The Frullani integrals

Notes by G.J.O. Jameson

We consider integrals of the form

$$I_f(a,b) = \int_0^\infty \frac{f(ax) - f(bx)}{x}\,dx,$$

where f is a continuous function (real or complex) on $(0,\infty)$ and a, $b > 0$. If $f(x)$ tends to a non-zero limit at 0, then the separate integrals of $f(ax)/x$ and $f(bx)/x$ diverge at 0, and a similar comment applies at infinity. The point is that under suitable conditions, the integral of the *difference* converges.

The basic theorem is as follows. I do not know many books or articles where proofs are given: two references (for which I am grateful to Nick Lord) are [Fer, p. 134–135] and [Tr].

FRU1 THEOREM. *Consider the following conditions:*

(C1) $f(x) \to c_0$ *as* $x \to 0^+$*;*

(C2) $f(x) \to c_\infty$ *as* $x \to \infty$*;*

(C3) there exists K *such that* $|F(x)| \le K$ *for all* $x > 0$*, where* $F(x) = \int_0^x f(t)\,dt$*.*

Under conditions (C1) and (C2), we have

$$I_f(a,b) = (c_0 - c_\infty)(\log b - \log a). \tag{1}$$

Under conditions (C1) and (C3), we have

$$I_f(a,b) = c_0(\log b - \log a). \tag{2}$$

Proof. We may assume that $b > a$: interchanging a and b then gives the case $b < a$. Let $0 < \delta < X$. The substitution $ax = y$ gives

$$\int_\delta^X \frac{f(ax)}{x}\,dx = \int_{a\delta}^{aX} \frac{f(y)}{y}\,dy.$$

So

$$\begin{aligned}\int_\delta^X \frac{f(ax)-f(bx)}{x}\,dx &= \int_{a\delta}^{aX} \frac{f(y)}{y}\,dy - \int_{b\delta}^{bX} \frac{f(y)}{y}\,dy \\ &= \int_{a\delta}^{b\delta} \frac{f(y)}{y}\,dy - \int_{aX}^{bX} \frac{f(y)}{y}\,dy \\ &= I(\delta) - I(X),\end{aligned}$$

where we write

$$I(r) = \int_{ar}^{br} \frac{f(y)}{y}\,dy$$

for any $r > 0$. Let $\varepsilon > 0$ be given. Under condition (C1), there exists $\delta_0 > 0$ such that if $0 < y \le b\delta_0$, then $|f(y) - c_0| \le \varepsilon$. Then for $\delta \le \delta_0$, we have $I(\delta) = c_0(\log b - \log a) + r_1(\delta)$, where

$$r_1(\delta) = \int_{a\delta}^{b\delta} \frac{f(y) - c_0}{y}\,dy,$$

hence

$$|r_1(\delta)| \le \int_{a\delta}^{b\delta} \frac{\varepsilon}{y}\,dy = \varepsilon(\log b - \log a).$$

So $I(\delta) \to c_0(\log b - \log a)$ as $\delta \to 0^+$.

In exactly the same way, condition (C2) implies that $I(X) \to c_\infty(\log b - \log a)$ as $X \to \infty$.

Under condition (C3), integration by parts gives

$$I(X) = \left[\frac{F(y)}{y}\right]_{aX}^{bX} + \int_{aX}^{bX} \frac{F(y)}{y^2}\,dy,$$

hence

$$|I(X)| \le \frac{2K}{aX} + K\int_{aX}^{bX} \frac{1}{y^2}\,dy < \frac{3K}{aX},$$

so $I(X) \to 0$ as $X \to \infty$. (Of course, this also shows that $\int_1^\infty \frac{f(x)}{x}\,dx$ converges; this statement could be taken as the hypothesis instead of (C3)). □

In particular, $I_f(1,b)$ equals $(c_0 - c_\infty)\log b$ in case (1), and $c_0 \log b$ in case (2).

We record a number of particular examples which are transparently cases of (1) or (2), without repeatedly writing out the integral expressions:

$f(x)$	$I_f(a,b)$
e^{-x}	$\log b - \log a$
$1/(1+x^2)$	$\log b - \log a$
$\cos x$	$\log b - \log a$
$e^{-x}\cos x$	$\log b - \log a$
$\sin x/x$	$\log b - \log a$
$\tan^{-1} x$	$\frac{\pi}{2}(\log a - \log b)$
$\tanh x$	$\log a - \log b$

Among these examples, $\cos x$ is the only one satisfying (C3), but not (C2). A simple alternative proof for this case is given in [Jam, p. 280].

Many further examples of Frullani integrals are given in [AABM].

We now state a simple extension of the Theorem.

FRU2. *Let* $G(x) = \sum_{j=1}^{n} m_j f(a_j x)$, *where* $a_j > 0$ *for* $1 \le j \le n$ *and* $\sum_{j=1}^{n} m_j = 0$. *If* f *satisfies (C1) and (C2), then*

$$\int_0^\infty \frac{G(x)}{x}\,dx = (c_\infty - c_0)\sum_{j=1}^{n} m_j \log a_j. \tag{3}$$

If f *satisifes (C1) and (C3), the same applies with* c_∞ *replaced by 0.*

Proof. Write $M_j = m_1 + \cdots + m_j$. By Abel summation, since $M_n = 0$,

$$G(x) = \sum_{j=1}^{n-1} M_j[f(a_j x) - f(a_{j+1}x)].$$

By (1),

$$\int_0^\infty \frac{G(x)}{x}\,dx = (c_\infty - c_0)\sum_{j=1}^{n-1} M_j(\log a_j - \log a_{j+1}) = (c_\infty - c_0)\sum_{j=1}^{n} m_j \log a_j. \qquad \square$$

Double integral method for (1).

