e^{-ax})(1 - e^{-bx}) \right]_0^\infty,
\]
\[
J_2 = \int_0^\infty \frac{1}{x} \left( ae^{-ax} + be^{-bx} - (a + b)e^{-(a+b)x} \right).
\]
For $x > 0$, we have $1 - e^{-x} = \int_0^x e^{-t} \, dt < x$, from which it is clear that $J_1 = 0$. Writing the bracketed factor in $J_2$ as $a(e^{-ax} - e^{-(a+b)x}) + b(e^{-bx} - e^{-(a+b)x})$, we see that
\[
J = J_2 = a[\log(a + b) - \log a] + b[\log(a + b) - \log b]
= (a + b) \log(a + b) - a \log a - b \log b.
\]

Next, consider the function
\[
E(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt.
\]
By reversal of the implied double integral, one sees easily that $\int_0^\infty E(x) \, dx = 1$. Now (re-using the notation $J$ for an integral to be evaluated) let
\[
J = \int_0^\infty e^{-ax} E(x) \, dx,
\]
where $a > 0$. Reversing the double integral and applying (4), we find
\[
J = \int_0^\infty \int_x^\infty \frac{e^{-t}}{t} \, dt \, dx
= \int_0^\infty \frac{e^{-t}}{t} \int_0^t e^{-ax} \, dx \, dt
= \int_0^\infty e^{-t}(1 - e^{-at}) \, dt
= \frac{1}{a} \log(1 + a).
\]
Another application of (4) occurs in the derivation of the integral representation for the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$ [AAR, p. 26–27].

Further examples of Frullani integrals are given in [AABM].

References