For monotonic f, the following is an alternative method for (1) (but not (2)). The method has been in circulation for a long time, at least for the special case $f(x) = e^{-x}$. For example, it can be seen in [Cou, p. 240].

Note that for x, $y > 0$,

$$\frac{1}{y}\frac{d}{dx}f(xy) = \frac{1}{x}\frac{d}{dy}f(xy) = f'(xy).$$

Hence

$$\int_a^b f'(xy)\,dy = \Big[\frac{1}{x}f(xy)\Big]_{y=a}^{b} = \frac{f(bx) - f(ax)}{x}.$$

Since $f'(x)$ is of constant sign, reversal of the following double integral is justfied:

$$-I_f(a,b) = \int_0^\infty \int_a^b f'(xy)\,dy\,dx = \int_a^b \int_0^\infty f'(xy)\,dx\,dy.$$

But

$$\int_0^\infty f'(xy)\,dx = \Big[\frac{1}{y}f(xy)\Big]_{x=0}^{\infty} = \frac{1}{y}(c_\infty - c_0).$$

So

$$-I_f(a,b) = (c_\infty - c_0)\int_a^b \frac{1}{y}\,dy = (c_\infty - c_0)(\log b - \log a). \qquad \square$$

Some applications of the case $f(x) = e^{-x}$.

This is undoubtedly the best known Frullani integral. Written out explicitly, the statement is

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log b - \log a. \tag{4}$$

The substitution $x = -\log y$ transforms (4) to

$$\int_0^1 \frac{y^{b-1} - y^{a-1}}{\log y}\, dy = \log b - \log a.$$

So, for example,

$$\int_0^1 \frac{y-1}{\log y}\, dy = \log 2$$

(cf. [BM, p. 97]).

Now consider the integral

$$I_1 = \int_0^\infty \frac{1}{x^2}(1 - e^{-ax})(1 - e^{-bx})\, dx.$$

Integration by parts gives $I_1 = A_1 + J_1$, where

$$A_1 = \Big[-\frac{1}{x}(1 - e^{-ax})(1 - e^{-bx}) \Big]_0^\infty,$$

$$J_1 = \int_0^\infty \frac{1}{x}\left(ae^{-ax} + be^{-bx} - (a+b)e^{-(a+b)x} \right).$$

Since $0 < 1 - e^{-x} < x$ for $x > 0$, we have $\frac{1}{x}(1 - e^{-ax})(1 - e^{-bx}) < abx$, hence $A_1 = 0$. So by (3),

$$I_1 = J_1 = (a+b)\log(a+b) - a\log a - b\log b.$$

Next, consider the function

$$E(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

By reversal of the implied double integral, one sees easily that $\int_0^\infty E(x)\, dx = 1$. Now let

$$I_2 = \int_0^\infty e^{-ax} E(x)\, dx,$$

where $a > 0$. Reversing the double integral and applying (4), we find

$$\begin{aligned} I_2 &= \int_0^\infty e^{-ax} \int_x^\infty \frac{e^{-t}}{t}\, dt\, dx \\ &= \int_0^\infty \frac{e^{-t}}{t} \int_0^t e^{-ax}\, dx\, dt \\ &= \int_0^\infty \frac{e^{-t}(1 - e^{-at})}{at}\, dt \\ &= \frac{1}{a}\log(1+a). \end{aligned}$$

Applications of the case $f(x) = \cos x$

We describe two applications of this case. First, since $2\sin ax \sin bx = \cos(a-b)x - \cos(a+b)x$, we have, for $a > b > 0$,

$$\int_0^\infty \frac{\sin ax \sin bx}{x}\, dx = \tfrac{1}{2} \log \frac{a+b}{a-b}.$$

Second, consider the integral

$$I_3 = \int_0^\infty \frac{\sin^5 x}{x^2}\, dx.$$

Since $16\sin^5 x = 10\sin x - 5\sin 3x + \sin 5x$, we have by (3)

$$\begin{aligned} 16I_3 &= \Big[-\frac{\sin^5 x}{x}\Big]_0^\infty + \int_0^\infty \frac{1}{x}(10\cos x - 15\cos 3x + 5\cos 5x)\, dx \\ &= -(10\log 1 - 15\log 3 + 5\log 5) \\ &= 15\log 3 - 5\log 5. \end{aligned}$$

Integrals of this type are discussed more generally in [Tr].

References

[AABM] Matthew Albano, Tewodros Amdeberhan, Erin Beyerstedt and Victor H. Moll, The integrals in Gradshteyn and Ryzhik. Part 15: Frullani integrals, *Scientica, Series A: Mathematical Sciences* **19** (2010), 113–119.

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[Jam] G. J. O. Jameson, Sine, cosine and exponential integrals, *Math. Gazette* **99** (2015), 276–289.

[Tr] J. Trainin, Integrating expressions of the form $\frac{\sin^n x}{x^m}$ and others, *Math. Gazette* **94** (2010), 216–223.

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